



# Degenerate equations with double resonance

Peng Jin<sup>a</sup>, Nikolaos S. Papageorgiou<sup>b,c</sup>,  
Vicențiu D. Rădulescu<sup>d,e,f,g,\*</sup>

<sup>a</sup> School of Mathematics and Statistics, HNP-LAMA, Central South University, Changsha 410083, Hunan, People's Republic of China

<sup>b</sup> National Technical University, Department of Mathematics, Zografou Campus, Athens 15780, Greece

<sup>c</sup> Center for Applied Mathematics of Guangxi, Yulin Normal University, Yulin 537000, People's Republic of China

<sup>d</sup> AGH University of Kraków, Faculty of Applied Mathematics, al. Mickiewicza 30, 30-059 Kraków, Poland

<sup>e</sup> Brno University of Technology, Faculty of Electrical Engineering and Communication, Technická 3058/10, Brno 61600, Czech Republic

<sup>f</sup> Simion Stoilow Institute of Mathematics of the Romanian Academy, 010702 Bucharest, Romania

<sup>g</sup> Scientific Research Center, Baku Engineering University, AZ0102 Baku, Azerbaijan



## ARTICLE INFO

### Article history:

Received 6 July 2025

Available online 19 May 2026

### MSC:

35J20

35J75

47J10

58C40

58E05

### Keywords:

Muckenhoupt weights

Generalized Orlicz spaces

Resonant reaction

## ABSTRACT

We consider a Dirichlet problem driven by the nonautonomous, degenerate  $p$ -Laplacian, with a reaction resonant at  $\pm\infty$  and at 0. Using variational tools and critical groups (Morse theory), we show that the problem has at least two nontrivial bounded solutions.

© 2026 Elsevier Masson SAS. All rights are reserved, including those for text and data mining, AI training, and similar technologies.

\* Corresponding author at: Faculty of Applied Mathematics, AGH University of Kraków, al. Mickiewicza 30, 30-059 Kraków, Poland.

E-mail addresses: [mathpengjin@163.com](mailto:mathpengjin@163.com) (P. Jin), [npapg@math.ntua.gr](mailto:npapg@math.ntua.gr) (N.S. Papageorgiou), [radulescu@inf.ucv.ro](mailto:radulescu@inf.ucv.ro) (V.D. Rădulescu).

Critical groups  
 Strong deformation retract  
 Long exact sequence

---

## 1. Introduction

Let  $\Omega \subseteq \mathbb{R}^N$  be a bounded domain with a  $C^2$ -boundary  $\partial\Omega$ . In this paper, we study the following Dirichlet problem

$$-\Delta_p^a u(z) = f(z, u(z)) \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0, \quad 1 < p < N. \quad (1)$$

If  $a \in C^{0,1}(\bar{\Omega})$ ,  $a(z) > 0$  for all  $z \in \Omega$ , then by  $\Delta_p^a$  we denote the weighted  $p$ -Laplace differential operator defined by

$$\Delta_p^a u = \operatorname{div}(a(z)|Du|^{p-2}Du).$$

If  $\min_{\bar{\Omega}} a > 0$ , then this operator is a nonautonomous version the standard  $p$ -Laplace differential operator and the boundary value problem can be treated using essentially the same tools and methods. The main reason behind this similarity, is that both operators exhibit balanced growth and so we can use the global regularity theory (that is, regularity up to the boundary) of Lieberman [7]. This theory provides powerful tools for the study of the problem.

The interest in the study of this type of problem is twofold. On the one hand, there are physical motivations, since the weighted differential operator has been applied to describe steady-state solutions of reaction-diffusion problems in biophysics, plasma physics, and chemical reaction analysis. The prototype equation for these models can be written in the form

$$u_t = \Delta_p^a u(z) + g(x, u).$$

In this framework, the function  $u$  generally stands for a concentration, the term  $\Delta_p^a u(z)$  corresponds to the diffusion with coefficient  $a(z)|Du|^{p-2}$ , while  $g(x, u)$  represents the reaction term related to source and loss processes; see Cherfilil-Il'yasov [2] and Singer [12]. On the other hand, such operators provide a valuable framework for explaining the behavior of highly anisotropic materials whose hardening properties, which are linked to the exponent governing the propagation of the gradient variable and the modulating coefficient  $a(\cdot)$  differ considerably with the point, where  $a(\cdot)$  dictates the geometry of an anisotropic material.

A feature of this paper is that the potential that describes the differential operator satisfies general regularity assumptions and it belongs to the  $p$ -Muckenhoupt class. Accordingly, the thorough spectral and the qualitative analysis contained in the present paper are developed in the general abstract setting of Musielak-Orlicz-Sobolev spaces.

Another feature of this paper is that we assume  $\min_{\Omega} a = 0$ . For such problems (degenerate equations), there is no global regularity theory and this makes the study of degenerate equations more difficult. In fact, little work has been done in this direction. We mention the books of Drábek-Kufner-Nicolosi [3] and of Heinonen-Kilpeläinen-Martio [4] and the references therein. In particular there are no works on degenerate  $p$ -Laplacian equations with a  $(p - 1)$ -sublinear and resonant reaction, because there was no spectral analysis of the Dirichlet  $-\Delta_p^a$ . Recently, Papageorgiou-Pudélko-Rădulescu [9] developed such a spectral description of  $-\Delta_p^a$  with Dirichlet boundary condition. In the present paper, we establish a multiplicity theorem for problem (1), when the reaction  $f(z, x)$  is both resonant as  $x \rightarrow \pm\infty$  and as  $x \rightarrow 0$  (double resonance). In addition to the spectral analysis developed in [9], our approach uses variational tools and the theory of critical groups (Morse theory).

## 2. Mathematical background and hypotheses

Another consequence of the degeneracy of the differential operator is that the standard Lebesgue and Sobolev spaces do not provide an adequate functional framework and we need to pass to weighted spaces (a special case of generalized Orlicz spaces).

Let  $L^0(\Omega)$  be the space of measurable functions  $u : \Omega \rightarrow \mathbb{R}$ . As usual we identify two such functions which differ only on a Lebesgue-null set. We say that  $a \in L^0(\Omega)$  is a weight, if

$$a \in L_{\text{loc}}^1(\Omega) \text{ and } a(z) > 0 \text{ for a.a. } z \in \Omega.$$

Let  $p > 1$ . We denote by  $\mathcal{A}_p$  the family of  $p$ -Muckenhoupt weights, that is,  $a \in \mathcal{A}_p$  if and only if  $a(\cdot)$  is a weight and for all balls  $B \subseteq \mathbb{R}^N$

$$\left( \frac{1}{|B|_N} \int_B a \, dz \right) \left( \frac{1}{|B|_N} \int_B a^{-\frac{1}{p-1}} \, dz \right)^{p-1} \leq M(p, a)$$

with  $|\cdot|_N$  being the Lebesgue measure on  $\mathbb{R}^N$  (see Muckenhoupt [8] and [4, p.297]). A typical example of such a weight is

$$a(z) = |z|^\mu \text{ for all } z \in \Omega, \quad -N < \mu < N(p-1).$$

The importance of this class of weights comes from the fact that in this case, the Hardy-Littlewood maximal operator is bounded on the corresponding weighted Lebesgue space (see [4, p.303]). We will use elements from a subclass of  $\mathcal{A}_p$ . So, let  $T = \left[ \frac{1}{p-1}, \infty \right) \cap \left( \frac{N}{p}, \infty \right)$  and  $s \in T$ . We consider the following subset of  $\mathcal{A}_p$

$$\mathcal{A}_p^s = \{a \in \mathcal{A}_p : a^{-1} \in L^s(\Omega)\}.$$

If  $a(z) = |z|^\mu$  for all  $z \in \Omega$  and  $-\frac{N}{s} < \mu < \frac{N}{s}$ , then  $a \in \mathcal{A}_p^s$ . Note that if  $1 < q < p$ , then  $\mathcal{A}_q \subseteq \mathcal{A}_p$ .

Our hypotheses on the weight  $a(\cdot)$  are the following

( $H_0$ )  $a \in C^{0,1}(\bar{\Omega}) \cap \mathcal{A}_p^s$  with  $1 < p < N$ ,  $s \in T$ .

Let  $\theta(z, t) = a(z)t^p$  for all  $z \in \Omega$ , all  $t \geq 0$ . The function  $\theta(z, \cdot)$  is uniformly convex for all  $z \in \Omega$ . Using  $\theta(z, t)$ , we can define the Lebesgue-Orlicz space  $L^\theta(\Omega)$  by

$$L^\theta(\Omega) = \left\{ u \in L^0(\Omega) : \rho_\theta(u) = \int_{\Omega} \theta(z, |u|) dz < \infty \right\}.$$

We call  $\rho_\theta(\cdot)$  the modular function corresponding to  $\theta(\cdot, \cdot)$ . We equip  $L^\theta(\Omega)$  with the so-called “Luxemburg norm”  $\|\cdot\|_\theta$  defined by

$$\|u\|_\theta = \inf \left\{ \lambda > 0 : \rho_\theta\left(\frac{u}{\lambda}\right) \leq 1 \right\} \text{ for all } u \in L^\theta(\Omega).$$

Alternatively, we can consider the weighted space  $L_a^p(\Omega)$  equipped with the norm  $|\cdot|_a$  defined by

$$|u|_a = \left( \int_{\Omega} a(z) |u|^p dz \right)^{\frac{1}{p}} \text{ for all } u \in L_a^p(\Omega).$$

Evidently  $L^\theta(\Omega) = L_a^p(\Omega)$  and the two norms are equal, that is  $\|u\|_\theta = |u|_a$  for all  $u \in L^\theta(\Omega) = L_a^p(\Omega)$ . Indeed, let  $\lambda_n \downarrow \|u\|_\theta$  with  $\rho_\theta\left(\frac{u}{\lambda_n}\right) \leq 1$  for all  $n \in \mathbb{N}$ . Then

$$\begin{aligned} \theta\left(z, \frac{|u|}{\lambda_n}\right) &\leq \frac{\lambda_n}{\lambda_{n+1}} \theta\left(z, \frac{|u|}{\lambda_n}\right) \\ &\leq \theta\left(z, \frac{|u|}{\lambda_{n+1}}\right) \quad (\text{using the convexity of } \theta(z, \cdot)), \\ &\Rightarrow \left\{ \theta\left(\cdot, \frac{|u|}{\lambda_n}\right) \right\}_{n \in \mathbb{N}} \text{ is nondecreasing.} \end{aligned}$$

The monotone convergence theorem implies that

$$\begin{aligned} \rho_\theta\left(\frac{u}{\|u\|_\theta}\right) &\leq 1, \\ \Rightarrow |u|_a^p &\leq \|u\|_\theta^p. \end{aligned}$$

On the other hand, note that

$$\int_{\Omega} a(z) \left( \frac{|u|}{|u|_a} \right)^p dz \leq 1,$$

$$\Rightarrow \|u\|_{\theta} \leq |u|_a.$$

We conclude that

$$\|\cdot\|_{\theta} = |\cdot|_a. \quad (2)$$

Normed this way  $L^{\theta}(\Omega) = L_a^p(\Omega)$  becomes a Banach space which is separable and reflexive (in fact uniformly convex). Using  $L^{\theta}(\Omega)$  we can define the corresponding Sobolev-Orlicz space

$$W^{1,\theta}(\Omega) = \{u \in L^{\theta}(\Omega) : |Du| \in L^{\theta}(\Omega)\},$$

with  $Du$  being the weak gradient of  $u$ . We equip this space with the norm  $\|\cdot\|_{1,\theta}$  defined by

$$\|u\|_{1,\theta} = \|u\|_{\theta} + \|Du\|_{\theta} \quad \text{for all } u \in W^{1,\theta}(\Omega),$$

with  $\|Du\|_{\theta} = \| |Du| \|_{\theta}$ . Of course we can also consider the norm  $|\cdot|_{1,a}$  defined by

$$|u|_{1,a} = |u|_a + |Du|_a \quad \text{for all } u \in W^{1,\theta}(\Omega).$$

Evidently we have  $\|\cdot\|_{1,\theta} = |\cdot|_{1,a}$  (see (2)). Let

$$W_0^{1,\theta}(\Omega) = \overline{C_c^{\infty}(\Omega)}^{\|\cdot\|_{1,\theta}},$$

with  $C_c^{\infty}(\Omega)$  being the space of  $C^{\infty}$ -functions on  $\Omega$  with compact support. The spaces  $W^{1,\theta}(\Omega)$ ,  $W_0^{1,\theta}(\Omega)$  are Banach spaces which are separable and reflexive (in fact uniformly convex). On  $W_0^{1,\theta}(\Omega)$  the Poincare inequality is valid, namely we have

$$\|u\|_{\theta} \leq c \|Du\|_{\theta}, \quad \text{for some } c > 0, \text{ all } u \in W_0^{1,\theta}(\Omega) \quad (\text{see [4, p.307]}). \quad (3)$$

This means that on  $W_0^{1,\theta}(\Omega)$ , we can consider the equivalent norm  $\|\cdot\|$  defined by

$$\|u\| = \|Du\|_{\theta} = |Du|_a = \left[ \int_{\Omega} a(z) |Du|^p dz \right]^{\frac{1}{p}}, \quad \text{for all } u \in W_0^{1,\theta}(\Omega).$$

We have the following embeddings between these spaces. In what follows,  $p_s = \frac{sp}{s+1} < p$  and  $p_s^* = \frac{Np_s}{N-p_s}$ . Also by  $\hookrightarrow$  we denote continuous embedding and by  $\hookrightarrow^c$  compact embedding.

**Proposition 1.**

- (a) If  $a \in C^{0,1}(\bar{\Omega}) \cap \mathcal{A}_p$ , then  $W_0^{1,\theta}(\Omega) \xhookrightarrow{c} L^\theta(\Omega)$  (Papageorgiou-Rădulescu-Zhang [11]).  
 (b) If hypotheses  $H_0$  hold, then  $W_0^{1,\theta}(\Omega) \hookrightarrow W_0^{1,p_s}(\Omega)$ ,  $W_0^{1,\theta}(\Omega) \hookrightarrow L^r(\Omega)$  for all  $1 \leq r \leq p_s^*$  and  $W_0^{1,\theta}(\Omega) \xhookrightarrow{c} L^r(\Omega)$  for all  $1 \leq r < p_s^*$  (see [3]).

If  $u \in L^0(\Omega)$ , then we write  $0 \prec u$  if and only if all  $K \subseteq \Omega$  compact,  $0 < c_K \leq u(z)$  for a.a.  $z \in K$ .

Consider the following nonlinear eigenvalue problem

$$-\Delta_p^a u(z) = \hat{\lambda} a(z) |u(z)|^{p-2} u(z) \text{ in } \Omega, \quad u|_{\partial\Omega} = 0. \quad (4)$$

This problem was investigated by Papageorgiou-Pudélko-Rădulescu [9] under the assumption that  $a \in C^{0,1}(\bar{\Omega}) \cap \mathcal{A}_p$ . They proved that (4) admits a smallest eigenvalue  $\hat{\lambda}_1^a > 0$  which has the following variational characterization

$$\hat{\lambda}_1^a = \inf \left\{ \frac{\rho_\theta(Du)}{\rho_\theta(u)} : u \in W_0^{1,\theta}(\Omega), u \neq 0 \right\}. \quad (5)$$

In addition, this eigenvalue has the following properties

- (a) If  $\hat{\sigma}^a(p)$  denotes the spectrum of (4), then  $\hat{\lambda}_1^a \in \hat{\sigma}^a(p)$  is isolated, that is, there exists  $\varepsilon > 0$  such that

$$(\hat{\lambda}_1^a, \hat{\lambda}_1^a + \varepsilon) \cap \hat{\sigma}^a(p) = \emptyset.$$

- (b)  $\hat{\lambda}_1^a > 0$  is simple, that is, if  $u, v$  are eigenfunctions corresponding to  $\hat{\lambda}_1^a$ , then  $u = \mu v$  for some  $\mu \neq 0$ .  
 (c) The eigenfunction corresponding to  $\hat{\lambda}_1^a$  has fixed sign and if  $u$  is such an eigenfunction, it realizes the infimum in (5) and  $0 \prec |u|$ .

In the sequel by  $\hat{u}_1$  we denote the  $L^\theta$ -normalized (that is,  $\|\hat{u}_1\|_\theta = 1$ ) positive eigenfunction corresponding to  $\hat{\lambda}_1^a$ . Then

$$\hat{u}_1 \in W_0^{1,\theta}(\Omega) \cap L^\infty(\Omega), \quad \|\hat{u}_1\|_\theta = 1, \quad 0 \prec \hat{u}_1.$$

We mention that if  $\hat{u}$  is an eigenfunction corresponding to an eigenvalue  $\hat{\lambda} \neq \hat{\lambda}_1^a$ , then  $\hat{u} \in W_0^{1,\theta}(\Omega) \cap L^\infty(\Omega)$  and it is nodal (sign changing). Since  $\hat{\sigma}^a(p)$  is closed, on account of (a) above, we can define  $\hat{\lambda}_2^a$  the second eigenvalue of (4) by

$$\hat{\lambda}_2^a = \inf \{ \hat{\lambda} \in \hat{\sigma}^a(p) : \hat{\lambda} > \hat{\lambda}_1^a \}.$$

From [9] we know that  $\hat{\lambda}_2^a$  has the following variational characterization

$$\hat{\lambda}_2^a = \inf_{\gamma \in \Gamma} \max_{-1 \leq t \leq 1} \rho_\theta(D\gamma(t)) \quad (6)$$

with

$$\Gamma = \{\gamma \in C([-1, 1], M) : \gamma(-1) = -\hat{u}_1, \gamma(1) = \hat{u}_1\}$$

and

$$M = \left\{ u \in W_0^{1,\theta}(\Omega) : \|u\|_\theta = |u|_a = 1 \right\}.$$

Using the above properties of the first eigenvalue, we can have the following useful inequality (see [9])

**Lemma 1.** *If  $a \in C^{0,1}(\bar{\Omega}) \cap \mathcal{A}_p$ ,  $\beta \in L^\infty(\Omega) \setminus \{0\}$ ,  $\beta(z) \leq \hat{\lambda}_1^a$  for a.a.  $z \in \Omega$  and the inequality is strict on a set of positive Lebesgue measure, then there exists  $\hat{c} > 0$  such that*

$$\hat{c}\|u\|^p \leq \rho_\theta(Du) - \int_{\Omega} \beta(z)a(z)|u|^p dz \quad \text{for all } u \in W_0^{1,\theta}(\Omega).$$

We introduce the following closed subspace of  $W_0^{1,\theta}(\Omega)$

$$V = \left\{ u \in W_0^{1,\theta}(\Omega) : \int_{\Omega} a(z)\hat{u}_1^{p-1}u dz = 0 \right\}.$$

We have  $W_0^{1,\theta}(\Omega) = \mathbb{R}\hat{u}_1 \oplus V$  and we define

$$\hat{\lambda}_V^a = \inf \left\{ \frac{\rho_\theta(Du)}{\rho_\theta(u)} : u \in V, u \neq 0 \right\}. \quad (7)$$

**Proposition 2.** *If hypotheses  $H_0$  hold, then  $\hat{\lambda}_1^a < \hat{\lambda}_V^a \leq \hat{\lambda}_2^a$ .*

**Proof.** From (5) and (7), we see that  $\hat{\lambda}_1^a \leq \hat{\lambda}_V^a$ . Arguing by contradiction, suppose that  $\hat{\lambda}_1^a = \hat{\lambda}_V^a$ . Then we can find  $\{v_n\}_{n \in \mathbb{N}} \subseteq V$  such that

$$\rho_\theta(v_n) = 1 \quad \text{for all } n \in \mathbb{N}, \quad \rho_\theta(Dv_n) \downarrow \hat{\lambda}_V^a = \hat{\lambda}_1^a.$$

Evidently  $\{v_n\}_{n \in \mathbb{N}} \subseteq W_0^{1,\theta}(\Omega)$  is bounded and so using Proposition 1, we may assume that

$$v_n \xrightarrow{w} v \text{ in } W_0^{1,\theta}(\Omega), \quad v_n \rightarrow v \text{ in } L^\theta(\Omega). \quad (8)$$

From (8) it follows that  $\rho_\theta(v) = 1$ . Moreover, on account of the sequential weak lower semicontinuity of the modular function and using (8), we have

$$\begin{aligned} \rho_\theta(Dv) &\leq \hat{\lambda}_1^a, \\ \Rightarrow \rho_\theta(Dv) &= \hat{\lambda}_1^a \quad (\text{see (5) and recall that } \rho_\theta(v) = 1), \\ \Rightarrow v &= \pm \hat{u}_1, \text{ and so } 0 \prec |v|, \ v \in V, \text{ a contradiction.} \end{aligned}$$

Therefore we have

$$\hat{\lambda}_1^a < \hat{\lambda}_V^a.$$

Next we show the inequality  $\hat{\lambda}_V^a \leq \hat{\lambda}_2^a$ . Again we argue indirectly. So, suppose that  $\hat{\lambda}_2^a < \hat{\lambda}_V^a$ . Then using (6), we see that we can find  $\gamma \in \Gamma$  such that

$$\rho_\theta(D\gamma(t)) < \hat{\lambda}_V^a \quad \text{for all } t \in [-1, 1]. \quad (9)$$

We consider the continuous function  $\tau : [-1, 1] \rightarrow \mathbb{R}$  defined by

$$\tau(t) = \int_{\Omega} a(z) \hat{u}_1^{p-1} \gamma(t) dz.$$

We have

$$\tau(-1) = -\rho_\theta(\hat{u}_1) < 0 \quad \text{and} \quad \tau(1) = \rho_\theta(\hat{u}_1) > 0.$$

So, by Bolzano's theorem, there exists  $t_0 \in (-1, 1)$  such that

$$\begin{aligned} \tau(t_0) &= \int_{\Omega} a(z) \hat{u}_1^{p-1} \gamma(t_0) dz = 0, \\ \Rightarrow \gamma(t_0) &\in V, \\ \Rightarrow \hat{\lambda}_V^a &\leq \rho_\theta(D\gamma(t_0)), \end{aligned}$$

which contradicts (9). Therefore

$$\hat{\lambda}_1^a < \hat{\lambda}_V^a \leq \hat{\lambda}_2^a,$$

which concludes the proof.  $\square$

**Remark 1.** Can we say that  $\hat{\lambda}_V^a = \hat{\lambda}_2^a$ ? This is an open problem.

As we already mentioned in the Introduction, our approach uses items from the theory of critical groups (Morse theory). A detailed presentation can be found in the book of Papageorgiou-Rădulescu-Repovš [10].

Let  $X$  be a Banach space,  $\varphi \in C^1(X)$  and  $c \in \mathbb{R}$ . We introduce the following sets

$$K_\varphi = \{u \in X : \varphi'(u) = 0\} \text{ (critical set of } \varphi),$$

$$\varphi^c = \{u \in X : \varphi(u) \leq c\}.$$

Let  $Y_2 \subseteq Y_1 \subseteq X$  and  $k \in \mathbb{N}_0$ . By  $H_k(Y_1, Y_2)$  we denote the  $k^{\text{th}}$  singular homology group with integer coefficients. If  $u \in K_\varphi$  is isolated and  $c = \varphi(u)$ , then the critical groups of  $\varphi(\cdot)$  at  $u$  are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{u\}) \text{ for all } k \in \mathbb{N}_0.$$

In this definition,  $U$  is a neighborhood of  $u$  such that  $K_\varphi \cap \varphi^c \cap U = \{u\}$ . The excision property of singular homology implies that the critical groups are well-defined in the sense that the above definition is independent of the choice of the isolating neighborhood  $U$ .

**Definition 1.** Let  $\varphi \in C^1(X)$ ,  $0 \in K_\varphi$  with  $\varphi(0) = 0$ ,  $m, n \in \mathbb{N}$ . We say that  $\varphi(\cdot)$  has  $(m, n)$ -local linking near the origin, if there exists  $U$  a neighborhood of the origin and nonempty sets  $E_0 \subseteq E \subseteq U$ ,  $D \subseteq X$  with  $E_0 \cap D = \emptyset$  and

- (a)  $K_\varphi \cap \varphi^0 \cap U = \{0\}$ ;
- (b)  $\text{rank im } i_* - \text{rank im } j_* \geq n$ , where  $i_* : H_{m-1}(E_0) \rightarrow H_{m-1}(X \setminus D)$  and  $j_* : H_{m-1}(E_0) \rightarrow H_{m-1}(E)$  are the group homomorphisms induced by the inclusion maps  $i : E_0 \rightarrow X \setminus D$  and  $j : E_0 \rightarrow E$ .
- (c)  $\varphi|_E \leq 0 < \varphi|_{U \cap D \setminus \{0\}}$ .

This notion leads to explicit information about the critical groups of  $\varphi(\cdot)$  at the origin (see Theorem 6.6.17 of [10, p.538]).

**Proposition 3.** If  $\varphi \in C^1(X)$ ,  $0 \in K_\varphi$  is isolated and  $\varphi(\cdot)$  has an  $(m, n)$ -local linking near the origin, then  $\text{rank } C_m(\varphi, 0) \geq n$ .

Let  $u \in L^0(\Omega)$ ,  $u(z) \geq 0$  for a.a.  $z \in \Omega$ . For every  $k > 0$  we introduce the set  $E_k = \{z \in \Omega : u(z) > k\}$ . Then from Ladyzhenskaya-Ural'tseva [6] (pp. 66, 73-74), we have the following result.

**Proposition 4.** If  $\tau, s > 1$ ,  $u \in L^\tau(\Omega)$ ,  $u \geq 0$  and

$$\left[ \int_{E_k} (u - k)^\tau dz \right]^{\frac{1}{s}} \leq c \left[ \int_{E_k} (u - k)^\tau dz + k^\tau |E_k|_N \right], \text{ for all } k \geq k_0$$

for some  $c, k_0 > 0$  (by  $|\cdot|_N$  we denote the Lebesgue measure on  $\mathbb{R}^N$ ), then  $u \in L^\infty(\Omega)$ .

Finally we introduce the hypotheses on the reaction  $f(z, x)$ .

( $H_1$ )  $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  is a Carathéodory function such that

(i) for every  $\rho > 0$ , there exists  $\hat{a}_\rho \in L^\infty(\Omega)$  such that

$$|f(z, x)| \leq \hat{a}_\rho(z) \text{ for a.a. } z \in \Omega, \text{ all } |x| \leq \rho;$$

(ii) there exists  $\beta_0 > 0$  such that

$$-\beta_0 \leq \liminf_{x \rightarrow \pm\infty} \frac{f(z, x)}{a(z)|x|^{p-2}x} \leq \limsup_{x \rightarrow \pm\infty} \frac{f(z, x)}{a(z)|x|^{p-2}x} \leq \hat{\lambda}_1^a, \text{ uniformly for a.a. } z \in \Omega$$

and

$$\lim_{x \rightarrow \pm\infty} [f(z, x)x - pF(z, x)] = +\infty \text{ uniformly for a.a. } z \in \Omega,$$

where  $F(z, x) = \int_0^x f(z, s)ds$ ;

(iii) there exist  $\delta > 0$  and  $\lambda_0 \in (\hat{\lambda}_1^a, \hat{\lambda}_V^a)$  such that

$$\hat{\lambda}_1^a a(z)|x|^p \leq f(z, x)x \leq \lambda_0 a(z)|x|^p$$

for a.a.  $z \in \Omega$  and all  $|x| \leq \delta$ .

**Remark 2.** Hypothesis  $H_1$  (iii) implies that

$$\hat{\lambda}_1^a \leq \liminf_{x \rightarrow 0} \frac{f(z, x)}{a(z)|x|^{p-2}x} \text{ uniformly for a.a. } z \in \Omega.$$

This fact together with hypothesis  $H_1$  (ii), reveal that our formulation covers the doubly resonant case (that is, resonance at  $\pm\infty$  and at 0).

A function  $f(z, x)$  satisfying the above hypotheses is given by

$$f(z, x) = \begin{cases} a(z)[\lambda|x|^{p-2}x - |x|^{\tau-2}x] & \text{if } |x| \leq 1 \\ a(z)[\hat{\lambda}_1^a|x|^{p-2}x - (\hat{\lambda}_1^a - \lambda + 1)|x|^{q-2}x] & \text{if } 1 < |x|, \end{cases}$$

with  $1 < q < p < \tau$  and  $\hat{\lambda}_1^a < \lambda < \min\{\hat{\lambda}_1^a + 1, \hat{\lambda}_V^a\}$ .

Another example is provided by the following function:

$$f(z, x) = \begin{cases} \hat{\lambda}_1^a a(z) |x|^{p-2} x & \text{if } |x| \leq 1 \\ \hat{\lambda}_1^a a(z) |x|^{p-2} x - \ln |x| & \text{if } 1 < |x|. \end{cases}$$

### 3. Two nontrivial bounded solutions

In this section, combining variational tools from the critical point theory with critical groups, we show the existence of at least two nontrivial bounded solutions for problem (1).

Consider the energy function  $\varphi : W_0^{1,\theta}(\Omega) \rightarrow \mathbb{R}$  for problem (1), defined by

$$\varphi(u) = \frac{1}{p} \rho_\theta(Du) - \int_{\Omega} F(z, u) dz \quad \text{for all } u \in W_0^{1,\theta}(\Omega).$$

Recall that  $F(z, x) = \int_0^x f(z, s) ds$ . Evidently  $\varphi \in C^1(W_0^{1,\theta}(\Omega))$ .

**Proposition 5.** *If hypotheses  $H_0$  and  $H_1$  hold, then the functional  $\varphi(\cdot)$  is coercive.*

**Proof.** On account of hypothesis  $H_1$ (ii), we have

$$-\hat{\beta}_0 \leq \liminf_{x \rightarrow \pm\infty} \frac{pF(z, x)}{a(z)|x|^p} \leq \limsup_{x \rightarrow \pm\infty} \frac{pF(z, x)}{a(z)|x|^p} \leq \hat{\lambda}_1^a, \quad \text{uniformly for a.a. } z \in \Omega, \quad (10)$$

where  $\hat{\beta}_0 > 0$ .

From the second part of hypothesis  $H_1$ (ii), we see that given any  $\eta > 0$ , we can find  $M_\eta > 0$  such that

$$\eta \leq f(z, x)x - pF(z, x), \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \geq M_\eta. \quad (11)$$

For  $|x| \geq M_\eta$ , we have

$$\frac{d}{dx} \left[ \frac{F(z, x)}{|x|^p} \right] = \frac{f(z, x)x - pF(z, x)}{|x|^p x} \begin{cases} \leq \frac{\eta}{|x|^p x}, & \text{if } x \leq -M_\eta, \\ \geq \frac{\eta}{x^{p+1}}, & \text{if } M_\eta \leq x, \end{cases}$$

for a.a.  $z \in \Omega$  (see (11)).

Integrating both inequalities, we obtain

$$\frac{F(z, y)}{|y|^p} - \frac{F(z, x)}{|x|^p} \geq -\frac{\eta}{p} \left[ \frac{1}{|y|^p} - \frac{1}{|x|^p} \right] \quad \text{for a.a. } z \in \Omega, \text{ all } |y| > |x| \geq M_\eta.$$

We let  $|y| \rightarrow \infty$  and use (10), then

$$\begin{aligned} \frac{1}{p} \hat{\lambda}_1^a a(z) - \frac{F(z, x)}{|x|^p} &\geq \frac{\eta}{p} \frac{1}{|x|^p} \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \geq M_\eta, \\ \Rightarrow \hat{\lambda}_1^a a(z) |x|^p - pF(z, x) &\geq \eta \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \geq M_\eta. \end{aligned}$$

Since  $\eta > 0$  is arbitrary, we conclude that

$$\lim_{x \rightarrow \pm\infty} \left[ \hat{\lambda}_1^a a(z) |x|^p - pF(z, x) \right] = +\infty, \quad \text{uniformly for a.a. } z \in \Omega. \quad (12)$$

Arguing by contradiction, suppose that  $\varphi(\cdot)$  is not coercive. Then we can find  $\{u_n\}_{n \in \mathbb{N}} \subseteq W_0^{1, \theta}(\Omega)$  such that

$$\|u_n\| \rightarrow \infty, \quad \varphi(u_n) \leq c \quad \text{for some } c > 0, \text{ all } n \in \mathbb{N}. \quad (13)$$

Let  $y_n = \frac{u_n}{\|u_n\|}$ ,  $n \in \mathbb{N}$ . Then  $\|y_n\| = 1$  for all  $n \in \mathbb{N}$  and so we may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1, \theta}(\Omega), \quad y_n \rightarrow y \text{ in } L^\theta(\Omega) \quad (\text{see Proposition 1}). \quad (14)$$

From (13), we have

$$\frac{1}{p} \rho_\theta(Dy_n) - \int_\Omega \frac{F(z, u_n)}{\|u_n\|^p} dz \leq \frac{c}{\|u_n\|^p}, \quad n \in \mathbb{N}. \quad (15)$$

Hypotheses  $H_1$ (i) (ii) imply that

$$\begin{aligned} |F(z, x)| &\leq c_0[1 + a(z)|x|^p] \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ some } c_0 > 0, \\ \Rightarrow \frac{|F(z, u_n)|}{\|u_n\|^p} &\leq \varepsilon_n + c_0 a(z) |y_n|^p \quad \text{for a.a. } z \in \Omega, \text{ with } \varepsilon_n \rightarrow 0^+, \\ \Rightarrow \left\{ \frac{F(\cdot, u_n(\cdot))}{\|u_n\|^p} \right\}_{n \in \mathbb{N}} &\subseteq L^1(\Omega) \text{ is uniformly integrable (see (14)).} \end{aligned}$$

By the Dunford-Pettis theorem, at least for a subsequence, we have

$$\frac{F(\cdot, u_n(\cdot))}{\|u_n\|^p} \xrightarrow{w} k \text{ in } L^1(\Omega) \quad (16)$$

with  $k(z) = \frac{\mu(z)}{p} a(z) |y(z)|^p$ ,  $-\hat{\beta}_0 \leq \mu(z) \leq \hat{\lambda}_1^a$  for a.a.  $z \in \Omega$  (see (10) and Aizicovici-Papageorgiou-Staicu [1], proof of Proposition 16). For the convenience of the reader, we provide an outline of the proof of this fact. Recall that on account of hypotheses  $H_1$ (ii), we have

$$-\frac{1}{p}\hat{\beta}_0 a(z) \leq \liminf_{x \rightarrow \pm\infty} \frac{F(z, x)}{|x|^p} \leq \limsup_{x \rightarrow \pm\infty} \frac{F(z, x)}{|x|^p} \leq \frac{1}{p}\hat{\lambda}_1^a a(z),$$

uniformly for a.a.  $z \in \Omega$  and for some  $\hat{\beta}_0 > 0$  (see (10)).

For the moment fix  $\varepsilon > 0$  and for every  $n \in \mathbb{N}$  introduce the set

$$D_{\varepsilon, n} = \left\{ z \in \Omega : \left( -\frac{\hat{\beta}_0}{p} - \varepsilon \right) a(z) \leq \frac{N_F(u_n)}{|u_n|^p} \leq \left( \frac{1}{p}\hat{\lambda}_1^a + \varepsilon \right) a(z) \right\},$$

where  $N_F(u)(\cdot) = F(\cdot, u(\cdot))$  for all  $u \in W_0^{1, \theta}(\Omega)$  (the Nemytskii map for the potential function  $F(z, x)$ ). We have

$$\chi_{D_{\varepsilon, n}}(z) \rightarrow 1 \quad \text{for a.a. } z \in \Omega, \text{ as } n \rightarrow \infty.$$

Therefore

$$\chi_{D_{\varepsilon, n}} \frac{N_F(u_n)}{|u_n|^p} \rightharpoonup k \quad \text{in } L^1(\Omega), \text{ as } n \rightarrow \infty \text{ (see (16)).}$$

From the definition of  $D_{\varepsilon, n}$ , we have

$$\chi_{D_{\varepsilon, n}}(z) \left( -\frac{\hat{\beta}_0}{p} - \varepsilon \right) a(z) |y_n(z)|^p \leq \chi_{D_{\varepsilon, n}}(z) \frac{N_F(u_n)}{|u_n|^p} \leq \chi_{D_{\varepsilon, n}}(z) \left( \frac{1}{p}\hat{\lambda}_1^a + \varepsilon \right) a(z) |y_n(z)|^p,$$

for a.a.  $z \in \Omega$ , all  $n \in \mathbb{N}$ . By Mazur's theorem, we obtain

$$\left( -\frac{\hat{\beta}_0}{p} - \varepsilon \right) a(z) |y(z)|^p \leq k(z) \leq \left( \frac{1}{p}\hat{\lambda}_1^a + \varepsilon \right) a(z) |y(z)|^p.$$

Since  $\varepsilon > 0$  is arbitrary, we let  $\varepsilon \rightarrow 0^+$  and obtain

$$-\frac{\hat{\beta}_0}{p} a(z) |y(z)|^p \leq k(z) \leq \frac{1}{p}\hat{\lambda}_1^a a(z) |y(z)|^p \quad \text{for a.a. } z \in \Omega,$$

hence

$$k(z) = \frac{\mu(z)}{p} a(z) |y(z)|^p \quad \text{for a.a. } z \in \Omega,$$

with  $-\hat{\beta}_0 \leq \mu(z) \leq \hat{\lambda}_1^a$  for a.a.  $z \in \Omega$ .

In (15), we pass to the limit as  $n \rightarrow \infty$  and use (14), (16) and the sequential weak lower semicontinuity of the modular function  $\rho_\theta(\cdot)$ , we obtain

$$\rho_\theta(Dy) \leq \int_{\Omega} \mu(z) a(z) |y|^p dz. \quad (17)$$

If  $\mu \neq \hat{\lambda}_1^a$  (see (16)), then using Lemma 1, we have

$$\begin{aligned}\hat{c}\|y\|^p &\leq 0, \\ \Rightarrow y &= 0.\end{aligned}$$

But then from (15) and (16), we see that

$$\begin{aligned}\rho_\theta(Dy_n) &\rightarrow 0, \\ \Rightarrow y_n &\rightarrow 0 \text{ in } W_0^{1,\theta}(\Omega) \text{ as } n \rightarrow \infty,\end{aligned}$$

a contradiction since  $\|y_n\| = 1$  for all  $n \in \mathbb{N}$ .

If  $\mu(z) = \hat{\lambda}_1^a$  for a.a.  $z \in \Omega$ , then (17) says that

$$\begin{aligned}\rho_\theta(Dy) &\leq \hat{\lambda}_1^a \rho_\theta(y), \\ \Rightarrow \rho_\theta(Dy) &= \hat{\lambda}_1^a \rho_\theta(y) \text{ (see (5))}, \\ \Rightarrow y &= \tau \hat{u}_1 \text{ with } \tau \in \mathbb{R}.\end{aligned}$$

If  $\tau = 0$ , then  $y = 0$  and as above, we reach a contradiction. If  $\tau \neq 0$ , then  $|y(z)| \neq 0$  for a.a.  $z \in \Omega$  (recall  $0 \prec \hat{u}_1$ ) and so we have

$$|u_n(z)| \rightarrow \infty \text{ for a.a. } z \in \Omega.$$

On account of (12), we have

$$\hat{\lambda}_1^a a(z) |u_n(z)|^p - pF(z, u_n(z)) \rightarrow +\infty \text{ for a.a. } z \in \Omega. \quad (18)$$

From (13), we know that

$$\begin{aligned}\frac{1}{p} \rho_\theta(Du_n) - \int_{\Omega} F(z, u_n) dz &\leq c \text{ for all } n \in \mathbb{N}, \\ \Rightarrow \hat{\lambda}_1^a \rho_\theta(u_n) - \int_{\Omega} pF(z, u_n) dz &\leq pc \text{ for all } n \in \mathbb{N} \text{ (see (5))}, \\ \Rightarrow \int_{\Omega} \left[ \hat{\lambda}_1^a a(z) |u_n|^p - pF(z, u_n) \right] dz &\leq pc \text{ for all } n \in \mathbb{N}.\end{aligned} \quad (19)$$

But from (18) and Fatou's lemma, we have

$$\lim_{n \rightarrow \infty} \int_{\Omega} \left[ \hat{\lambda}_1^a a(z) |u_n|^p - pF(z, u_n) \right] dz = +\infty,$$

which contradicts (19).

We conclude that  $\varphi(\cdot)$  is coercive.  $\square$

**Proposition 6.** *If hypotheses  $H_0$  and  $H_1$  hold, then there exists  $\rho > 0$  such that*

$$\varphi(0) = 0 < \varphi(v) \quad \text{for all } v \in V, \quad 0 < \|v\| \leq \rho.$$

**Proof.** From hypotheses  $H_1$ , we see that if  $r \in (p, p_s^*)$ , then we can find  $c_1 = c_1(r) > 0$  such that

$$F(z, x) \leq \frac{\lambda_0 a(z)}{p} |x|^p + c_1 |x|^r \quad \text{for a.a. } z \in \Omega, \quad \text{all } x \in \mathbb{R}. \quad (20)$$

Indeed, on account of hypothesis  $H_1$ (iii) we have

$$F(z, x) \leq \frac{\lambda_0 a(z)}{p} |x|^p \quad \text{for a.a. } z \in \Omega, \quad \text{all } |x| \leq \delta.$$

Moreover, hypothesis  $H_1$ (i) implies that we can find  $c_1 = c_1(r) > 0$  such that

$$F(z, x) \leq c_1 |x|^r \quad \text{for a.a. } z \in \Omega, \quad \text{all } |x| > \delta.$$

These two estimates lead to (20).

Let  $v \in V$ . Then

$$\begin{aligned} \varphi(v) &\geq \frac{1}{p} \rho_\theta(Dv) - \frac{\lambda_0}{p} \rho_\theta(v) - c_1 \|v\|_r^r \quad (\text{see (20)}) \\ &\geq \frac{1}{p} \left( 1 - \frac{\lambda_0}{\hat{\lambda}_V^a} \right) \rho_\theta(Dv) - c_2 \|v\|^r \quad \text{for some } c_2 > 0 \\ &\quad (\text{see (7) and Proposition 1}) \\ &= \frac{1}{p} \frac{\hat{\lambda}_V^a - \lambda_0}{\hat{\lambda}_V^a} \|v\|^p - c_2 \|v\|^r. \end{aligned}$$

Since  $\lambda_0 < \hat{\lambda}_V^a$  and  $p < r$ , we see that for  $\rho \in (0, 1)$  small, we have

$$\varphi(v) > 0 = \varphi(0) \quad \text{for all } v \in V, \quad 0 < \|v\| \leq \rho.$$

The proof is now complete.  $\square$

**Remark 3.** The above proposition says that  $u = 0$  is a strict local minimizer of the functional  $\varphi|_V$ .

**Proposition 7.** *If hypotheses  $H_0$  and  $H_1$  hold, then we can find  $\hat{\rho} > 0$  such that*

$$\varphi(y) \leq 0 \quad \text{for all } y \in Y = \mathbb{R} \hat{u}_1 \quad \text{with } \|y\| \leq \hat{\rho}.$$

**Proof.** We know that  $Y \subseteq L^\infty(\Omega)$  and since  $\dim Y = 1$ , all norms are equivalent and so we can find  $\hat{\rho} > 0$  such that

$$y \in Y, \|y\| \leq \hat{\rho} \quad \Rightarrow \quad |y(z)| \leq \delta \quad \text{for a.a. } z \in \Omega, \quad (21)$$

with  $\delta > 0$  as postulated by hypothesis  $H_1(\text{iii})$ . Then for such a  $y \in Y$  (see (21)), we have  $y = t\hat{u}_1$ ,  $t \in \mathbb{R}$ . Then

$$\begin{aligned} \varphi(y) &= \varphi(t\hat{u}_1) \leq \frac{|t|^p}{p} \left[ \rho_\theta(D\hat{u}_1) - \hat{\lambda}_1^a \rho_\theta(\hat{u}_1) \right] = 0, \\ \Rightarrow \quad \varphi(y) &\leq 0 \quad \text{for all } y \in Y \text{ with } \|y\| \leq \hat{\rho}. \end{aligned}$$

The proof is now complete.  $\square$

Now we can state and prove the multiplicity theorem.

**Theorem 1.** *If hypotheses  $H_0$  and  $H_1$  hold, then problem (1) has at least two nontrivial solutions*

$$u_0, \hat{u} \in W_0^{1,\theta}(\Omega) \cap L^\infty(\Omega).$$

**Proof.** Let  $\rho_0 = \min\{\rho, \hat{\rho}\}$ . We introduce the following sets:

$$\begin{aligned} X &= W_0^{1,\theta}(\Omega), \quad E_0 = \{-\rho_0\hat{u}_1, \rho_0\hat{u}_1\} = Y \cap \partial B_{\rho_0}, \quad E = \{t\hat{u}_1 : 0 \leq |t| \leq \rho_0\} = Y \cap \bar{B}_{\rho_0}, \\ D &= V \text{ and } U = \bar{B}_{\rho_0} = \left\{ u \in W_0^{1,\theta}(\Omega) : \|u\|_\theta \leq \rho_0 \right\}. \end{aligned}$$

Then  $E_0 \subseteq E \subseteq \bar{B}_{\rho_0} = U$ ,  $E_0 \cap D = \emptyset$ ,  $0 \notin E_0$ .

Clearly we can assume that  $K_\varphi$  is finite or otherwise we already have an infinity of solutions for problem (1) and so we are done.

Let  $i : E_0 \rightarrow X \setminus D$  and  $j : E_0 \rightarrow E$  be the inclusion maps. The set  $E$  is contractible hence from Proposition 6.1.29 of [11], we have

$$\begin{aligned} H_0(E, E_0) &= H_{-1}(E_0, *) = 0 \quad \left( \text{with } * \in W_0^{1,\theta}(\Omega) \right), \\ \Rightarrow j_* : H_0(E_0) &\rightarrow H_0(E) \text{ is surjective (see [10], p.462),} \\ \Rightarrow \text{rank im } j_* &= 1. \end{aligned} \quad (22)$$

If  $u \in X \setminus D$ , then  $u = y + v$  with  $y \in Y$ ,  $v \in V$  uniquely. So, if we consider the deformation  $h : [0, 1] \times (W_0^{1,\theta}(\Omega) \setminus D) \rightarrow (W_0^{1,\theta}(\Omega) \setminus D)$  defined by

$$h(t, y + v) = (1 - t)(y + v) + t\rho_0 \frac{y}{\|y\|},$$

then we see that  $E_0$  is a strong deformation retract of  $W_0^{1,\theta}(\Omega) \setminus D$  (see Definition 5.3.10 in [10, p.385]). Indeed,  $h$  is continuous,

$$h(1, u) = t\rho_0 \frac{y}{\|y\|} \in E_0 \text{ and } h(t, y) = y \text{ for all } t \in [0, 1], \text{ all } y \in E_0.$$

It follows that

$$\begin{aligned} i_* : H_0(E_0) &\rightarrow H_0(W_0^{1,\theta}(\Omega) \setminus D) \text{ is a bijection (see [10, p.463]),} \\ \Rightarrow \text{rank im } i_* &= 2. \end{aligned} \tag{23}$$

From (22) and (23), it follows that

$$\begin{aligned} \text{rank im } i_* - \text{rank im } j_* &= 1, \\ \Rightarrow \varphi(\cdot) &\text{ has a } (1, 1) - \text{local linking near the origin (see Definition 1),} \\ \Rightarrow C_1(\varphi, 0) &\neq 0 \text{ (see Proposition 3).} \end{aligned} \tag{24}$$

From Proposition 5, we know that  $\varphi(\cdot)$  is coercive. Also using Proposition 1, we see that  $\varphi(\cdot)$  is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem, we can find  $u_0 \in W_0^{1,\theta}(\Omega)$  such that

$$\begin{aligned} \varphi(u_0) &= \inf \left\{ \varphi(u) : u \in W_0^{1,\theta}(\Omega) \right\} = \hat{m}, \\ \Rightarrow C_k(\varphi, u_0) &= \delta_{k,0}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0 \text{ (see [10]).} \end{aligned} \tag{25}$$

From (24) and (25), it follows that  $u_0 \neq 0$ .

Suppose that  $K_\varphi = \{0, u_0\}$  and let  $a, b \in \mathbb{R}$  such that

$$\hat{m} < a < 0 < b.$$

From these choices of the levels  $a, b$  and the Second Deformation Theorem (see [10, p.386]), we have that  $\{\varphi^a\}$  is a strong deformation retract of  $\varphi^a$ . Therefore

$$H_k(\varphi^a, \{u_0\}) = 0 \text{ for all } k \in \mathbb{N}_0 \text{ (see [10, p.462]).} \tag{26}$$

Similarly  $\varphi^b$  is a strong deformation retract of  $W_0^{1,\theta}(\Omega)$ , hence

$$H_k(\varphi^b, \{u_0\}) = H_k(W_0^{1,\theta}(\Omega), \{u_0\}) = 0 \text{ for all } k \in \mathbb{N}_0 \text{ (see [10, pp.468,469]).} \tag{27}$$

We consider the following long exact sequence

$$\cdots \rightarrow H_1(\varphi^a, \{u_0\}) \rightarrow H_1(\varphi^b, \{u_0\}) \rightarrow H_1(\varphi^b, \varphi^a) \rightarrow H_0(\varphi^a, \{u_0\}) \rightarrow \cdots \tag{28}$$

Then from (26), (27) and (28) it follows that

$$H_1(\varphi^b, \varphi^a) = 0.$$

Applying Proposition 6.7.7 of [10, p.550], we infer that there exists  $\hat{u} \in K_\varphi$  such that

$$a < \varphi(\hat{u}) < 0 \quad \text{and} \quad C_0(\varphi, \hat{u}) \neq 0$$

or

$$0 < \varphi(\hat{u}) < b \quad \text{and} \quad C_2(\varphi, \hat{u}) \neq 0.$$

Therefore  $\hat{u}$  is a solution of (1) and  $\hat{u} \notin \{0, u_0\}$ .

Let  $u \in W_0^{1,\theta}(\Omega)$  be a nontrivial solution of (1). We have

$$\int_{\Omega} a(z) |Du|^{p-2} (Du, Dh)_{\mathbb{R}^N} dz = \int_{\Omega} f(z, u) h dz \quad \text{for all } h \in W_0^{1,\theta}(\Omega). \quad (29)$$

On account of hypotheses  $H_1(\text{i})(\text{ii})$ , if  $r \in (p, p_s^*)$ , then

$$|f(z, x)| \leq c_3(1 + |x|^{r-1}) \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ some } c_3 > 0. \quad (30)$$

For  $k > 1$ , we introduce the set

$$E_k = \{z \in \Omega : |u(z)| > k\}.$$

Choose  $k > 1$  large, so that  $\|(|u| - k)^+\| \leq 1$ . On  $E_k$ , we have

$$|u(z)|^r \leq (|u(z)| - k + k)^r \leq 2^{r-1}(|u(z)| - k)^r + 2^{r-1}k^r. \quad (31)$$

In (29), we choose the test function  $h = (|u| - k)^+ \in W_0^{1,\theta}(\Omega)$ . Then

$$\begin{aligned} \|(|u| - k)^+\|^p &= \rho_\theta(D(|u| - k)^+) \quad (\text{by Poincaré's inequality}) \\ &= \int_{\Omega} f(z, u)(|u| - k)^+ dz \\ &\leq \int_{\Omega} 2c_3|u|^{r-1}(|u| - k)^+ dz \quad (\text{see (30) and recall } k > 1). \end{aligned}$$

Since  $W_0^{1,\theta}(\Omega) \hookrightarrow L^r(\Omega)$  (see Proposition 1), it follows that

$$\left( \int_{E_k} (|u| - k)^r dz \right)^{\frac{p}{r}} \leq c_4 \left[ \int_{E_k} (|u| - k)^r dz + k^r |E_k|_N \right],$$

$$\Rightarrow u \in L^\infty(\Omega) \quad (\text{see Proposition 4 and note that } \frac{r}{p} > 1).$$

We conclude that  $u_0, \hat{u} \in W_0^{1,\theta}(\Omega) \cap L^\infty(\Omega)$ .  $\square$

**Remark 4.** There is an alternative to show the boundedness of the solutions of (1).

In (29), we choose the test function  $h = (|u| - k)^+ \in W_0^{1,\theta}(\Omega)$ . Then we have

$$\begin{aligned} \|(|u| - k)^+\|^p &= \rho_\theta(D(|u| - k)^+) \\ &= \int_{\Omega} f(z, u)(|u| - k)^+ dz \\ &\leq \int_{\Omega} c_3[1 + |u|^{r-1}](|u| - k)^+ dz \\ &\leq c_5 |E_k|_N^{\frac{1}{r'}} \|(|u| - k)^+\| \quad \text{for some } c_5 > 0 \quad (\text{by Hölder's inequality}), \\ &\Rightarrow \|(|u| - k)^+\|^{p-1} \leq c_5 |E_k|_N^{\frac{1}{r'}}. \end{aligned} \tag{32}$$

For  $m > k$ , we have

$$\begin{aligned} (m - k)^p |E_m|_N^{\frac{p}{r}} &\leq \left[ \int_{E_m} ((|u| - k)^+)^r dz \right]^{\frac{p}{r}} \\ &\leq \left[ \int_{E_k} ((|u| - k)^+)^r dz \right]^{\frac{p}{r}} \quad (\text{since } E_m \subseteq E_k) \\ &\leq c_6 \|(|u| - k)^+\|^p \quad \text{for some } c_6 > 0 \quad (\text{see Proposition 1}), \\ &\Rightarrow (m - k)^{p-1} |E_m|_N^{\frac{p-1}{r}} \leq c_7 \|(|u| - k)^+\|^{p-1} \quad \text{for some } c_7 > 0, \\ &\Rightarrow (m - k)^{p-1} |E_m|_N^{\frac{p-1}{r}} \leq c_8 |E_k|_N^{\frac{1}{r'}} \quad \text{for some } c_8 > 0, \\ &\Rightarrow |E_m|_N \leq \frac{c_9}{(m - k)^r} |E_k|_N^{\frac{r-1}{p-1}} \quad \text{for some } c_9 > 0. \end{aligned}$$

Since  $\frac{r-1}{p-1} > 1$ , from Lemma B.1 of Kinderlehrer-Stampacchia [5, p.63], it follows that there exists  $M > 0$  large such that

$$\begin{aligned} |E_M|_N &= 0, \\ &\Rightarrow u \in L^\infty(\Omega). \end{aligned}$$

Therefore again we conclude that  $u_0, \hat{u} \in L^\infty(\Omega)$ .

### **CRedit authorship contribution statement**

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

### **Ethical approval**

Not applicable.

### **Funding**

The research of Peng Jin has been supported by the Hunan Province Graduate Research Innovation Project (No. CX20240163) and the China Scholarship Council (No. 202406370154). The research of V.D. Rădulescu was supported by the AGH University of Kraków under grant no. 16.16.420.054, funded by the Polish Ministry of Science and Higher Education.

### **Declaration of competing interest**

The authors have no relevant financial or non-financial interests to disclose.

### **Acknowledgements**

The authors wish to thank an anonymous reviewer for his/her corrections and constructive remarks.

### **Data availability**

Data sharing not applicable to this article as no data sets were generated or analysed during the current study.

### **References**

- [1] S. Aizicovici, N.S. Papageorgiou, V. Staicu, Degree theory for operators of monotone type and nonlinear elliptic equations with inequality constraints, *Mem. Am. Math. Soc.* 196 (915) (November 2008).
- [2] L. Cherfils, Y. Il'yasov, On the stationary solutions of generalized reaction diffusion equations with  $p$ -&  $q$ -Laplacian, *Commun. Pure Appl. Anal.* 4 (1) (2005) 9–22.
- [3] P. Drábek, A. Kufner, F. Nicolosi, *Quasilinear Elliptic Equations with Degenerations and Singularities*, De Gruyter, Berlin, 1997.
- [4] J. Heinonen, T. Kilpeläinen, O. Martio, *Nonlinear Potential Theory of Degenerate Elliptic Equations*, Clarendon Press, Oxford, 1993.

- [5] D. Kinderlehrer, G. Stampacchia, An Introduction to Variational Inequalities and Their Applications, Academic Press, New York-London, 1980.
- [6] O.A. Ladyzhenskaya, N.N. Ural'tseva, Linear and Quasilinear Elliptic Equations, Academic Press, New York-London, 1968.
- [7] G.M. Lieberman, The natural generalization of the natural conditions of Ladyzhenskaya and Ural'tseva for elliptic equations, Commun. Partial Differ. Equ. 16 (1991) 311–361.
- [8] B. Muckenhoupt, Weighted norm inequalities for the Hardy maximal function, Trans. Am. Math. Soc. 165 (1972) 207–226.
- [9] N.S. Papageorgiou, A. Pudelko, V.D. Rădulescu, Non-autonomous  $(p, q)$ -equations with unbalanced growth, Math. Ann. 385 (2023) 1707–1745.
- [10] N.S. Papageorgiou, V.D. Rădulescu, D.D. Repovš, Nonlinear Analysis: Theory and Methods, Springer, Cham, 2019.
- [11] N.S. Papageorgiou, V.D. Rădulescu, Y.P. Zhang, Resonant double phase equations, Nonlinear Anal., Real World Appl. 64 (2022) 103454.
- [12] T. Singer, Existence of weak solutions of parabolic systems with  $p, q$ -growth, Manuscr. Math. 151 (2016) 87–112.