Discrete boundary value problems involving oscillatory nonlinearities: small and large solutions

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We consider the discrete boundary value problem (P): $-\Delta(\Delta u(k-1)) = f(u(k)), \ k \in [1, T]$, $u(0) = u(T+1) = 0$, where the nonlinear term $f : [0, \infty) \to \mathbb{R}$ has an oscillatory behaviour near the origin or at infinity. By a direct variational method we show that (P) has a sequence of non-negative, distinct solutions which converges to 0 (resp. $+\infty$) in the sup-norm whenever $f$ oscillates at the origin (resp. at infinity).

Keywords: difference equations; oscillatory nonlinearities; small solutions; large solutions

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1. Introduction and main results

In many cases a problem in a continuous framework can be handled by using a suitable method from discrete mathematics and conversely; a beautiful description of such phenomena can be found in Lovász [12]. The modeling/simulation of certain nonlinear problems from economics, biological neural networks, optimal control and others enforced in a natural manner the rapid development of the theory of difference equations. The reader may consult the comprehensive monographs of Agarwal [1], Kelley-Peterson [10], Lakshmikantham-Trigiante [11].

Within the theory of difference equations, a large class of problems is the nonlinear discrete boundary value problems. To be more precise, we consider the problem

\[
\begin{cases}
-\Delta(\Delta u(k-1)) = f(u(k)), \ k \in [1, T], \\
u(0) = u(T+1) = 0,
\end{cases}
\]

where $T \geq 2$ is an integer, $[1, T]$ is the discrete interval $\{1,...,T\}$, $\Delta u(k) = u(k+1) - u(k)$ is the forward difference operator, and $f$ is a continuous non-linearity. In order to establish existence/multiplicity of solutions for (P) under specific restrictions on $f$ (sublinear or superlinear at infinity), the authors exploited various abstract methods as fixed point theorems, sub- and super-solution
arguments, Brouwer degree and critical point theory. We refer the reader to the recent papers of Agarwal-Perera-O’Regan [2, 3], Bereanu-Mawhin [4, 5], Bereanu-Thompson [6], Bonanno-Candito [7], Cabada-Iannizzotto-Tersian [8], Cai-Yu [9], Mihăilescu-Rădulescu-Tersian [13], Yu-Guo [15], Tang-Luo-Li-Ma [14], Zhang-Liu [16], and references therein.

The main purpose of the present paper is to trait problem (P) when the nonlinear term $f : [0, \infty) \to \mathbb{R}$ has a suitable oscillatory behaviour. A direct variational argument provides two results (see Theorems 1.1 and 1.2), guaranteeing sequences of non-negative solutions with further asymptotic properties whenever $f$ oscillates near the origin or at infinity. Before to state our results, we mention that solutions of (P) are going to be sought in the function space

$$X = \{ u : [0, T + 1] \to \mathbb{R}; u(0) = u(T + 1) = 0 \}.$$  

Clearly, $X$ is a $T$-dimensional Hilbert space (see [2]) with the inner product

$$\langle u, v \rangle = \sum_{k=1}^{T+1} \Delta u(k-1) \Delta v(k-1), \quad \forall \ u, v \in X.$$  

The associated norm is defined by

$$\|u\| = \left( \sum_{k=1}^{T+1} |\Delta u(k-1)|^2 \right)^{1/2}.$$  

The space $X$ being finite-dimensional, the sup-norm $\| \cdot \|_\infty$ is equivalent to $\| \cdot \|$; here, we denote $\|u\|_\infty = \max_{k\in[1,T]} |u(k)|$, $u \in X$.

In the sequel, we state our results. Let $F(s) = \int_0^s f(t)dt$, $s \in [0, \infty)$.

Our first result concerns the case when $f$ has a certain type of oscillation near the origin. To be more precise, we assume

$$(H^0) \quad \liminf_{s \to 0^+} \frac{f(s)}{s} < 0; \limsup_{s \to 0^+} \frac{F(s)}{s^2} > \frac{1}{T}.$$  

**Theorem 1.1.** Let $f \in C^0([0, \infty); \mathbb{R})$ verifying $(H^0)$. Then there exists a sequence $\{u_n^0\}_n \subset X$ of non-negative, distinct solutions of (P) such that

$$\lim_{n \to \infty} \|u_n^0\|_\infty = \lim_{n \to \infty} \|u_n^0\| = 0. \quad (1)$$

A perfect counterpart of Theorem 1.1 can be stated when the nonlinear term oscillates at infinity. Instead of $(H^0)$, we assume

$$(H^\infty) \quad \liminf_{s \to \infty} \frac{f(s)}{s} < 0; \limsup_{s \to \infty} \frac{F(s)}{s^2} > \frac{1}{T}.$$  

**Theorem 1.2.** Let $f \in C^0([0, \infty); \mathbb{R})$ verifying $(H^\infty)$ and $f(0) = 0$. Then there exists a sequence $\{u_n^\infty\}_n \subset X$ of non-negative, distinct solutions of (P) such that

$$\lim_{n \to \infty} \|u_n^\infty\|_\infty = \lim_{n \to \infty} \|u_n^\infty\| = \infty. \quad (2)$$

**Example 1.3** (a) Let $\alpha, \beta, \gamma \in \mathbb{R}$ such that $0 < \alpha < 1 < \alpha + \beta$, and $\gamma \in (0, 1)$. Then, the function $f : [0, \infty) \to \mathbb{R}$ defined by $f(0) = 0$ and $f(s) = s^\alpha(\gamma + \sin s^{-\beta})$, $s > 0$, verifies hypothesis $(H^0)$.

(b) Let $\alpha, \beta, \gamma \in \mathbb{R}$ such that $1 < \alpha$, $|\alpha - \beta| < 1$, and $\gamma \in (0, 1)$. Then, the function $f : [0, \infty) \to \mathbb{R}$ defined by $f(s) = s^\alpha(\gamma + \sin s^\beta)$ verifies $(H^\infty)$. 

A. Kristály, M. Mihăilescu and V. Rădulescu
Small and large solutions for discrete BVPs

The paper is divided as follows. In the next section we consider a related difference equation to (P); the existence of a non-negative solution is proved under some generic assumptions. This result is used in Sections 3 and 4, where Theorems 1.1 and 1.2 are proved by a careful analysis of certain energy levels associated to (P).

2. A key result

For a fixed $c > 0$, we consider the problem

$$\begin{align*}
-\Delta(\Delta u(k-1)) + cu(k) &= g(u(k)), \quad k \in [1, T], \\
u(0) &= u(T+1) = 0,
\end{align*}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a continuous function. Moreover, let $E_c : X \to \mathbb{R}$ be the energy functional associated to problem $(P_c)$ defined by

$$E_c(u) = \frac{1}{2} \|u\|^2 + \frac{c}{2} \sum_{k=1}^{T} (u(k))^2 - G(u), \quad u \in X,$$

where

$$G(u) = \sum_{k=1}^{T} G(u(k)), \quad \text{and} \quad G(s) = \int_{0}^{s} g(t)dt, \quad s \in \mathbb{R}.$$

It is immediate to show that $E_c$ is well-defined, it belongs to $C^1(X; \mathbb{R})$ and

$$E'_c(u)(v) = \langle u, v \rangle + c \sum_{k=1}^{T} u(k)v(k) - \sum_{k=1}^{T} g(u(k))v(k), \quad \forall u, v \in X.$$

Since we have

$$\langle u, v \rangle = -\sum_{k=1}^{T+1} \Delta(\Delta u(k-1))v(k),$$

an element $u \in X$ is a solution for $(P_c)$ if $E'_c(u)(v) = 0$ for every $v \in X$, i.e., $u$ is a critical point of $E_c$.

Let $d < 0 < a < b$ some fixed numbers. We introduce the set

$$N^b = \{u \in X : d \leq u(k) \leq b \quad \text{for every} \quad k \in [1, T]\}.$$  \hspace{1cm} (3)

We assume on $g : \mathbb{R} \to \mathbb{R}$ that

$$(H_g) \quad g(s) = 0 \quad \text{for} \quad s \leq 0, \quad \text{and} \quad g(s) \leq 0 \quad \text{for every} \quad s \in [a,b].$$

The main result of this section is as follows.

**Proposition 2.1.** Assume that $g : \mathbb{R} \to \mathbb{R}$ verifies $(H_g)$. Then

(a) $E_c$ is bounded from below on $N^b$ attaining its infimum at $\tilde{u} \in N^b$;
(b) $\tilde{u}(k) \in [0, a]$ for every $k \in [1, T]$;
(c) $\tilde{u}$ is a solution of $(P_c)$. 

Proof. (a) Since the norms $\|\cdot\|_\infty$ and $\|\cdot\|$ are equivalent in the finite-dimensional space $X$, the set $N^b$ is compact in $X$. Combining this fact with the continuity of $E_c$, we infer that $E_c|_{N^b}$ attains its infimum at $\bar{u} \in N^b$.

(b) Let $K = \{k \in [1,T] : \bar{u}(k) \notin [0,a]\}$ and suppose that $K \neq \emptyset$. Define the truncation function $\gamma : \mathbb{R} \to \mathbb{R}$ by $\gamma(s) = \min(s, s_+, a)$, where $s_+ = \max(s, 0)$. Now, set $w = \gamma \circ \bar{u}$. Since $\gamma(0) = 0$ we have that $w(0) = w(T + 1) = 0$, so $w \in X$. Moreover, $w(k) \in [0,a]$ for every $k \in [1,T]$; thus $w \in N^a \subset N^b$.

We introduce the sets

$$K_- = \{k \in K : \bar{u}(k) < 0\} \quad \text{and} \quad K_+ = \{k \in K : \bar{u}(k) > a\}.$$  

Thus, $K = K_- \cup K_+$, and we have that $w(k) = \bar{u}(k)$ for all $k \in [1,T] \setminus K$, $w(k) = 0$ for all $k \in K_-$, and $w(k) = a$ for all $k \in K_+$. Moreover, we have

$$E_c(w) - E_c(\bar{u}) = \frac{1}{2}(\|w\|^2 - \|\bar{u}\|^2) + \frac{c}{2} \sum_{k=1}^T [(w(k))^2 - (\bar{u}(k))^2] - [G(w) - G(\bar{u})]$$

$$=: \frac{1}{2}I_1 + \frac{c}{2}I_2 - I_3. \quad (4)$$

Since $\gamma$ is a Lipschitz function with Lipschitz-constant 1, and $w = \gamma \circ \bar{u}$, we have

$$I_1 = \|w\|^2 - \|\bar{u}\|^2 = \sum_{k=1}^{T+1} |\Delta w(k-1)|^2 - |\Delta \bar{u}(k-1)|^2$$

$$= \sum_{k=1}^{T+1} |w(k) - w(k-1)|^2 - |\bar{u}(k) - \bar{u}(k-1)|^2$$

$$\leq 0. \quad (5)$$

Moreover, we have

$$I_2 = \sum_{k=1}^T [(w(k))^2 - (\bar{u}(k))^2] = \sum_{k \in K} [(w(k))^2 - (\bar{u}(k))^2]$$

$$= \sum_{k \in K_-} -(\bar{u}(k))^2 + \sum_{k \in K_+} [a^2 - (\bar{u}(k))^2]$$

$$\leq 0. \quad (6)$$

Next, we estimate $I_3$. First, let us point out that $G(s) = 0$ for $s \leq 0$; thus, $\sum_{k \in K_+} [G(w(k)) - G(\bar{u}(k))] = 0$. By the mean value theorem, for every $k \in K_+$, there exists $n_k \in [a, \bar{u}(k)] \subset [a, b]$ such that $G(w(k)) - G(\bar{u}(k)) = G(a) - G(\bar{u}(k)) = g(n_k)(a - \bar{u}(k))$. Taking into account hypothesis $(H_g)$, we have that $G(w(k)) - G(\bar{u}(k)) \geq 0$ for every $k \in K_+$. Consequently,

$$I_3 = G(w) - G(\bar{u}) = \sum_{k \in K} [G(w(k)) - G(\bar{u}(k))] = \sum_{k \in K_+} [G(w(k)) - G(\bar{u}(k))]$$

$$\geq 0. \quad (7)$$
Combining relations (5)-(7) with (4), we have that

\[ E_c(w) - E_c(\bar{u}) \leq 0. \]

On the other hand, since \( w \in N^b \), then \( E_c(w) \geq E_c(\bar{u}) = \inf_{N^b} E_c \). So, every term in \( E_c(w) - E_c(\bar{u}) \) should be zero. In particular, from \( I_2 \), we have

\[ \sum_{k \in K_-} (\bar{u}(k))^2 = \sum_{k \in K_+} [a^2 - (\bar{u}(k))^2] = 0, \]

which imply that \( \bar{u}(k) = 0 \) for every \( k \in K_- \) and \( \bar{u}(k) = a \) for every \( k \in K_+ \). By definition of the sets \( K_- \) and \( K_+ \), we must have \( K_- = K_+ = \emptyset \), which contradicts \( K_- \cup K_+ = K \neq \emptyset \).

(c) Let us fix \( v \in X \) arbitrarily. Due to (b), it is clear that \( \bar{u} + \varepsilon v \in N^b \) for \( |\varepsilon| \) small enough. Consequently, due to (a), the function \( j(\varepsilon) = E_c(\bar{u} + \varepsilon v) \) has its minimum at 0; being differentiable at 0, we have that \( j'(0) = 0 \), i.e., \( E_c'(\bar{u})(v) = 0 \), which means that \( \bar{u} \) is a solution of \( (P_\varepsilon) \). This completes the proof. \( \square \)

3. Proof of Theorem 1.1

We assume hypothesis \((H^0)\) holds. In particular, we have \( f(0) = 0 \). One may fix \( c_0 > 0 \) such that \( \lim \inf_{s \to 0^+} \frac{f(s)}{s} < -c_0 < 0 \). Consequently, there is a sequence \( \{\sigma_n\}_n \subset (0, 1) \) converging (decreasingly) to 0, such that

\[ f(\sigma_n) < -c_0 \sigma_n. \]  

(8)

Let us define the functions \( g_0, G_0 : \mathbb{R} \to \mathbb{R} \) by

\[ g_0(s) = f(s_+) + c_0 s_+ \quad \text{and} \quad G_0(s) = \int_0^s g_0(t) dt, \quad s \in \mathbb{R}, \]  

(9)

where \( s_+ = \max(s, 0) \). Due to (8), \( g_0(\sigma_n) < 0 \); so, there are two sequences \( \{a_n\}_n \), \( \{b_n\}_n \subset (0, 1) \), both converging to 0, such that \( b_{n+1} < a_n < \sigma_n < b_n \) for every \( n \in \mathbb{N} \) and

\[ g_0(s) \leq 0 \quad \text{for all} \quad s \in [a_n, b_n]. \]

In this way, hypothesis \((H_a)\) is verified for \( g_0 \) on every interval \([a_n, b_n], n \in \mathbb{N} \). Applying Proposition 2.1 to every interval \([a_n, b_n], n \in \mathbb{N} \), the problem

\[
\begin{cases}
-\Delta(\Delta u(k - 1)) + c_0 u(k) = g_0(u(k)), & k \in [1, T], \\
 u(0) = u(T + 1) = 0,
\end{cases}
\]

(P_n)

has a sequence of non-negative solutions \( \{u_n^0\}_n \subset X \), where \( u_n^0 \) is a relative minimum of the functional \( E_{c_0} \) associated to \((P_n)\) on the set \( N^{b_n} \), \( n \in \mathbb{N} \). Furthermore, since \( g_0(s) = f(s) + c_0 s \) on the interval \((0, 1)\), the elements \( u_n^0 \) are also solutions of problem (P). Moreover, due to Proposition 2.1 (b), we also have

\[ 0 \leq u_n^0(k) \leq a_n \quad \text{for every} \quad k \in [1, T], \quad n \in \mathbb{N}. \]  

(10)
In the sequel, carrying out an energy-level analysis, we prove that there are infinitely many distinct elements in the sequence \( \{u_n^0\}_{n \in \mathbb{N}} \subset X \). Due to \((H^0)\) and (9), we have that 
\[
\limsup_{s \to 0^+} \frac{G_0(s)}{s^2} > \frac{1}{T} + \frac{c_0}{2}.
\]
In particular, there exists a sequence \( \{s_n\}_{n \in \mathbb{N}} \) with \( 0 < s_n \leq a_n, n \in \mathbb{N} \), and
\[
G_0(s_n) > \left( \frac{1}{T} + \frac{c_0}{2} \right) s_n^2.
\]
Define the function \( w_n \in X \) by \( w_n(k) = s_n \) for every \( k \in [1, T] \). Then, we have
\[
E_{c_0}(w_n) = \frac{1}{2} \sum_{k=1}^{T+1} |\Delta w_n(k-1)|^2 + \frac{c_0}{2} \sum_{k=1}^{T} (w_n(k))^2 - \sum_{k=1}^{T} G_0(w_n(k))
\]
\[
= s_n^2 + \frac{c_0 T}{2} s_n^2 - T G_0(s_n)
\]
\[
< s_n^2 + \frac{c_0 T}{2} s_n^2 - T \left( \frac{1}{T} + \frac{c_0}{2} \right) s_n^2
\]
\[
= 0.
\]
The above estimation and \( w_n \in N^{a_n} \subset N^{b_n} \) show that
\[
E_{c_0}(u_n^0) = \min_{N^{b_n}} E_{c_0} \leq E_{c_0}(w_n) < 0 \quad \text{for all } n \in \mathbb{N}.
\] (11)

Once we prove that
\[
\lim_{n \to \infty} E_{c_0}(u_n^0) = 0,
\] (12)
our claim holds. Indeed, (11) and (12) imply that there are infinitely many distinct elements in the sequence \( \{u_n^0\}_{n \in \mathbb{N}} \subset X \). We clearly have
\[
E_{c_0}(u_n^0) = \frac{1}{2} \sum_{k=1}^{T+1} |\Delta u_n^0(k-1)|^2 + \frac{c_0}{2} \sum_{k=1}^{T} (u_n^0(k))^2 - \sum_{k=1}^{T} G_0(u_n^0(k))
\]
\[
\geq - \sum_{k=1}^{T} G_0(u_n^0(k)) \geq - \sum_{k=1}^{T} u_n^0(k) \max_{s \in [0, u_n^0(k)]} |g_0(s)|
\]
\[
\geq - \max_{s \in [0, a_n]} |g_0(s)| \sum_{k=1}^{T} u_n^0(k)
\]
\[
\geq - a_n T \max_{s \in [0, 1]} |g_0(s)|.
\]
Since \( \lim_{n \to \infty} a_n = 0 \), the above estimate and (11) yield (12).

Relation (1) is an immediate consequence of (10), \( \lim_{n \to \infty} a_n = 0 \), and to the fact that the norms \( \| \cdot \|_\infty \) and \( \| \cdot \| \) are equivalent. The proof of Theorem 1.1 is complete.
4. Proof of Theorem 1.2

The proof is similar to that of Theorem 1.1. We assume hypothesis \((H^\infty)\) holds. We choose \(c_\infty > 0\) such that \(\liminf_{s \to \infty} \frac{f(s)}{s} < -c_\infty < 0\). Consequently, we may fix a sequence \(\{s_n\}_n \subset (0, \infty)\) such that \(\lim_{n \to \infty} s_n = \infty\) and

\[
f(s_n) < -c_\infty s_n. \tag{13}
\]

We define the functions \(g_\infty, G_\infty : \mathbb{R} \to \mathbb{R}\) by

\[
g_\infty(s) = f(s_+) + c_\infty s_+ \quad \text{and} \quad G_\infty(s) = \int_0^s g_\infty(t) \, dt, \quad s \in \mathbb{R}. \tag{14}
\]

Due to the right hand side inequality of \((H^\infty)\) and (14), we have that

\[
\limsup_{s \to \infty} \frac{G_\infty(s)}{s^2} > \frac{1}{T} + \frac{c_\infty}{2}. \tag{15}
\]

Since \(\lim_{n \to \infty} s_n = \infty\), one can fix a subsequence \(\{s_{m_n}\}_n \subset \{s_n\}_n\) such that \(s_n \leq s_{m_n}\) for every \(n \in \mathbb{N}\). On account of (13), \(g_\infty(s_{m_n}) < 0\); thus, we may fix two sequences \(\{a_n\}_n, \{b_n\}_n \subset (0, \infty)\) such that \(a_n < s_{m_n} < b_n < a_{n+1}\) for every \(n \in \mathbb{N}\), \(\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = \infty\), and

\[
g_\infty(s) \leq 0 \quad \text{for all} \quad s \in [a_n, b_n].
\]

Consequently, the function \(g_\infty\) fulfills \((H_g)\) on every interval \([a_n, b_n], \, n \in \mathbb{N}\). We apply Proposition 2.1 to every interval \([a_n, b_n], \, n \in \mathbb{N}\), obtaining that the problem

\[
\begin{cases}
-\Delta(\Delta u(k - 1)) + c_\infty u(k) = g_\infty(u(k)), & k \in [1, T], \\
u(0) = u(T + 1) = 0,
\end{cases} \tag{P_{c_\infty}}
\]

has a sequence of non-negative solutions \(\{u_n^\infty\}_n \subset X\), where \(u_n^\infty\) is a relative minimum of the functional \(E_{c_\infty}\) associated to \((P_{c_\infty})\) on the set \(N^{b_n}, \, n \in \mathbb{N}\). Since \(g_\infty(s) = f(s) + c_\infty s\) on \([0, \infty)\), the elements \(u_n^\infty\) are solutions not only for \((P_{c_\infty})\) but also for \((P)\).

Now, we are going to prove that there are infinitely many distinct elements in the sequence \(\{u_n^\infty\}_n \subset X\). To do this, it is enough to show that

\[
\lim_{n \to \infty} E_{c_\infty}(u_n^\infty) = -\infty. \tag{16}
\]

Define the function \(w_n \in X\) by \(w_n(k) = s_n\) for every \(k \in [1, T]\). Then, by using
(15), we have

\[
E_{c_\infty}(w_n) = \frac{1}{2} \sum_{k=1}^{T+1} |\Delta w_n(k-1)|^2 + \frac{c_\infty}{2} \sum_{k=1}^{T} (w_n(k))^2 - \sum_{k=1}^{T} G_\infty(w_n(k))
\]

\[
= s_n^2 + \frac{c_\infty T}{2} s_n^2 - T G_\infty(s_n)
\]

\[
< s_n^2 + \frac{c_\infty T}{2} s_n^2 - T \left( \frac{1}{T} + \frac{c_\infty}{2} + \varepsilon_\infty \right) s_n^2
\]

\[
= -T \varepsilon_\infty s_n^2.
\]

By construction, we know that \(w_n \in N^{s_n} \subset N^{b_n}\), thus

\[
E_{c_\infty}(u_\infty^n) = \min_{N^{b_n}} E_{c_\infty} \leq E_{c_\infty}(w_n) < -T \varepsilon_\infty s_n^2 \quad \text{for all } n \in \mathbb{N}.
\] (17)

Since \(\lim_{n \to \infty} s_n = \infty\), relation (17) implies (16).

It remains to prove (2). Since the norms \(\| \cdot \|_\infty\) and \(\| \cdot \|\) are equivalent, it is enough to prove the former limit, i.e., \(\lim_{n \to \infty} \| u_\infty^n \|_\infty = \infty\). By contradiction, we assume that for a subsequence of \(\{ u_\infty^n \}_n\), still denoted by \(\{ u_\infty^n \}_n\), one can find a constant \(C > 0\) such that \(\| u_\infty^n \|_\infty \leq C\) for every \(n \in \mathbb{N}\). Therefore, we have

\[
E_{c_\infty}(u_\infty^n) \geq -\sum_{k=1}^{T} G_\infty(u_\infty^n(k)) \geq -T \max_{s \in [0,C]} |G_\infty(s)| \quad \text{for every } n \in \mathbb{N}.
\]

This inequality contradicts relation (16) which completes the proof of Theorem 1.2.

**Remark 1.** When \(T = 2\), the conclusions of Theorems 1.1 and 1.2 may be obtained in a very simple way. In this case, it is enough to solve the system

\[
\begin{align*}
2a - b &= f(a), \\
2b - a &= f(b), \\
a, b &> 0.
\end{align*}
\]

\((P')\)

Indeed, a solution of \((P)\) is any function \(u : [0, 3] \to \mathbb{R}\) defined by \(u(0) = u(3) = 0, u(1) = a, u(2) = b\). As one can observe, if there is a sequence of distinct fixed points for \(f\), say \(\{c_n\}_n \subset (0, \infty)\), we have infinitely many solutions for problem \((P')\) of the form \((a, b) = (c_n, c_n)\). Let us assume the contrary, i.e., there is at most finite number of distinct fixed points for \(f\). Combining this assumption with the left hand side of \((H^0)\), there exists a \(\delta > 0\) such that \(f(s) < s\) for every \(s \in (0, \delta)\). After an integration we obtain that

\[\limsup_{s \to 0^+} \frac{F(s)}{s^2} \leq \frac{1}{2} = \frac{1}{T}\]

which contradicts the right hand side of \((H^0)\). In a similar manner, when \((H^\infty)\) holds, we can fix a compact set \(L \subset [0, \infty)\) such that \(f(s) < s\) for every \(s \in (0, \infty) \setminus L\), which contradicts the right hand side of \((H^\infty)\).

The above arguments also suggest that the constant \(\frac{1}{T}\) in \((H^0)\) and \((H^\infty)\) is optimal.
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