COMBINED EFFECTS AND DEGENERATE PHENOMENA
IN NONLINEAR STATIONARY PROBLEMS

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In this survey paper we are concerned with several nonlinear stationary problems involving nonhomogeneous differential operators. We report on some recent qualitative results related with various nonlinear problems in Orlicz-Sobolev spaces. Our analysis combines spectral analysis techniques with variational methods.

1. Basic properties of Orlicz-Sobolev spaces

Let \( \Omega \subset \mathbb{R}^N \) be an open set with smooth boundary. In Orlicz [31], the standard Lebesgue spaces \( L^p(\Omega) \) were replaced by more general function spaces denoted \( L_\Phi(\Omega) \) and which are now called Orlicz spaces. The spaces \( L_\Phi(\Omega) \) were thoroughly studied in the monograph by Krasnosel'skii & Rutickii [18] and also in the doctoral thesis of Luxemburg [23]. If the role played by \( L^p(\Omega) \) in the definition of the Sobolev spaces \( W^{m,p}(\Omega) \) is assigned instead to an Orlicz space \( L_\Phi(\Omega) \), the resulting space is denoted by \( W^{m,p_\Phi}(\Omega) \) and called an Orlicz-Sobolev space. Many properties of Sobolev spaces have been extended to Orlicz-Sobolev spaces, mainly by Donaldson & Trudinger [12] and O’Neill [30]. Orlicz-Sobolev spaces have been used in the last decades to model various
We recall in what follows the definition and the main properties of Orlicz-Sobolev spaces. Consider the mapping $\phi : \mathbb{R} \to \mathbb{R}$ defined by $\phi(t) := \log(1 + |t|^q) \cdot |t|^{p-2}$. Set $\Phi(t) := \int_0^t \phi(s)ds$. A straightforward computation yields

$$\Phi(t) = \frac{1}{p} \log(1 + |t|^q) \cdot |t|^p - \int_0^{|t|} \frac{s^{p+q-1}}{1 + s^q} ds,$$

for all $t \in \mathbb{R}$. We observe that $\phi$ is an odd, increasing homeomorphism of $\mathbb{R}$ into $\mathbb{R}$, while $\Phi$ is convex and even on $\mathbb{R}$ and increasing from $\mathbb{R}_+$ to $\mathbb{R}_+$.

Set

$$\Phi^*(t) := \int_0^t \phi^{-1}(s) ds,$$

for all $t \in \mathbb{R}$.

The functions $\Phi$ and $\Phi^*$ are complementary $N$-functions (see Kranosel’kii & Rutickii [18]).

Define the Orlicz class

$$K_\Phi(\Omega) := \{u : \Omega \to \mathbb{R}, \text{measurable}; \int_\Omega \Phi(|u(x)|) \, dx < \infty\}$$

and the Orlicz space

$$L_\Phi(\Omega) := \text{the linear hull of } K_\Phi(\Omega).$$

The space $L_\Phi(\Omega)$ is a Banach space endowed with the Luxemburg norm

$$\|u\|_\Phi := \inf \left\{ k > 0; \int_\Omega \Phi\left(\frac{u(x)}{k}\right) \, dx \leq 1 \right\}$$

or the equivalent norm (the Orlicz norm)

$$\|u\|_{\Phi} := \sup \left\{ \left| \int_\Omega uv \, dx \right|; v \in K_{\Phi^*}(\Omega), \int_\Omega \Phi(|v|) \, dx \leq 1 \right\},$$

where $\Phi^*$ denotes the conjugate Young function of $\Phi$, that is,

$$\Phi^*(t) = \sup\{ts - \Phi(s); s \in \mathbb{R}\}.$$

By Lemma 2.4 and Example 2 in Clément, de Pagter, Sweers & de Thélin [11, p. 243] we have

$$1 < \liminf_{t \to \infty} \frac{t\phi(t)}{\Phi(t)} \leq \sup_{t > 0} \frac{t\phi(t)}{\Phi(t)} < \infty. \quad (1)$$
These inequalities imply that $\Phi$ satisfies the $\Delta_2$-condition. By Lemma C.4 in [11] it follows that $\Phi^*$ also satisfies the $\Delta_2$-condition. Then, according to Adams [2, p. 234], it follows that $L_\Phi(\Omega) = K_\Phi(\Omega)$. Moreover, by Theorem 8.19 in Adams [2], $L_\Phi(\Omega)$ is reflexive.

We denote by $W^1 L_\Phi(\Omega)$ the Orlicz-Sobolev space defined by

$$W^1 L_\Phi(\Omega) := \left\{ u \in L_\Phi(\Omega); \frac{\partial u}{\partial x_i} \in L_\Phi(\Omega), i = 1, \ldots, N \right\}.$$  

Then $W^1 L_\Phi(\Omega)$ is a Banach space with respect to the norm

$$\|u\|_{1,\Phi} := \|u\|_\Phi + \|\nabla u\|_\Phi.$$  

We also define the Orlicz-Sobolev space $W^1_0 L_\Phi(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^1 L_\Phi(\Omega)$. By Lemma 5.7 in [16] we obtain that on $W^1_0 L_\Phi(\Omega)$ we may consider an equivalent norm $\|u\| := \|\nabla u\|_\Phi$. The space $W^1_0 L_\Phi(\Omega)$ is also a reflexive Banach space.

We refer to Adams [2], Luxemburg [23], and Kranosel’kii & Rutickii [18] for more details.

2. Crucial role of nonlinearities sign

Let $2^*$ denote the critical Sobolev exponent, that is, $2^* := 2N/(N-2)$ if $N \geq 3$ and $2^* := +\infty$ if $N \in \{1, 2\}$. If $2 < r < 2^*$, consider the Dirichlet problems

$$\begin{cases} 
-\Delta u = -\lambda u + u^{r-1}, & \text{in } \Omega \\
u = 0, & \text{on } \partial \Omega \\
u > 0, & \text{in } \Omega 
\end{cases}$$  

and

$$\begin{cases} 
-\Delta u = \lambda u - u^{r-1}, & \text{in } \Omega \\
u = 0, & \text{on } \partial \Omega \\
u > 0, & \text{in } \Omega.
\end{cases}$$  

A direct application of the mountain pass theorem implies that problem (2) has at least one solution for any $\lambda > 0$. By multiplication with the first eigenfunction $\phi_1 > 0$ of the Laplace operator in (3) we obtain

$$\lambda_1 \int_\Omega u \phi_1 dx = \lambda \int_\Omega u \phi_1 dx - \int_\Omega u^{r-1} \phi_1 dx.$$  

Thus, a necessary condition that problem (3) has a solution is that $\lambda$ is sufficiently large.
In this section, we describe the corresponding setting in the framework of nonhomogeneous differential operators (see Mihăilescu & Rădulescu [26]).

We first consider the boundary value problem
\begin{equation}
\begin{aligned}
- \text{div} \left( \log(1 + |\nabla u|^q) |\nabla u|^{p-2} \nabla u \right) &= -\lambda |u|^{p-2} u + |u|^{r-2} u, & \text{in } \Omega \\
u &= 0, & \text{on } \partial \Omega.
\end{aligned}
\end{equation}

We say that \( u \in W^1_0 L_\Phi(\Omega) \) is a weak solution of problem (4) if
\[
\int_\Omega \log(1 + |\nabla u(x)|^q) |\nabla u(x)|^{p-2} \nabla u \cdot \nabla v \, dx + \lambda \int_\Omega |u(x)|^{p-2} u(x) v(x) \, dx \\
- \int_\Omega |u(x)|^{r-2} u(x) v(x) \, dx = 0
\]
for all \( v \in W^1_0 L_\Phi(\Omega) \).

The property corresponding to problem (2) is the following multiplicity result.

**Theorem 2.1.** Assume that \( p, q > 1 \), \( p + q < N \), \( p + q < r \) and \( r < (Np - N + p)/(N - p) \). Then, for every \( \lambda > 0 \) problem (4), has infinitely many weak solutions.

We remark that in the particular case \( q = 1 \), \( \lambda = 0 \), \( 1 < p < N - 1 \), and \( p < r \leq [N(p - 1) + p]/(N - p) \), problem (4) has a nontrivial weak solution, by means of Theorem 1.2 in Clément, García-Huidobro, Manásevich & Schmitt [10]. On the other hand, Theorem 1.2 in [10] also applies for solving equations involving more general differential operators \( \text{div}(a(|\nabla u|) |\nabla u|) \).

Next, we consider the problem
\begin{equation}
\begin{aligned}
- \text{div} \left( \log(1 + |\nabla u|^q) |\nabla u|^{p-2} \nabla u \right) &= \lambda |u|^{p-2} u - |u|^{r-2} u, & \text{in } \Omega \\
u &= 0, & \text{on } \partial \Omega.
\end{aligned}
\end{equation}

We say that \( u \in W^1_0 L_\Phi(\Omega) \) is a weak solution of problem (5) if
\[
\int_\Omega \log(1 + |\nabla u(x)|^q) |\nabla u(x)|^{p-2} \nabla u \cdot \nabla v \, dx - \lambda \int_\Omega |u(x)|^{p-2} u(x) v(x) \, dx \\
+ \int_\Omega |u(x)|^{r-2} u(x) v(x) \, dx = 0
\]
for all \( v \in W^1_0 L_\Phi(\Omega) \).

The following result shows that problem (5) has a solution provided that \( \lambda \) is large enough.

**Theorem 2.2.** Assume that the hypotheses of Theorem 2.1 are fulfilled. Then there exists \( \lambda_* > 0 \) such that for any \( \lambda \geq \lambda_* \), problem (5) has a nontrivial weak solution.
We sketch in what follows the proof of Theorem 2.1. The key argument is the following $\mathbb{Z}_2$-symmetric version (for even functionals) of the Mountain Pass Lemma (see Theorem 9.12 in Rabinowitz [35]).

**Mountain Pass Lemma.** Let $X$ be an infinite dimensional real Banach space and let $I \in C^1(X, \mathbb{R})$ be even, satisfying the Palais-Smale condition (that is, any sequence $\{x_n\} \subset X$ such that $\{I(x_n)\}$ is bounded and $I'(x_n) \to 0$ in $X^*$ has a convergent subsequence) and $I(0) = 0$. Suppose that

1. there exist two constants $\rho, b > 0$ such that $I(x) \geq b$ if $\|x\| = \rho$;
2. for each finite dimensional subspace $X_1 \subset X$, the set $\{x \in X_1; I(x) \geq 0\}$ is bounded.

Then $I$ has an unbounded sequence of critical values.

Let $E$ denote the Orlicz-Sobolev space $W^{1,0}_0L_\Phi(\Omega)$. Let $\lambda > 0$ be arbitrary but fixed.

The energy functional associated to problem (4) is $J_\lambda : E \to \mathbb{R}$ defined by

$$J_\lambda(u) := \int_\Omega \Phi(|\nabla u(x)|) \, dx + \frac{\lambda}{p} \int_\Omega |u(x)|^p \, dx - \frac{1}{r} \int_\Omega |u(x)|^r \, dx.$$

We split the proof of Theorem 2.1 into several steps.

**Step 1.** There exist $\eta > 0$ and $\alpha > 0$ such that $J_\lambda(u) \geq \alpha > 0$ for any $u \in E$ with $\|u\| = \eta$.

**Step 2.** Assume that $E_1$ is a finite dimensional subspace of $E$. Then the set $S = \{u \in E_1; J_\lambda(u) \geq 0\}$ is bounded.

**Step 3.** Assume that $\{u_n\} \subset E$ is a sequence which satisfies the properties

$$|J_\lambda(u_n)| < M \quad (6)$$
$$J_\lambda'(u_n) \to 0 \quad \text{as} \quad n \to \infty, \quad (7)$$

where $M$ is a positive constant. Then $\{u_n\}$ possesses a convergent subsequence.

**Proof of Theorem 2.1 completed.** The energy functional $J_\lambda$ is even and verifies $J_\lambda(0) = 0$. Step 3 implies that $J_\lambda$ satisfies the Palais-Smale condition. On the other hand, Steps 1 and 2 show that conditions (I1) and (I2) are satisfied. Thus, the mountain pass lemma can be applied to the functional $J_\lambda$. We conclude that equation (4) has infinitely many weak solutions in $E$. The proof of Theorem 2.1 is complete. \hfill \Box

We point out that the Orlicz-Sobolev space $E$ cannot be replaced by a classical Sobolev space. Indeed, in such a case, condition (I1) in the mountain
pass lemma cannot be satisfied (see the proof of Remark 4 in Clément, García-Huidobro, Manásevich & Schmitt [10, p. 56-57]).

Fix \( \lambda > 0 \) and consider the energy functional associated to problem (5), that is,

\[
I_\lambda(u) := \int_\Omega \Phi(|\nabla u(x)|) \, dx - \frac{\lambda}{p} \int_\Omega |u(x)|^p \, dx + \frac{1}{r} \int_\Omega |u(x)|^r \, dx
\]
for all \( u \in E \).

Standard arguments show that \( I_\lambda \) is coercive and lower semi-continuous. Thus, there exists a global minimizer \( u_\lambda \in E \) of \( I_\lambda \), hence a weak solution of problem (5). We show that \( u_\lambda \) is not trivial for \( \lambda \) large enough. Indeed, letting \( t_0 > 1 \) be a fixed real and \( \Omega_1 \) be an open subset of \( \Omega \) with \( |\Omega_1| > 0 \) we deduce that there exists \( u_1 \in \mathcal{C}_0^\infty(\Omega) \subset E \) such that \( u_1(x) = t_0 \) for any \( x \in \overline{\Omega}_1 \) and \( 0 \leq u_1(x) \leq t_0 \) in \( \Omega \setminus \Omega_1 \). We have

\[
I_\lambda(u_1) = \int_\Omega \Phi(|\nabla u_1(x)|) \, dx - \frac{\lambda}{p} \int_\Omega |u_1(x)|^p \, dx + \frac{1}{r} \int_\Omega |u_1(x)|^r \, dx
\leq L - \frac{\lambda}{p} \int_{\Omega_1} |u_1(x)|^p \, dx
\leq L - \frac{\lambda}{p} \cdot t_0^p \cdot |\Omega_1|
\]
where \( L \) is a positive constant. Thus, there exists \( \lambda_* > 0 \) such that \( I_\lambda(u_1) < 0 \) for any \( \lambda \in [\lambda_*, \infty) \). It follows that \( I_\lambda(u_\lambda) < 0 \) for any \( \lambda \geq \lambda_* \) and thus \( u_\lambda \) is a nontrivial weak solution of problem (5) for \( \lambda \) large enough. The proof of Theorem 2.2 is complete.

A careful analysis of the proofs shows that Theorems 2.1 and 2.2 still remain valid for more general classes of differential operators. Indeed, we can replace \( \text{div}(\log(1 + |\nabla u(x)|^q)|\nabla u(x)|^{p-2}\nabla u(x)) \) by \( \text{div}(a(|\nabla u(x)|)|\nabla u(x)|) \), where \( a(t) \) is so that the assumption (1) is fulfilled. Some potentials \( a(t) \) satisfying this hypothesis are \( a(t) = |t|^\alpha - 1 \) (\( \alpha > 0 \)) and \( a(t) = |t|^\alpha / \log(1 + |t|^\beta) \) (0 < \( \beta < \alpha \).

3. Eigenvalue problems in Orlicz-Sobolev spaces

In this section we are concerned with a related nonlinear eigenvalue problem in a new framework, corresponding to Orlicz-Sobolev spaces. The main result establishes a curious phenomenon, which does not hold in the standard setting corresponding to the Laplace operator. More precisely, we prove that there exist two constants \( 0 < \lambda_0 \leq \lambda_1 \) such that any \( \lambda \in [\lambda_1, \infty) \) is an eigenvalue, while any \( \lambda \in (0, \lambda_0) \) is not an eigenvalue of our problem.

Consider the nonlinear eigenvalue problem

\[
\begin{aligned}
-\text{div}(a_1(|\nabla u|) + a_2(|\nabla u|))\nabla u &= \lambda |u|^{q(x)-2}u, & \text{in } \Omega \\
u &= 0, & \text{on } \partial \Omega.
\end{aligned}
\]
We assume that for any \( i = 1, 2 \), the functions \( a_i : (0, \infty) \to \mathbb{R} \) are such that the mappings \( \phi_i : \mathbb{R} \to \mathbb{R} \) defined by

\[
\phi_i(t) = \begin{cases} 
a_i(|t|)t, & \text{for } t \neq 0 \\
o, & \text{for } t = 0,
\end{cases}
\]

are odd, increasing homeomorphisms from \( \mathbb{R} \) onto \( \mathbb{R} \). We also suppose throughout this section that \( \lambda > 0 \) and \( q : \overline{\Omega} \to (0, \infty) \) is a continuous function.

We work with functions \( \Phi_i \) and \( (\Phi_i)_* \), \( i = 1, 2 \), satisfying the \( \Delta_2 \)-condition (at infinity), namely

\[
1 < \liminf_{t \to \infty} \frac{t\phi_i(t)}{\Phi_i(t)} \leq \limsup_{t > 0} \frac{t\phi_i(t)}{\Phi_i(t)} < \infty.
\]

Then \( L_{\Phi_i}(\Omega) \) and \( W_{0}^{1}L_{\Phi_i}(\Omega) \), \( i = 1, 2 \), are reflexive Banach spaces.

Now we introduce the Orlicz-Sobolev conjugate \( (\Phi_i)_* \) of \( \Phi_i \), \( i = 1, 2 \), defined as

\[
(\Phi_i)^{-1}_*(t) = \int_{0}^{t} \left( \frac{1}{s^{(N+1)/N}} \right) ds.
\]

We assume that

\[
\lim_{t \to 0} \int_{1}^{t} \frac{(\Phi_i)^{-1}_*(s)}{s^{(N+1)/N}} ds < \infty, \quad \text{and} \quad \lim_{t \to \infty} \int_{1}^{t} \frac{(\Phi_i)^{-1}_*(s)}{s^{(N+1)/N}} ds = \infty, \quad i = 1, 2.
\]

Finally, we define

\[
(p_i)_0 := \inf_{t > 0} \frac{t\phi_i(t)}{\Phi_i(t)} \quad \text{and} \quad (p_i)^0 := \sup_{t > 0} \frac{t\phi_i(t)}{\Phi_i(t)}, \quad i = 1, 2.
\]

We study problem (8) under the following basic assumptions:

\[
1 < (p_2)_0 \leq (p_2)^0 < q(x) < (p_1)_0 \leq (p_1)^0, \quad \forall x \in \overline{\Omega}
\]

and

\[
\lim_{t \to \infty} \frac{|t|^{q^+}}{(\Phi_2)_*(kt)} = 0, \quad \text{for all } k > 0.
\]

We say that \( \lambda \in \mathbb{R} \) is an eigenvalue of problem (8) if there exists \( u \in W_{0}^{1}L_{\Phi_1}(\Omega) \setminus \{0\} \) such that

\[
\int_{\Omega} (a_1(|\nabla u|) + a_2(|\nabla u|)) \nabla u \nabla v dx = \lambda \int_{\Omega} |u|^{q(x)-2} uv dx = 0,
\]

for all \( v \in W_{0}^{1}L_{\Phi_1}(\Omega) \). We point out that if \( \lambda \) is an eigenvalue of problem (4) then the corresponding \( u \in W_{0}^{1}L_{\Phi_1}(\Omega) \setminus \{0\} \) is a weak solution of (8).
Define
\[ \lambda_1 := \inf_{u \in W_0^1L_{\Phi_1}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} \Phi_1(|\nabla u|) \, dx + \int_{\Omega} \Phi_2(|\nabla u|) \, dx}{\int_{\Omega} \frac{1}{q(x)} |u|^{q(x)} \, dx}. \]

The main result in this section is the following (see Mihăilescu & Rădulescu \[27\]).

**Theorem 3.1.** Assume that conditions (9), (10) and (11) are fulfilled. Then \( \lambda_1 > 0 \). Moreover, any \( \lambda \in [\lambda_1, \infty) \) is an eigenvalue of problem (8). Furthermore, there exists a positive constant \( \lambda_0 \) such that \( \lambda_0 \leq \lambda_1 \) and any \( \lambda \in (0, \lambda_0) \) is not an eigenvalue of problem (8).

**Proof.** Let \( E \) denote the generalized Sobolev space \( W_0^1L_{\Phi_1}(\Omega) \). Denote by \( \| \cdot \|_1 \) the norm on \( W_0^1L_{\Phi_1}(\Omega) \) and by \( \| \cdot \|_2 \) the norm on \( W_0^1L_{\Phi_2}(\Omega) \).

Define the energy functionals \( J, I, J_1, I_1 : E \to \mathbb{R} \) by
\[
J(u) = \int_{\Omega} \Phi_1(|\nabla u|) \, dx + \int_{\Omega} \Phi_2(|\nabla u|) \, dx,
\]
\[
I(u) = \int_{\Omega} \frac{1}{q(x)} |u|^{q(x)} \, dx,
\]
\[
J_1(u) = \int_{\Omega} a_1(|\nabla u|)|\nabla u|^2 \, dx + \int_{\Omega} a_2(|\nabla u|)|\nabla u|^2 \, dx,
\]
\[
I_1(u) = \int_{\Omega} |u|^{q(x)} \, dx.
\]

Then \( J, I \in C^1(E, \mathbb{R}) \) and for all \( u, v \in E \),
\[
\langle J'(u), v \rangle = \int_{\Omega} (a_1(|\nabla u|) + a_2(|\nabla u|)) \nabla u \nabla v \, dx,
\]
\[
\langle I'(u), v \rangle = \int_{\Omega} |u|^{q(x) - 2} uv \, dx.
\]

We split the proof of Theorem 3.1 into four steps.

**Step 1.** We have \( \lambda_1 > 0 \).

A straightforward computation combined with relation (10) implies
\[
2 \cdot c \cdot (\Phi_1(|\nabla u(x)|) + \Phi_2(|\nabla u(x)|)) \geq 2 \cdot (|\nabla u(x)|^{(p_1)_0} + |\nabla u(x)|^{(p_2)_0}) \geq |\nabla u(x)|^q + |\nabla u(x)|^{q^-}
\]
and
\[
|u(x)|^q + |u(x)|^{q^-} \geq |u(x)|^{q(x)}.
\]
Integrating these inequalities we find

\[ 2c \cdot \int_{\Omega} (\Phi_1(|\nabla u(x)|) + \Phi_2(|\nabla u(x)|)) \, dx \geq \int_{\Omega} (|\nabla u|_p^q + |\nabla u|_1^q) \, dx, \quad \forall u \in E \]  

(12)

and

\[ \int_{\Omega} (|u|_p^q + |u|_1^q) \, dx \geq \int_{\Omega} |u|^{q(x)} \, dx, \quad \forall u \in E. \]  

(13)

On the other hand, there exist two positive constants \( \lambda_{q^+} \) and \( \lambda_{q^-} \) such that

\[ \int_{\Omega} |\nabla u|_p^q \, dx \geq \lambda_{q^+} \int_{\Omega} |u|_p^q \, dx, \quad \forall u \in W^{1,q^+}(\Omega) \]  

(14)

and

\[ \int_{\Omega} |\nabla u|_1^q \, dx \geq \lambda_{q^-} \int_{\Omega} |u|_1^q \, dx, \quad \forall u \in W^{1,q^-}(\Omega). \]  

(15)

Using again the fact that \( q^- \leq q^+ < (p_i)_0 \), we deduce that \( E \) is continuously embedded both in \( W^{1,q^+}(\Omega) \) and in \( W^{1,q^-}(\Omega) \). Thus, inequalities (14) and (15) hold true for any \( u \in E \).

Using inequalities (14), (15) and (13) we obtain a positive constant \( \mu \) such that

\[ \int_{\Omega} (|\nabla u|_p^q + |\nabla u|_1^q) \, dx \geq \mu \int_{\Omega} |u|^{q(x)} \, dx, \quad \forall u \in E. \]  

(16)

Next, inequalities (16) and (12) yield

\[ \int_{\Omega} (\Phi_1(|\nabla u(x)|) + \Phi_2(|\nabla u(x)|)) \, dx \geq \frac{\mu}{2c} \int_{\Omega} |u|^{q(x)} \, dx, \quad \forall u \in E. \]  

(17)

The above inequality implies

\[ J(u) \geq \frac{\mu \cdot q^-}{2c} I(u) \quad \forall u \in E. \]  

(18)

The last inequality assures that \( \lambda_1 > 0 \) and thus, step 1 is verified.

We point out that by the definitions of \( (p_i)_0, i = 1, 2 \), we have

\[ a_i(t) \cdot t^2 = \phi_i(t) \cdot t \geq (p_i)_0 \Phi_i(t), \quad \forall t > 0. \]

The above inequality and relation (17) imply

\[ \lambda_0 = \inf_{v \in E \setminus \{0\}} \frac{J_1(v)}{I_1(v)} > 0. \]  

(19)

\textit{Step 2.} We show that \( \lambda_1 \) is an eigenvalue of problem (8).

We start with some auxiliary results.
Lemma 3.2. The following relations hold true:

\[ \lim_{\|u\| \to \infty} \frac{J(u)}{I(u)} = \infty \]  
(20)

and

\[ \lim_{\|u\| \to 0} \frac{J(u)}{I(u)} = \infty. \]  
(21)

Proof of lemma. Since \( E \) is continuously embedded in \( L^{q^+}(\Omega) \) it follows that there exist two positive constants \( c_1 \) and \( c_2 \) such that

\[ \|u\|_1 \geq c_1 \cdot |u|_{q^+}, \ \forall \ u \in E \]  
(22)

and

\[ \|u\|_1 \geq c_2 \cdot |u|_{q^-}, \ \forall \ u \in E. \]  
(23)

For any \( u \in E \) with \( \|u\|_1 > 1 \), relations (13), (22), (23) imply that

\[ \frac{J(u)}{I(u)} \geq \frac{\|u\|_1^{(p_1)0}}{|u|_{q^+}^{q^+} + |u|_{q^-}^{q^-}} \geq \frac{c_1^{-q^+} \|u\|_1^{q^+} + c_2^{-q^-} \|u\|_1^{q^-}}{q^-}. \]

Since \((p_1)0 > q^+ \geq q^-\), passing to the limit as \( \|u\|_1 \to \infty \) in the above inequality we deduce that relation (20) holds true.

Next, the space \( W^{1}_{0} L_{\Phi_1}(\Omega) \) is continuously embedded in \( W^{1}_{0} L_{\Phi_2}(\Omega) \). Thus, \( \|u\|_1 < 1 \) is small enough, then \( \|u\|_2 < 1 \). On the other hand, since (11) holds true we deduce that \( W^{1}_{0} L_{\Phi_2}(\Omega) \) is continuously embedded in \( L^{q^+}(\Omega) \). It follows that there exist two positive constants \( d_1 \) and \( d_2 \) such that

\[ \|u\|_2 \geq d_1 \cdot |u|_{q^+}, \ \forall \ u \in W^{1}_{0} L_{\Phi_2}(\Omega) \]  
(24)

and

\[ \|u\|_2 \geq d_2 \cdot |u|_{q^-}, \ \forall \ u \in W^{1}_{0} L_{\Phi_2}(\Omega). \]  
(25)

Thus, for any \( u \in E \) with \( \|u\|_1 < 1 \) small enough, relations (13), (24), (25) imply

\[ \frac{J(u)}{I(u)} \geq \frac{\int_{\Omega} \Phi_2(|\nabla u|) \, dx}{|u|_{q^+}^{q^+} + |u|_{q^-}^{q^-}} \geq \frac{\|u\|_2^{(p_2)0}}{d_1^{-q^+} \|u\|_2^{q^+} + d_2^{-q^-} \|u\|_2^{q^-}}. \]

Since \((p_2)0 < q^- \leq q^+\), passing to the limit as \( \|u\|_1 \to 0 \) (and thus, \( \|u\|_2 \to 0 \)) in the above inequality we deduce that relation (21) holds true. The proof of Lemma 3.2 is complete. \( \square \)
Lemma 3.3. There exists $u \in E \setminus \{0\}$ such that $\frac{J(u)}{I(u)} = \lambda_1$.

Proof of lemma. Let $\{u_n\} \subset E \setminus \{0\}$ be a minimizing sequence for $\lambda_1$, that is,

$$\lim_{n \to \infty} \frac{J(u_n)}{I(u_n)} = \lambda_1 > 0. \tag{26}$$

By relation (20) we deduce that $\{u_n\}$ is bounded in $E$. Since $E$ is reflexive it follows that there exists $u \in E$ such that $u_n$ converges weakly to $u$ in $E$. On the other hand, the functional $J$ is weakly lower semi-continuous. Therefore

$$\liminf_{n \to \infty} J(u_n) \geq J(u). \tag{27}$$

By Remark 1 it follows that $E$ is compactly embedded in $L^{q(x)}(\Omega)$. Thus, $u_n$ converges strongly in $L^{q(x)}(\Omega)$, hence

$$\lim_{n \to \infty} I(u_n) = I(u). \tag{28}$$

Relations (27) and (28) imply that if $u \neq 0$ then

$$\frac{J(u)}{I(u)} = \lambda_1.$$

Thus, in order to conclude that the lemma holds true it is enough to show that $u$ cannot be trivial. Assume by contradiction the contrary. Then $u_n$ converges weakly to 0 in $E$ and strongly in $L^{q(x)}(\Omega)$. In other words, we have

$$\lim_{n \to \infty} I(u_n) = 0. \tag{29}$$

Letting $\varepsilon \in (0, \lambda_1)$ be fixed by relation (26) we deduce that for $n$ large enough we have

$$|J(u_n) - \lambda_1 I(u_n)| < \varepsilon I(u_n),$$

or

$$(\lambda_1 - \varepsilon) I(u_n) < J(u_n) < (\lambda_1 + \varepsilon) I(u_n).$$

Passing to the limit in the above inequalities and taking into account that relation (29) holds true we find $\lim_{n \to \infty} J(u_n) = 0$. That implies that actually $u_n$ converges strongly to 0 in $E$, that is, $\lim_{n \to \infty} \|u_n\|_1 = 0$. So, by (21),

$$\lim_{n \to \infty} \frac{J(u_n)}{I(u_n)} = \infty,$$

and this is a contradiction. Thus, $u \neq 0$. The proof of Lemma 3.3 is complete. \qed
By Lemma 3.3 we conclude that there exists \( u \in E \setminus \{0\} \) such that
\[
\frac{J(u)}{I(u)} = \lambda_1 = \inf_{w \in E \setminus \{0\}} \frac{J(w)}{I(w)}.
\] (30)

Then, for any \( v \in E \) we have
\[
\frac{d}{d\varepsilon} \left( J(u + \varepsilon v) \right) \big|_{\varepsilon = 0} = 0.
\]

A simple computation yields
\[
\int_{\Omega} \left( |\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2} \right) \nabla u \nabla v \, dx \cdot I(u) - J(u) \cdot \int_{\Omega} |u|^{q(x)-2} uv \, dx = 0, \\
\forall v \in E.
\] (31)

Relation (31) combined with the fact that \( J(u) = \lambda_1 I(u) \) and \( I(u) \neq 0 \) implies the fact that \( \lambda_1 \) is an eigenvalue of problem (8). Thus, step 2 is verified.

**Step 3.** Any \( \lambda \in (\lambda_1, \infty) \) is an eigenvalue of problem (8).

Fix \( \lambda \in (\lambda_1, \infty) \). Define \( T_{\lambda} : E \to \mathbb{R} \) by
\[
T_{\lambda}(u) = J(u) - \lambda I(u).
\]

Thus, \( \lambda \) is an eigenvalue of problem (8) if and only if there exists \( u_{\lambda} \in E \setminus \{0\} \) a critical point of \( T_{\lambda} \).

With similar arguments as in the proof of relation (20) we deduce that \( T_{\lambda} \) is coercive, that is, \( \lim_{\|u\| \to \infty} T_{\lambda}(u) = \infty \). On the other hand, \( T_{\lambda} \) is weakly lower semi-continuous. Thus, there exists \( u_{\lambda} \in E \) a global minimum point of \( T_{\lambda} \) and hence, a critical point of \( T_{\lambda} \). It remains to show that \( u_{\lambda} \) is not trivial. Indeed, since \( \lambda_1 = \inf_{u \in E \setminus \{0\}} \frac{J(u)}{I(u)} \) and \( \lambda > \lambda_1 \) it follows that there exists \( v_{\lambda} \in E \) such that \( J(v_{\lambda}) < \lambda I(v_{\lambda}) \), or, equivalently, \( T_{\lambda}(v_{\lambda}) < 0 \). Thus, \( \inf_E T_{\lambda} < 0 \) and we conclude that \( u_{\lambda} \) is a nontrivial critical point of \( T_{\lambda} \), that is, \( \lambda \) is an eigenvalue of problem (8). Thus, step 3 is verified.

**Step 4.** Any \( \lambda \in (0, \lambda_0) \), where \( \lambda_0 \) is given by relation (19), is not an eigenvalue of problem (8).

Indeed, assuming by contradiction that there exists \( \lambda \in (0, \lambda_0) \) an eigenvalue of problem (8) it follows that there exists \( u_{\lambda} \in E \setminus \{0\} \) such that
\[
\langle J'(u_{\lambda}), v \rangle = \lambda \langle I'(u_{\lambda}), v \rangle, \quad \forall v \in E.
\]

Thus, for \( v = u_{\lambda} \) we find
\[
\langle J'(u_{\lambda}), u_{\lambda} \rangle = \lambda \langle I'(u_{\lambda}), u_{\lambda} \rangle,
\]
or
\[ J_1(u_\lambda) = \lambda I_1(u_\lambda). \]

The fact that \( u_\lambda \in E \setminus \{0\} \) assures that \( I_1(u_\lambda) > 0 \). Since \( \lambda < \lambda_0 \), the above information implies
\[ J_1(u_\lambda) \geq \lambda_0 I_1(u_\lambda) > \lambda I_1(u_\lambda) = J_1(u_\lambda). \]

Clearly, the above inequalities lead to a contradiction. Thus, step 4 is verified.

By steps 2, 3 and 4 we deduce that \( \lambda_0 \leq \lambda_1 \). The proof of Theorem 3.1 is now complete. \( \square \)

4. Neumann problems in Orlicz-Sobolev spaces

In this section we study the nonhomogeneous Neumann problem

\[
\begin{cases}
-\text{div}(a(x,|\nabla u(x)||\nabla u(x))) + a(x,|u(x)||u(x)) = \lambda g(x,u(x)), & \text{for } x \in \Omega \\
\frac{\partial u}{\partial \nu}(x) = 0, & \text{for } x \in \partial \Omega,
\end{cases}
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) with smooth boundary \( \partial \Omega \) and \( \nu \) is the outward unit normal to \( \partial \Omega \). We assume that the function \( a(x,t) : \overline{\Omega} \times \mathbb{R} \rightarrow \mathbb{R} \) is such that \( \varphi(x,t) : \overline{\Omega} \times \mathbb{R} \rightarrow \mathbb{R} \),

\[
\varphi(x,t) = \begin{cases} a(x,|t|)t, & \text{for } t \neq 0 \\ 0, & \text{for } t = 0, \end{cases}
\]

and satisfies

(\( \varphi \)) for all \( x \in \Omega \), \( \varphi(x,\cdot) : \mathbb{R} \rightarrow \mathbb{R} \) is an odd, increasing homeomorphism from \( \mathbb{R} \) onto \( \mathbb{R} \);

and \( \Phi(x,t) : \overline{\Omega} \times \mathbb{R} \rightarrow \mathbb{R} \),

\[
\Phi(x,t) = \int_0^t \varphi(x,s) \, ds, \quad \forall \, x \in \overline{\Omega}, \ t \geq 0,
\]

belongs to \textit{class} \( \Phi \), that is, \( \Phi \) satisfies the following conditions

(\( \Phi_1 \)) for all \( x \in \Omega \), \( \Phi(x,\cdot) : [0,\infty) \rightarrow \mathbb{R} \) is a nondecreasing continuous function, with \( \Phi(x,0) = 0 \) and \( \Phi(x,t) > 0 \) whenever \( t > 0 \); \( \lim_{t \to \infty} \Phi(x,t) = \infty \);

(\( \Phi_2 \)) for every \( t \geq 0 \), \( \Phi(\cdot,t) : \Omega \rightarrow \mathbb{R} \) is a measurable function.

We also assume that there exist two positive constants \( \varphi_0 \) and \( \varphi^0 \) such that

\[
1 < \varphi_0 \leq \frac{t\varphi(x,t)}{\Phi(x,t)} \leq \varphi^0 < \infty, \quad \forall \, x \in \overline{\Omega}, \ t \geq 0.
\]
Furthermore, we assume that $\Phi$ satisfies the following condition:

for each $x \in \overline{\Omega}$, the function $[0, \infty) \ni t \mapsto \Phi(x, \sqrt{t})$ is convex. 

Relation (16) assures that $L^\Phi(\Omega)$ is an uniformly convex space and thus, a reflexive space.

We study problem (32) in the particular case when $\Phi$ satisfies

\[ M \cdot |t|^{p(x)} \leq \Phi(x, t), \quad \forall x \in \overline{\Omega}, \; t \geq 0, \]

where $p(x) \in C(\overline{\Omega})$ with $p(x) > 1$ for all $x \in \overline{\Omega}$ and $M > 0$ is a constant.

On the other hand, we assume that the function $g$ from problem (32) satisfies the hypotheses

\[ |g(x, t)| \leq C_0 \cdot |t|^{q(x)-1}, \quad \forall x \in \Omega, \; t \in \mathbb{R} \]

and

\[ C_1 \cdot |t|^{q(x)} \leq G(x, t) := \int_0^t g(x, s) \, ds \leq C_2 \cdot |t|^{q(x)}, \quad \forall x \in \Omega, \; t \in \mathbb{R}, \]

where $C_0, C_1$ and $C_2$ are positive constants and $q(x) \in C(\overline{\Omega})$ satisfies $1 < q(x) < \frac{Np}{N-p}$ for all $x \in \overline{\Omega}$.

We say that $u \in W^{1,\Phi}(\Omega)$ is a weak solution of problem (32) if

\[ \int_{\Omega} a(x, |\nabla u|) \nabla u \nabla v \, dx + \int_{\Omega} a(x, |u|) uv \, dx - \lambda \int_{\Omega} g(x, u) v \, dx = 0, \]

for all $v \in W^{1,\Phi}(\Omega)$.

The main results of this section are the following (see Mihăilescu & Rădulescu [28]).

**Theorem 4.1.** Assume $\varphi$ and $\Phi$ verify conditions $(\varphi)$, $(\Phi_1)$, $(\Phi_2)$, (33), (34) and (35) and the functions $g$ and $G$ satisfy conditions (36) and (37). Furthermore, we assume that $q^- < \varphi_0$. Then there exists $\lambda_* > 0$ such that for any $\lambda \in (0, \lambda_*)$ problem (32) has a nontrivial weak solution.

**Theorem 4.2.** Assume $\varphi$ and $\Phi$ verify conditions $(\varphi)$, $(\Phi_1)$, $(\Phi_2)$, (33), (34) and (35) and the functions $g$ and $G$ satisfy conditions (36) and (37). Furthermore, we assume that $q^+ < \varphi_0$. Then there exists $\lambda_* > 0$ and $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda_*) \cup (\lambda^*, \infty)$ problem (32) has a nontrivial weak solution.

Let $E$ denote the generalized Orlicz-Sobolev space $W^{1,\Phi}(\Omega)$.

For each $\lambda > 0$ we define the energy functional $J_\lambda : E \rightarrow \mathbb{R}$ by

\[ J_\lambda(u) = \int_{\Omega} [\Phi(x, |\nabla u|) + \Phi(x, |u|)] \, dx - \lambda \int_{\Omega} G(x, u) \, dx. \]
Then $J_\lambda$ is well-defined on $E$, $J_\lambda \in C^1(E, \mathbb{R})$, and
\[
\langle J'_\lambda(u), v \rangle = \int_\Omega a(x, |\nabla u|) \nabla u \cdot \nabla v \, dx + \int_\Omega a(x, |u|)uv \, dx - \lambda \int_\Omega g(x,u)v \, dx,
\]
for all $u, v \in E$. Standard arguments show that $J_\lambda$ is weakly lower semi-continuous.

We also define the functional $\Lambda : E \to \mathbb{R}$ by
\[
\Lambda(u) = \int_\Omega \left[ \Phi(x, |\nabla u|) + \Phi(x, |u|) \right] \, dx.
\]
Then $\Lambda$ is well-defined on $E$, $\Lambda \in C^1(E, \mathbb{R})$ is weakly lower semi-continuous, and for all $u, v \in E$,
\[
\langle \Lambda'(u), v \rangle = \int_\Omega a(x, |\nabla u|) \nabla u \cdot \nabla v \, dx + \int_\Omega a(x, |u|)uv \, dx.
\]

Proof of Theorem 4.1. We split the proof into several steps.

Step 1. There exists $\lambda_\star > 0$ such that for all $\lambda \in (0, \lambda_\star)$, there are $\rho, \alpha > 0$ such that $J_\lambda(u) \geq \alpha > 0$, for any $u \in E$ with $\|u\| = \rho$. The value of $\lambda_\star$ is given by
\[
\lambda_\star = \frac{\rho^{q_0-q^-}}{2 \cdot C_2 \cdot c_1^{q^-}}.
\]  

Step 2. There exists $\theta \in E$ such that $\theta \geq 0$, $\theta \neq 0$ and $J_\lambda(t\theta) < 0$, for $t > 0$ small enough.

Step 3. Conclusion.

Fix $\lambda \in (0, \lambda_\star)$. Then, by Step 1, it follows that on the boundary of the ball centered in the origin and of radius $\rho$ in $E$, denoted by $B_\rho(0)$, we have $\inf_{\partial B_\rho(0)} J_\lambda > 0$. On the other hand, by Step 2, there exists $\theta \in E$ such that $J_\lambda(t \cdot \theta) < 0$ for all $t > 0$ small enough. Moreover, our hypotheses imply that for any $u \in B_\rho(0)$ we have
\[
J_\lambda(u) \geq \|u\|^{q_0} - \lambda \cdot C_2 \cdot c_1^{q^-} \|u\|^{q^-}.
\]
It follows that
\[
-\infty < \underline{c} := \inf_{B_\rho(0)} J_\lambda < 0.
\]

We let now $0 < \varepsilon < \inf_{\partial B_\rho(0)} J_\lambda - \inf_{B_\rho(0)} J_\lambda$. Applying Ekeland’s variational principle we find $u_\varepsilon \in B_\rho(0)$ such that
\[
J_\lambda(u_\varepsilon) < \inf_{B_\rho(0)} J_\lambda + \varepsilon
\]
and
\[
J_\lambda(u_\varepsilon) < J_\lambda(u) + \varepsilon \cdot \|u - u_\varepsilon\|, \quad u \neq u_\varepsilon.
\]
Since
\[
J_\lambda(u_\epsilon) \leq \inf_{B_\rho(0)} J_\lambda + \epsilon \leq \inf_{\partial B_\rho(0)} J_\lambda + \epsilon < \inf_{\partial B_\rho(0)} J_\lambda,
\]
we deduce that \(u_\epsilon \in B_\rho(0)\). Now, we define \(I_\lambda : B_\rho(0) \to \mathbb{R}\) by
\[
I_\lambda(u) = J_\lambda(u) + \epsilon \cdot \| u - u_\epsilon \|.
\]
Then \(u_\epsilon\) is a minimum point of \(I_\lambda\) and thus
\[
\frac{I_\lambda(u_\epsilon + t \cdot v) - I_\lambda(u_\epsilon)}{t} \geq 0
\]
for small \(t > 0\) and any \(v \in B_1(0)\). Therefore
\[
\frac{J_\lambda(u_\epsilon + t \cdot v) - J_\lambda(u_\epsilon)}{t} + \epsilon \cdot \| v \| \geq 0.
\]
Letting \(t \to 0\) it follows that \(\langle J_\lambda'(u_\epsilon), v \rangle + \epsilon \cdot \| v \| > 0\) and we infer that
\[
\| J_\lambda'(u_\epsilon) \| \leq \epsilon.
\]
We deduce that there exists a sequence \(\{w_n\} \subset B_\rho(0)\) such that
\[
J_\lambda(w_n) \to \zeta \quad \text{and} \quad J_\lambda'(w_n) \to 0. \tag{39}
\]
It is clear that \(\{w_n\}\) is bounded in \(E\). Thus, there exists \(w \in E\) such that, up to a subsequence, \(\{w_n\}\) converges weakly to \(w\) in \(E\). Since \(E\) is compactly embedded in \(L^{q(x)}(\Omega)\), it follows that \(\{w_n\}\) converges strongly to \(w\) in \(L^{q(x)}(\Omega)\).

Thus, by (36) and Hölder’s inequality,
\[
\left| \int_{\Omega} g(x, w_n \cdot (w_n - w)) \, dx \right| \leq C_0 \cdot \int_{\Omega} |w_n|^{q(x)-1} |w_n - w| \, dx
\leq C_0 \cdot \|w_n|^{q(x)-1} \|_{q(x)} \cdot |w_n - w|_{q(x)} \to 0, \tag{40}
\]
as \(n \to \infty\).

On the other hand, by (39) we have
\[
\lim_{n \to \infty} \langle J_\lambda'(w_n), w_n - w \rangle = 0. \tag{41}
\]
Relations (40) and (41) imply \(\lim_{n \to \infty} \langle \Lambda'(w_n), w_n - w \rangle = 0\). Thus, \(\{w_n\}\) converges strongly to \(w\) in \(E\). So, by (39), \(J_\lambda(w) = \zeta < 0\) and \(J_\lambda'(w) = 0\). We conclude that \(w\) is a nontrivial weak solution for problem (32) for any \(\lambda \in (0, \lambda_\star)\).

The proof of Theorem 4.1 is complete. \(\square\)

Proof of Theorem 4.2. Since \(q^+ < \varphi_0\) it follows that \(q^- < \varphi_0\). Thus, by Theorem 4.1, there exists \(\lambda_\star > 0\) such that for any \(\lambda \in (0, \lambda_\star)\) problem (32) has a nontrivial weak solution.
Next, we observe that $J_\lambda$ is coercive and weakly lower semi-continuous in $E$, for all $\lambda > 0$. Thus, there exists $u_\lambda \in E$ a global minimizer of $I_\lambda$, hence a weak solution of problem (32).

We show that $u_\lambda$ is not trivial for $\lambda$ large enough. Indeed, letting $t_0 > 1$ be a fixed real and $u_0(x) = t_0$, for all $x \in \Omega$ we have $u_0 \in E$ and

$$J_\lambda(u_0) = \Lambda(u_0) - \lambda \int_\Omega G(x, u_0) \, dx \leq \int_\Omega \Phi(x, t_0) \, dx - \lambda \cdot C_1 \cdot \int_\Omega |t_0|^{q(x)} \, dx \leq L - \lambda \cdot C_1 \cdot t_0^{q^*} \cdot |\Omega|,$$

where $L$ is a positive constant. Thus, there exists $\lambda^* > 0$ such that $J_\lambda(u_0) < 0$ for any $\lambda \in [\lambda^*, \infty)$. It follows that $J_\lambda(u_\lambda) < 0$ for any $\lambda \geq \lambda^*$ and thus $u_\lambda$ is a nontrivial weak solution of problem (32) for $\lambda$ large enough. The proof of Theorem 4.2 is complete.

We conclude this section with several examples of functions $\varphi$ and $\Phi$ for which the results in this section do apply.

**Example 4.3.** Define

$$\varphi(x, t) = p(x)|t|^{p(x)-2}t \quad \text{and} \quad \Phi(x, t) = |t|^{p(x)},$$

with $p(x) \in C(\overline{\Omega})$ satisfying $2 \leq p(x) < N$, for all $x \in \overline{\Omega}$.

**Example 4.4.** Define

$$\varphi(x, t) = p(x)|t|^{p(x)-2}t \quad \text{and} \quad \Phi(x, t) = |t|^{p(x)},$$

and

$$\Phi(x, t) = \frac{|t|^{p(x)}}{\log(1 + |t|)} + \int_0^{|t|} \frac{s^{p(x)}}{(1 + s)(\log(1 + s))^2} \, ds,$$

with $p(x) \in C(\overline{\Omega})$ satisfying $3 \leq p(x) < N$, for all $x \in \overline{\Omega}$.

**Example 4.5.** Define

$$\varphi(x, t) = p(x) \cdot \log(1 + \alpha + |t|) \cdot |t|^{p(x)-1}t,$$

and

$$\Phi(x, t) = \log(1 + \alpha + |t|) \cdot |t|^{p(x)} - \int_0^{|t|} \frac{s^{p(x)}}{1 + \alpha + s} \, ds,$$

where $\alpha > 0$ is a constant and $p(x) \in C(\overline{\Omega})$ satisfying $2 \leq p(x) < N$, for all $x \in \overline{\Omega}$.
5. Variational analysis versus nonlinear eigenvalue problems

Consider the eigenvalue problem

\[
\begin{cases}
-\text{div}(\alpha(|\nabla u|)\nabla u) + \alpha(|u|)u = \lambda f(x,u) & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega.
\end{cases}
\tag{N_{\alpha,\lambda}^f}
\]

We assume that \( f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R} \) is continuous and \( \alpha : (0,\infty) \to \mathbb{R} \) is such that the mapping \( \phi : \mathbb{R} \to \mathbb{R} \) defined by

\[
\phi(t) = \begin{cases}
\alpha(|t|)t, & \text{for } t \neq 0 \\
0, & \text{for } t = 0,
\end{cases}
\]

is an odd, strictly increasing homeomorphism from \( \mathbb{R} \) onto \( \mathbb{R} \).

The main result in this section (see Bonanno, Molica Bisci & Rădulescu [7]) establishes that if \( p > N + 1 \) and \( \lambda > 0 \) is arbitrary, then there exists a sequence of pairwise distinct solutions of problem \( (N_{\alpha,\lambda}^f) \) that converges to zero in \( W^{1,p}(\Omega) \). We also refer to Bonanno & Molica Bisci [6] for a related property for the \( p \)-Laplace operator.

Throughout this section we assume that \( \Phi \) satisfies the following hypotheses:

\[
(\Phi_0) \quad 1 < \liminf_{t \to \infty} \frac{t \phi(t)}{\Phi(t)} \leq p^0 := \sup_{t > 0} \frac{t \phi(t)}{\Phi(t)} < \infty;
\]

\[
(\Phi_1) \quad N < p_0 := \inf_{t > 0} \frac{t \phi(t)}{\Phi(t)} < \liminf_{t \to \infty} \frac{\log(\Phi(t))}{\log(t)}.
\]

Let

\[
A := \liminf_{\xi \to 0^+} \frac{\int_{\Omega} \max_{|x| \leq \xi} F(x,t) \, dx}{\xi^{p_0}}, \quad B := \limsup_{\xi \to 0^+} \frac{\int_{\Omega} F(x,\xi) \, dx}{\xi^{p_0}}.
\]

The following multiplicity result has been established in [7].

**Theorem 5.1.** Let \( f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R} \) be a continuous function, \( \Phi \) be a Young function satisfying the structural hypotheses \( (\Phi_0) - (\Phi_1) \) and let \( \rho \) be a positive constant such that

\[
(\Phi_\rho) \quad \lim_{t \to 0^+} \frac{\Phi(t)}{t^{p_0}} < \rho.
\]
Further, assume that
\[
(h_0) \quad \liminf_{\xi \to 0^+} \frac{\int_{\Omega} \max_{|t| \leq \xi} F(x, t) \, dx}{\xi^{p_0}} < \frac{1}{(2c)^{p_0} \rho |\Omega|} \limsup_{\xi \to 0^+} \frac{\int_{\Omega} F(x, \xi) \, dx}{\xi^{p_0}}.
\]

Then, for every \( \lambda \) belonging to
\[
\left( \rho \frac{|\Omega|}{B}, \frac{1}{(2c)^{p_0} A} \right],
\]
the problem \((N_{\alpha, \lambda}^f)\) admits a sequence of pairwise distinct weak solutions which strongly converges to zero in \( W^1 L_\Phi(\Omega) \).

The key ingredient in the proof of Theorem 5.1 is the following result of Bonanno \& Molica Bisci [5, Theorem 2.1], which is a refinement of Ricceri’s variational principle [37]. Ricceri’s result goes back to an elementary property established by Pucci and Serrin [33, 34], which asserts that if a functional of class \( C^1 \) defined on a real Banach space has two local minima, then it has a third critical point. At our best knowledge, the first three critical point property was found by Krasnoselskii [17]. He showed that if \( f \) is a coercive \( C^1 \) functional defined on a finite dimensional space having a nondegenerate critical point \( x_0 \) (that is, the topological index \( \text{ind} f' (x_0)(0) \) is different from zero) which is not a global minimum, then \( f \) admits a third critical point. This result was extended to infinite dimensional Banach spaces by Amann [3]. We refer to Bonanno \& Marano [4], Livrea \& Marano [22], and Marano \& Motreanu [24] for related results and applications of Ricceri’s variational principle. The recent book by Kristály, Rădulescu \& Varga [20] contains several applications of Ricceri’s variational principle.

**Theorem 5.2.** (Bonanno \& Molica Bisci [5, Theorem 2.1]). Let \( X \) be a reflexive real Banach space, let \( J, I : X \to \mathbb{R} \) be two Gâteaux differentiable functionals such that \( J \) is strongly continuous, sequentially weakly lower semicontinuous and coercive and \( I \) is sequentially weakly upper semicontinuous. For every \( r > \inf_X J \), put
\[
\phi(r) := \inf_{u \in J^{-1}([-\infty, r])} \frac{\left( \sup_{v \in J^{-1}([-\infty, r])} I(v) \right)}{r - J(u)},
\]
and \( \delta := \liminf_{r \to (\inf_X J)^+} \phi(r) \).

Then, if \( \delta < +\infty \), for each \( \lambda \in ]0, \frac{1}{\delta} [ \), the following alternative holds:
either

(c1) there is a global minimum of $J$ which is a local minimum of $g_\lambda := J - \lambda I$, or

(c2) there is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of $g_\lambda$ which weakly converges to a global minimum of $J$, with $\lim_{n \to +\infty} J(u_n) = \inf_X J$.

Define $$\phi(t) = \frac{|t|^{p-2}}{\log(1 + |t|)} t \quad \text{for} \quad t \neq 0, \quad \text{and} \quad \phi(0) = 0.$$ 

A straightforward computation shows that the assumptions $(\Phi_0)$, $(\Phi_1)$, and $(\Phi_\rho)$ are fulfilled. A direct application of Theorem 5.1 implies the following multiplicity property.

**Corollary 5.3.** Let $p > N + 1$ and $g : \mathbb{R} \to \mathbb{R}$ be a continuous non-negative function with potential $G(\xi) := \int_0^\xi g(t) \, dt$. Assume that

$$\liminf_{\xi \to 0^+} \frac{G(\xi)}{\xi^p} = 0, \quad \text{and} \quad \limsup_{\xi \to 0^+} \frac{G(\xi)}{\xi^{p-1}} = +\infty.$$ 

Let $h : \overline{\Omega} \to \mathbb{R}$ be a continuous and positive function.

Then, for each $\lambda > 0$, the Neumann problem

\[
\begin{cases}
-\text{div} \left( \frac{|\nabla u|^{p-2}}{\log(1 + |\nabla u|)} \nabla u \right) + \frac{|u|^{p-2}}{\log(1 + |u|)} u = \lambda h(x)g(u) & \text{in} \quad \Omega, \\
\frac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial \Omega,
\end{cases}
\]

admits a sequence of pairwise distinct weak solutions which strongly converges to zero in $W^{1, L_\Phi(\Omega)}$.

The reader interested in nonlinear PDE’s in Orlicz-Sobolev spaces may consult the following very related references: Byun, Yao & Zhou [8], Fukagai, Ito & Narukawa [13], Le [21], Kristály, Mihăilescu & Rădulescu [19], Mihăilescu, Rădulescu & Repovš [29], Pucci & Rădulescu [32], and Xing & Ding [39]. For many examples and related properties we also refer to the books by Ghergu & Rădulescu [14, 15].

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