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Applications of local linking to nonlocal Neumann problems

Giovanni Molica Bisci
Dipartimento P.A.U.
Università Degli Studi Mediterranea di Reggio Calabria
Salita Melissari - Feo di Vito
89100 Reggio Calabria, Italy
gmolica@unirc.it

Vicenţiu D. Rădulescu
Institute of Mathematics "Simion Stoilow" of the Romanian Academy
014700 Bucharest, Romania
Department of Mathematics, University of Craiova
200585 Craiova, Romania
vicentiu.radulescu@imar.ro

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We study a nonlocal Neumann problem driven by a nonhomogeneous elliptic differential operator. The reaction term is a nonlinearity function that exhibits p-superlinear growth but need not satisfy the Ambrosetti–Rabinowitz condition. By using an abstract linking theorem for smooth functionals, we prove a multiplicity result on the existence of weak solutions for such problems. An explicit example illustrates the main abstract result of this paper.

Keywords: Variational methods; linking; weak solutions; nonlocal equations.

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1. Introduction

Let X be a real Banach space with a direct decomposition

$$X = Y \oplus V,$$

for certain linear subspaces Y and V.

The condition of local linking was introduced in [20]. We say that $\varphi \in C^1(X, \mathbb{R})$ has a *local linking* if

$$\dim Y < \infty \tag{1}$$

and there are $\alpha, \rho > 0$ such that

$$\varphi(u) \geq \alpha \quad \forall\, u \in V, \ \ \|u\| = \rho. \tag{2}$$

Under these assumptions, Li and Liu [20] established the existence of nontrivial critical points for functionals in the following classes:

(a) bounded from below;
(b) super-quadratic;
(c) asymptotically quadratic.

The weaker notion of *local linking at the origin* is due to Li and Willem [21]. We say that the functional $\varphi \in C^1(X, \mathbb{R})$ has a *local linking at the origin* if there exists a positive constant ρ such that

$$\varphi(u) \leq 0 \quad \forall\, u \in Y, \ \ \|u\| \leq \rho \tag{3}$$

and

$$\varphi(u) \geq 0 \quad \forall\, u \in V, \ \ \|u\| \leq \rho. \tag{4}$$

The concept of local linking generalizes the notions of local minimum and local maximum. When 0_X is a nondegenerate critical point of a functional φ of class C^2 defined on a Hilbert space and $\varphi(0_X) = 0$, then φ has a local linking at zero.

Successively, Brezis and Nirenberg, in their quoted paper [10], proved the existence of a nontrivial critical point in case (a), assuming only (1), (3), (4), and the Palais-Smale condition.

2. Statement of the Problem

Let Ω be a bounded domain in $(\mathbb{R}^N, |\cdot|)$ with smooth boundary $\partial\Omega$. Assume that $p > 1$ is a real number. Let $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ denote the p-Laplace operator and let $\partial u/\partial\nu$ be the outer unit normal derivative.

In this paper, we are interested in the existence of weak solutions to the following nonlocal Neumann problem:

$$(N^p_{M,f}) \qquad \begin{cases} -\left[M\left(\displaystyle\int_\Omega |\nabla u(x)|^p dx\right)\right]^{p-1} \Delta_p u = f(x,u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

In the sequel we will assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and $M : [0, +\infty) \to [0, +\infty)$ is continuous such that

$$(C^1_M) \ \ 0 < m_0 \leq M(t), \quad \text{for every } t \in [0, +\infty).$$

Further, we require that

$$(C^2_M) \ \ \int_0^t [M(s)]^{p-1} ds \geq t[M(t)]^{p-1}, \quad \text{for every } t \in [0, +\infty).$$

Problem $(N^p_{M,f})$ is called *nonlocal* because of the presence of the term $M(\int_\Omega |\nabla u(x)|^p dx)$, which implies that the quasilinear partial differential equation in $(N^p_{M,f})$ is no longer a pointwise identity. This phenomenon creates several mathematical difficulties in the qualitative analysis of such equations.

Problem $(N^p_{M,f})$ is related to the stationary analogue of the hyperbolic equation

$$u_{tt} - \left(a + b\int_\Omega |\nabla u(x)|^2 dx\right)\Delta u = f(x,u) \quad \text{in } \Omega, \tag{5}$$

where Δ is the usual Laplace operator. Equation (5) is a general version of the Kirchhoff equation

$$\rho u_{tt} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L u_x^2 dx\right) u_{xx} = 0, \tag{6}$$

presented by Kirchhoff [18].

This relation is as an extension of the classical d'Alembert's wave equation by considering the effects of changes in the length of the strings during the vibrations. The parameters in Eq. (6) have the following meanings: E is the Young modulus of the material, ρ is the mass density, L is the length of the string, h is the area of cross-section, and P_0 is the initial tension.

The Kirchhoff's model takes into account the length changes of the string produced by transverse vibrations. The early classical studies dedicated to Kirchhoff equations were given by Bernstein [8] and Pohozaev [27]. However, Eq. (5) received much attention only after the paper by Lions [22], where an abstract framework to the problem was proposed. Some related results can be found, for example, in [4, 12, 13]. We also point out that Arosio and Panizzi [4] studied the Cauchy–Dirichlet problem related to (5) in the Hadamard sense as a special case of an abstract second-order Cauchy problem in a Hilbert space. D'Ancona and Spagnolo [13] considered Kirchhoff's equation as a quasi-linear hyperbolic Cauchy problem that describes the transverse oscillations of a stretched string. Perera and Zhang [26] obtained a nontrivial solution for nonlocal equations via Yang index and critical groups. Chipot and Lovat [11] pointed out that this kind of problems models several physical and biological systems, where u describes a process depending on the average of itself (for example, population density). Nonlocal problems have been studied in these last years by variational arguments; see the papers [3, 29, 30] and references therein. For completeness we refer the reader to some recent interesting results obtained by Autuori and Pucci in [5–7] studying Kirchhoff equations by using different approaches.

Recently, there have been results for nonlinear Neumann problems driven by the p-Laplacian differential operator. See, for instance, the works [1, 2, 14, 15, 24, 28, 31]. With the exceptions of [1, 24], in all the cited works, it is assumed that $p > N$ (low-dimensional problems) and the authors exploit the fact that in this setting, the Sobolev space $W^{1,p}(\Omega)$ is compactly embedded in $C^0(\bar{\Omega})$. Further, in [15, 24], the Euler energy functional is assumed to be coercive in $W^{1,p}(\Omega)$.

In this paper, motivated by this large interest on nonlocal Neumann equations and exploiting a slight variant of a result of Li and Willem [21] for functionals having a local linking at zero, we prove the existence of at least one nontrivial weak solution for problem $(N^p_{M,f})$. In contrast with the above cited papers, in our case the Euler functional is indefinite.

More precisely, we prove an existence theorem (see Theorem 4.3) for problem $(N^p_{M,f})$ when the right-hand side nonlinearity f is p-superlinear. In the context of the Dirichlet problems, this setting was investigated by Liu [23], who employed the Ambrosetti–Rabinowitz (simply (AR)) condition. His approach uses Morse theory. In particular, the (AR) condition was crucial in the computation of the critical groups of the Euler functional at infinity. In Theorem 4.3, we do not assume the (AR) condition on f and so the approach of [23] cannot be adopted.

In this paper, we use some ideas developed in [17], where the authors consider nonlinear Neumann problems driven by p-Laplacian type operators which are not necessarily homogeneous.

The plan of the paper is as follows. Section 3 is devoted to our abstract framework, while Sec. 4 is dedicated to the main results. A concrete example illustrates the main abstract result of this paper (see Example 4.8). We refer to Brezis [9] for basic analytic preliminaries in relationship with partial differential equations. We also cite the monographs [16, 19] as general references on the variational setting adopted in this paper.

3. Abstract Framework

Let $W^{1,p}(\Omega)$ be the usual Sobolev space, equipped with the norm

$$\|u\| := \left(\int_\Omega (|\nabla u(x)|^p + |u(x)|^p)dx\right)^{1/p}.$$

Let $\langle\cdot,\cdot\rangle$ denote the duality pairing between $(W^{1,p}(\Omega))^*$ and $W^{1,p}(\Omega)$. We denote by p^* the critical exponent of the Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$. Recall that if $p < N$ then $p^* = Np/(N-p)$ and for every $q \in [1, p^*]$ there exists a positive constant c_q such that

$$\|u\|_{L^q(\Omega)} \leq c_q\|u\|, \quad \forall\, u \in W^{1,p}(\Omega). \tag{7}$$

Moreover, when $p \geq N$, this inequality holds for all $q \in [1, +\infty[$, since $p^* = +\infty$.

We also recall the classical Poincaré–Wirtinger inequality (see [9, Chap. 9]): there exists $\eta > 0$ such that

$$\int_\Omega |u(x)|^p dx \leq \eta \int_\Omega |\nabla u(x)|^p dx, \quad \forall\, u \in W^{1,p}(\Omega) \text{ s.t. } \int_\Omega u(x)dx = 0. \tag{8}$$

For the sake of completeness, we recall that a C^1-functional $\varphi : X \to \mathbb{R}$ satisfies the Cerami condition at level $\mu \in \mathbb{R}$, (briefly $(\mathrm{C})_\mu$) if

$(\mathrm{C})_\mu$ *Every sequence* $\{u_n\}$ *in* X *such that*

$$\varphi(u_n) \to \mu \quad \text{and} \quad (1 + \|u_n\|)\|\varphi'(u_n)\|_{X^*} \to 0,$$

as $n \to \infty$*, possesses a convergent subsequence.*

We say that φ satisfies the Cerami condition (in short (C)) if $(\mathrm{C})_\mu$ holds for every $\mu \in \mathbb{R}$.

A basic tool in this paper is the following abstract local linking theorem due to Li and Willem [21]. We reformulate this result in the special case given in [17, Theorem 2.1].

Theorem 3.1. *Let* X *be a Banach space such that* $X = Y \oplus V$ *with* $\dim Y < +\infty$. *Assume that* $\varphi \in C^1(X)$ *satisfies the following conditions*:

(i) φ *has a local linking at zero*;
(ii) φ *satisfies the* (C) *condition*;
(iii) φ *maps bounded sets into bounded sets*;
(iv) *for every finite-dimensional subspace* $E \subseteq V$, *we have*

$$\varphi(u) \to -\infty \quad \text{as } \|u\| \to +\infty \quad \text{and} \quad u \in Y \oplus E.$$

Then φ *admits at least one nontrivial critical point.*

4. Main Result

Let $\Phi : W^{1,p}(\Omega) \to \mathbb{R}$ be the smooth functional defined by

$$\Phi(u) := \frac{1}{p}\widehat{M}\left(\int_\Omega |\nabla u(x)|^p dx\right), \tag{9}$$

where

$$\widehat{M}(t) := \int_0^t [M(s)]^{p-1} ds, \quad \forall\, t \in [0, \infty).$$

Then the Fréchet derivative of Φ is $\Phi' : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ defined by

$$\langle \Phi'(u), v\rangle = \left[M\left(\int_\Omega |\nabla u(x)|^p dx\right)\right]^{p-1} \int_\Omega |\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla v(x) dx,$$

for all $u, v \in W^{1,p}(\Omega)$.

Moreover, it is well-known that the operator $A : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ given by

$$\langle A(u), v\rangle = \int_\Omega |\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla v(x) dx,$$

satisfies the (S_+) property.

This means that for every sequence $\{u_n\} \subset W^{1,p}(\Omega)$ such that $u_n \rightharpoonup u$ (weakly) in $W^{1,p}(\Omega)$ and

$$\limsup_{n \to \infty} \langle A(u_n), u_n - u\rangle \le 0, \tag{10}$$

it follows that $u_n \to u$ (strongly) in $W^{1,p}(\Omega)$.

4.1. *Remarks on our assumptions*

The validity of the next lemma will be crucial in the sequel.

Lemma 4.1. *Suppose that conditions* (C_M^1) *and* (C_M^2) *are fulfilled. Then there are positive constants* m_1 *and* m_2 *such that*

$$\widehat{M}(t) \leq m_1 t + m_2, \tag{11}$$

for every $t \in [0, +\infty)$.

Proof. Let $t_1 > 0$. By our assumptions we have

$$\frac{[M(t)]^{p-1}}{\widehat{M}(t)} \leq \frac{1}{t},$$

for every $t \in]t_1, +\infty)$. Integrating this inequality we obtain

$$\int_{t_1}^{t} \frac{[M(s)]^{p-1}}{\widehat{M}(s)} ds = \log \frac{\widehat{M}(t)}{\widehat{M}(t_1)} \leq \log \frac{t}{t_1},$$

for every $t \in]t_1, +\infty)$. Therefore,

$$\widehat{M}(t) \leq \frac{\widehat{M}(t_1)}{t_1} t,$$

for every $t \in]t_1, +\infty)$. Hence the growth condition (11) holds taking, for instance, $m_1 := \frac{\widehat{M}(t_1)}{t_1}$ and $m_2 := \max_{t \in [0, t_1]} \widehat{M}(t)$. □

Owing to conditions (C_M^1) and (C_M^2), by Lemma 4.1 we deduce the following inequalities:

$(\widehat{C}_M)$

$$\frac{m_0^{p-1}}{p} \int_\Omega |\nabla u(x)|^p dx \leq \Phi(u) \leq \frac{m_1}{p} \|u\|^p + \frac{m_2}{p},$$

for every $u \in W^{1,p}(\Omega)$.

Moreover, from now on, we assume that the nonlinearity $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the following hypotheses $\mathbf{H}(f)$:

- (h_1) for every $t \in \mathbb{R}$, the function $x \mapsto f(x, t)$ is measurable;
- (h_2) for every $x \in \Omega$, the function $t \mapsto f(x, t)$ is continuous;
- (h_3) there exist $a_1 \in L^\infty(\Omega)_+$, $a_2 > 0$ and $r \in (p, p^*)$, such that for almost every $x \in \Omega$ and $t \in \mathbb{R}$, we have

$$|f(x, t)| \leq a_1(x) + a_2 |t|^{r-1};$$

- (h_4) if $F(x, \xi) := \int_0^\xi f(x, t) dt$ (for all $\xi \in \mathbb{R}$) then

$$\lim_{|\xi| \to \infty} \frac{F(x, \xi)}{|\xi|^p} = +\infty,$$

  uniformly with respect to a.e. $x \in \Omega$;

(h_5) there exists

$$\mu \in \left((r-p) \max\left\{1, \frac{N}{p}\right\}, r \right]$$

such that

$$\liminf_{|\zeta|\to\infty} \frac{f(x,\zeta)\xi - pF(x,\zeta)}{|\zeta|^\mu} > 0,$$

uniformly for almost every $x \in \Omega$;

(h_6) we have

$$\lim_{t\to 0} \frac{f(x,t)}{|t|^{p-2}t} = 0,$$

uniformly for almost every $x \in \Omega$;

(h_7) there exists $\delta > 0$ such that $F(x,\xi) \geq 0$ for almost every $x \in \Omega$ and $|\xi| \leq \delta$.

Remark 4.2. A typical example when the above conditions hold is given by the real function

$$f(t) := |t|^{p-2}t \log(1+t^2) + \frac{2t|t|^p}{p(1+t^2)}, \quad \forall\, t \in \mathbb{R}.$$

A direct computation ensures that the (AR) condition fails in this special case.

For the sake of completeness we recall that a *weak solution* of problem $(N^p_{M,f})$ is a function $u \in W^{1,p}(\Omega)$ such that

$$\langle \Phi'(u), v\rangle = \int_\Omega f(x,u(x))v(x)dx, \quad \forall\, v \in W^{1,p}(\Omega).$$

The main result in this paper is the following.

Theorem 4.3. *Assume that hypotheses $\boldsymbol{H}(f)$ are fulfilled. Then problem $(N^p_{M,f})$ has at least one nontrivial weak solution.*

4.2. *Auxiliary properties*

We first establish the boundedness of Cerami sequences for a related energy functional.

Lemma 4.4. *Suppose that conditions $\boldsymbol{H}(f)$ are fulfilled. Then every Cerami sequence for the functional*

$$\varphi(u) := \Phi(u) - \int_\Omega F(x,u(x))dx, \quad \forall\, u \in W^{1,p}(\Omega)$$

is bounded in $W^{1,p}(\Omega)$.

Proof. Let $\{u_n\} \subset W^{1,p}(\Omega)$ be a Cerami sequence, i.e.:

$$|\varphi(u_n)| \leq M_1, \quad \forall\, n \in \mathbb{N}, \tag{12}$$

for some positive constant M_1 and

$$(1 + \|u_n\|)\|\varphi'(u_n)\|_{(W^{1,p}(\Omega))^*} \to 0, \tag{13}$$

as $n \to \infty$.

We proceed arguing by contradiction. So, suppose that the conclusion is not true. Passing to a subsequence if necessary, we may assume that

$$\|u_n\| \to +\infty \quad \text{as } n \to \infty.$$

By relations (12) and (13), we have

$$|\langle \varphi'(u_n), v\rangle| = \left|\langle \Phi'(u_n), v\rangle - \int_\Omega f(x, u_n(x))v(x)dx\right| \leq \frac{\varepsilon_n\|v\|}{1 + \|u_n\|}, \tag{14}$$

for every $u \in W^{1,p}(\Omega)$, with $\varepsilon_n \searrow 0^+$.

Choosing as test function $v := u_n$ in (14), we easily obtain

$$-\langle \Phi'(u_n), u_n\rangle + \int_\Omega f(x, u_n(x))u_n(x)dx \leq \varepsilon_n, \quad \forall\, n \in \mathbb{N}. \tag{15}$$

Inequality (12) yields

$$\widehat{M}\left(\int_\Omega |\nabla u_n(x)|^p dx\right) - p\int_\Omega F(x, u_n(x))dx \leq pM_1, \quad \forall\, n \in \mathbb{N}. \tag{16}$$

Adding (15) and (16) we also have that

$$\begin{aligned} &\widehat{M}\left(\int_\Omega |\nabla u_n(x)|^p dx\right) - \langle \Phi'(u_n), u_n\rangle \\ &\qquad + \int_\Omega [f(x, u_n(x))u_n(x) - pF(x, u_n(x))]dx \\ &\leq M_2, \quad \forall\, n \in \mathbb{N}, \end{aligned} \tag{17}$$

for some $M_2 > 0$.

Next, we observe that, by condition (C_M^2), we have

$$\frac{\widehat{M}\left(\int_\Omega |\nabla u_n(x)|^p dx\right)}{\left[M\left(\int_\Omega |\nabla u_n(x)|^p dx\right)\right]^{p-1}} \geq \int_\Omega |\nabla u_n(x)|^p dx,$$

for every $n \in \mathbb{N}$. Therefore,

$$\widehat{M}\left(\int_\Omega |\nabla u_n(x)|^p dx\right) - \langle \Phi'(u_n), u_n \rangle \geq 0, \quad \forall\, n \in \mathbb{N}. \tag{18}$$

From now on, arguing as in [17], by virtue of hypothesis (h_5), we find two positive constants, denoted by β and $M_{3,\beta}$, such that

$$0 < \beta|\zeta|^\mu \leq f(x,\zeta)\zeta - pF(x,\zeta), \tag{19}$$

for almost every $x \in \Omega$ and every $|\zeta| \geq M_{3,\beta}$. Further, condition (h_3) implies that there exists a positive constant M_4 such that

$$|f(x,\zeta)\zeta - pF(x,\zeta)| \leq M_4, \tag{20}$$

for almost every $x \in \Omega$ and every $|\zeta| \geq M_{3,\beta}$. Hence, through (19) and (20), we obtain

$$\beta|\zeta|^\mu - (M_4 + \beta M_{3,\beta}^\beta) \leq f(x,\zeta)\zeta - pF(x,\zeta), \tag{21}$$

for almost every $x \in \Omega$ and every $\zeta \in \mathbb{R}$. Exploiting relation (17), by using (18) and (21), we have

$$\int_\Omega |u_n(x)|^\mu dx \leq M_5, \quad \forall\, n \in \mathbb{N},$$

for some $M_5 > 0$. Thus, it follows that the sequence $\{u_n\}$ is bounded in $L^\mu(\Omega)$.

Now, since $\mu \leq r < p^*$, let $\sigma \in [0,1)$ be such that

$$\frac{1}{r} = \frac{1-\sigma}{\mu} + \frac{t}{p^*}.$$

Thus, we have

$$\|u_n\|_{L^r(\Omega)} \leq \|u_n\|_{L^\mu(\Omega)}^{1-\sigma} \|u_n\|_{L^{p^*}(\Omega)}^\sigma,$$

for every $n \in \mathbb{N}$; see, for completeness [16, p. 905].

Then

$$\|u_n\|_{L^r(\Omega)} \leq M_6 \|u_n\|_{L^{p^*}(\Omega)}^{\sigma r}, \tag{22}$$

for every $n \in \mathbb{N}$ and for some $M_6 > 0$.

From (14), with $v = u_n$, we have

$$\left| \langle \Phi'(u_n), u_n \rangle - \int_\Omega f(x, u_n(x)) u_n(x) dx \right| \leq \varepsilon_n, \tag{23}$$

for every $n \in \mathbb{N}$.

Hypotheses (h_3) and (h_6) imply that for a given $\varepsilon > 0$, we can find $c_\varepsilon > 0$ such that

$$|f(x,t)t| \leq \varepsilon|t|^p + c_\varepsilon|t|^r, \tag{24}$$

for almost every $x \in \Omega$ and every $t \in \mathbb{R}$. Using (24) and (23), we obtain

$$\langle \Phi'(u_n), u_n \rangle \leq \varepsilon_n + \varepsilon \|u_n\|^p_{L^p(\Omega)} + c_\varepsilon \|u_n\|^r_{L^r(\Omega)},$$

for every $n \in \mathbb{N}$. Hence

$$m_0^{p-1} \int_\Omega |\nabla u_n(x)|^p dx \leq \varepsilon_n + \varepsilon \|u_n\|^p_{L^p(\Omega)} + c_\varepsilon M_6 \|u_n\|^{\sigma r}_{L^{p^*}(\Omega)}, \tag{25}$$

for every $n \in \mathbb{N}$.

Let us denote $y_n := \frac{u_n}{\|u_n\|}$. Then $\|y_n\| = 1$ and so we may assume that

$$y_n \rightharpoonup y \quad \text{in } W^{1,p}(\Omega), \tag{26}$$

and

$$y_n \to y \quad \text{in } L^p(\Omega). \tag{27}$$

Moreover, by (27), we also have

$$y_n(x) \to y(x) \quad \text{a.e. in } \Omega, \tag{28}$$

and there exists $h \in L^p(\Omega)$ such that

$$|y_n(x)| \leq h(x), \tag{29}$$

for almost all $x \in \Omega$.

Dividing by $\|u_n\|^p$ relation (25) we obtain

$$m_0^{p-1} \int_\Omega |\nabla y_n(x)|^p dx \leq \frac{\varepsilon_n}{\|u_n\|^p} + \varepsilon \|y_n\|^p_{L^p(\Omega)} + \frac{c_\varepsilon M_6}{\|u_n\|^{p-\sigma r}} \|y_n\|^{\sigma r}_{L^{p^*}(\Omega)}, \tag{30}$$

for every $n \in \mathbb{N}$.

Now, we observe that condition

$$\mu > (r-p) \max\left\{1, \frac{N}{p}\right\}$$

is equivalent to $\sigma r < p$.

Passing to the limit as $n \to +\infty$ in (30) and using (26), we have

$$m_0^{p-1} \int_\Omega |\nabla y(x)|^p dx \leq \varepsilon \|y\|^p_{L^p(\Omega)} \leq \varepsilon \|y\|^p \leq \varepsilon. \tag{31}$$

But

$$\|y\| \leq \liminf_{n\to\infty} \|y_n\| = 1.$$

Now, since $\varepsilon > 0$ is arbitrary, we deduce that $y = \kappa \in \mathbb{R}$. If $y = 0$, we have $\nabla y_n \to 0$ in $L^p(\Omega; \mathbb{R}^N)$ and so $y_n \to 0$ in $W^{1,p}(\Omega)$. This fact is a contradiction since $\|y_n\| = 1$, for all $n \in \mathbb{N}$. Consequently we obtain $y \neq 0$. This implies that

$$|u_n(x)| \to +\infty, \quad \text{for every } x \in \Omega.$$

By (12) we have

$$\left|\frac{\Phi(u_n)}{\|u_n\|^p} - \int_\Omega \frac{F(x,u_n(x))}{\|u_n\|^p}dx\right| \leq \frac{M_1}{\|u_n\|^p}, \tag{32}$$

for every $n \in \mathbb{N}$. Therefore, the right-hand side of condition $(\widehat{C}_M)$ implies that

$$\frac{\Phi(u_n)}{\|u_n\|^p} \leq \frac{m_1}{p} + \frac{m_2}{p\|u_n\|^p}, \tag{33}$$

for every $n \in \mathbb{N}$. Hence, by (32) and (33) we can write

$$\int_\Omega \frac{F(x,u_n(x))}{\|u_n\|^p}dx \leq \frac{M_1}{\|u_n\|^p} + \frac{m_1}{p} + \frac{m_2}{p\|u_n\|^p}, \tag{34}$$

for every $n \in \mathbb{N}$.

On the other hand, condition (h_4) shows that for a given $\eta > 0$, there exists $M_{7,\eta} > 0$ such that

$$\frac{F(x,\xi)}{|\xi|^p} \geq \eta > 0, \tag{35}$$

for almost every $x \in \Omega$ and every $|\xi| \geq M_{7,\eta}$. Set

$$M^+_{7,\eta} := \{x \in \Omega : |u_n(x)| \geq M_{7,\eta}\}, \quad M^-_{7,\eta} := \{x \in \Omega : |u_n(x)| < M_{7,\eta}\}.$$

By using (35) and (h_3) we can write

$$\begin{aligned}
\int_\Omega \frac{F(x,u_n(x))}{\|u_n\|^p}dx &= \int_{M^+_{7,\eta}} \frac{F(x,u_n(x))}{|u_n(x)|^p}|y_n(x)|^p dx \\
&\quad + \int_{M^-_{7,\eta}} \frac{F(x,u_n(x))}{\|u_n\|^p}dx \\
&\geq \int_{M^+_{7,\eta}} \eta |y_n(x)|^p dx \\
&\quad + \int_{M^-_{7,\eta}} \frac{F(x,u_n(x))}{\|u_n\|^p}dx \\
&\geq \eta \int_{M^+_{7,\eta}} |y_n(x)|^p dx - \frac{M_8}{\|u_n\|^p},
\end{aligned} \tag{36}$$

for some $M_8 > 0$ and for every $n \in \mathbb{N}$. On the other hand, observing that $|u_n(x)| \to +\infty$ for almost every $x \in \Omega$ and bearing in mind (28), we have

$$\chi_{M^+_{7,\eta}}(x) y_n(x) \to \chi_\Omega(x)\kappa,$$

for almost every $x \in \Omega$. Moreover, by (29),

$$\chi_{M^+_{7,\eta}}(x)|y_n(x)|^p \in L^1(\Omega).$$

Thus, as $n \to \infty$,

$$\int_{M_{7,\eta}^+} |y_n(x)|^p dx = \int_\Omega \chi_{M_{7,\eta}^+}(x)|y_n(x)|^p dx \to |\kappa|^p \operatorname{meas}(\Omega). \tag{37}$$

So, passing to the limit as $n \to \infty$ and using (37), we obtain

$$\liminf_{n\to\infty} \int_\Omega \frac{F(x, u_n(x))}{\|u_n\|^p} dx \geq \eta |\kappa|^p \operatorname{meas}(\Omega).$$

Since $\eta > 0$ is arbitrary and $\kappa \neq 0$, it follows that

$$\liminf_{n\to\infty} \int_\Omega \frac{F(x, u_n(x))}{\|u_n\|^p} dx = +\infty. \tag{38}$$

Comparing relations (38) and (34) we obtain a contradiction. The proof of Lemma 4.4 is now complete. □

The next result shows that the functional φ defined in Lemma 4.4 satisfies the Cerami condition.

Lemma 4.5. *Assume that conditions* $\boldsymbol{H}(f)$ *hold. Then* φ *satisfies the compactness condition* (C).

Proof. Let $\{u_n\} \subset W^{1,p}(\Omega)$ be a Cerami sequence. Thus, by Lemma 4.4, the sequence $\{u_n\}$ is bounded in $W^{1,p}(\Omega)$. Since $W^{1,p}(\Omega)$ is reflexive, we can suppose that, up to a subsequence, $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$. We prove that, in fact, $\{u_n\}$ strongly converges to $u \in W^{1,p}(\Omega)$. We first observe that

$$\langle \Phi'(u_n), u_n - u\rangle = \langle \varphi'(u_n), u_n - u\rangle + \int_\Omega f(x, u_n(x))(u_n - u)(x) dx.$$

Since $\|\varphi'(u_n)\|_{(W^{1,p}(\Omega))^*} \to 0$ and the sequence $\{u_n - u\}$ is bounded in $W^{1,p}(\Omega)$, we obtain

$$\langle \varphi'(u_n), u_n - u\rangle \to 0,$$

as $n \to \infty$.

Next, by (h_3) and taking into account that $u_n \to u$ in $L^r(\Omega)$ for all $r < p^*$, we deduce that

$$\int_\Omega |f(x, u_n(x))||u_n(x) - u(x)| dx \to 0,$$

as $n \to \infty$. Therefore, by (C_M^1), we obtain

$$\langle A(u_n), u_n - u\rangle \to 0,$$

as $n \to \infty$.

Since A has the (S_+) property, we conclude that $u_n \to u$ strongly in $W^{1,p}(\Omega)$. The proof of Lemma 4.5 is now complete. □

The next result establishes that φ has a local linking behavior.

Proposition 4.6. *Assume that hypotheses* $\boldsymbol{H}(f)$ *hold. Then* φ *has a local linking at the origin.*

Proof. Let $\delta > 0$ as in condition (h_7). Fix $\gamma \in \mathbb{R}$ such that $|\gamma| \leq \delta$. Then

$$\|\gamma\| = \delta\, \mathrm{meas}\,(\Omega)^{1/p}.$$

Set $\rho_1 := \delta\, \mathrm{meas}(\Omega)^{1/p}$. Using ($h_7$), since $\Phi(0_{W^{1,p}(\Omega)}) = 0$, we deduce that

$$\varphi(u) \leq 0, \quad \forall\, u \in W^{1,p}(\Omega), \|\gamma\| \leq \rho_1. \tag{39}$$

Next, combining the mean value theorem with (24) we obtain that

$$|F(x,\xi)| \leq \varepsilon|\xi|^p + c_\varepsilon|\xi|^r, \tag{40}$$

for almost every $x \in \Omega$ and every $\xi \in \mathbb{R}$. Let us consider

$$V := \left\{ u \in W^{1,p}(\Omega) : \int_\Omega u(x)dx = 0 \right\}.$$

Then, by (40) and the Poincaré–Wirtinger inequality (8), we have

$$\begin{aligned} \varphi(u) &= \Phi(u) - \int_\Omega F(x,u(x))dx \\ &\geq \frac{m_0^{p-1}}{p} \int_\Omega |\nabla u(x)|^p dx - \varepsilon\|u\|^p_{L^p(\Omega)} - c_\varepsilon\|u\|^r_{L^r(\Omega)} \\ &\geq \left(\frac{m_0^{p-1}}{p} - \varepsilon\eta\right) \int_\Omega |\nabla u(x)|^p dx - c_\varepsilon\|u\|^r_{L^r(\Omega)}. \end{aligned} \tag{41}$$

Fix

$$\varepsilon \in \left(0, \frac{m_0^{p-1}}{\eta p}\right).$$

Since $p < r < p^*$, the space $W^{1,p}(\Omega)$ is continuously embedded in $L^r(\Omega)$. Thus, by (7) and (8), it follows that

$$\|u\|_{L^r(\Omega)} \leq c_r(1+\eta^p)^{1/p} \left(\int_\Omega |\nabla u(x)|^p dx\right)^{1/p}, \quad \forall\, u \in V. \tag{42}$$

Set

$$M_9 := c_\varepsilon \left(c_r(1+\eta^p)^{1/p}\right)^r. \tag{43}$$

By (41), relations (42) and (43) yield

$$\varphi(u) \geq \left(\frac{m_0^{p-1}}{p} - \varepsilon\eta\right) \int_\Omega |\nabla u(x)|^p dx - M_9 \left(\int_\Omega |\nabla u(x)|^p dx\right)^{r/p},$$

for every $u \in V$.

Bearing in mind the choice of ε and taking into account that $p < r$, the above inequality implies that there is $\rho_2 > 0$ sufficiently small such that for all $u \in V$

with $\|u\| \le \rho_2$,

$$\varphi(u) \ge 0. \tag{44}$$

In conclusion, if $\rho \le \min\{\rho_1, \rho_2\}$, relations (44) and (39) imply that the energy functional φ has a local linking at the origin. $\square$

Proposition 4.7. *Assume that hypotheses* $\boldsymbol{H}(f)$ *hold and* $E \subseteq V$ *is a finite-dimensional linear subspace. Then*

$$\varphi(u) \to -\infty \quad \textit{as } \|u\| \to +\infty, \;\; u \in \mathbb{R} \oplus E.$$

Proof. We will prove that the restriction of the functional φ to $\mathbb{R} \oplus E$ is anticoercive. For our goal, exploiting hypothesis (h_4), for a given $\theta > 0$, we find a positive constant $M_{10,\theta}$ such that

$$F(x, \xi) \ge \theta |\xi|^p, \tag{45}$$

for almost every $x \in \Omega$ and every $|\xi| \ge M_{10,\theta}$. Moreover, owing to (h_3), there exists $M_{11} > 0$ such that

$$|F(x, \xi)| \le M_{11}, \tag{46}$$

for almost every $x \in \Omega$ and every $|\xi| < M_{11}$. Thus, thanks to (46) and (45), there exists $M_{12} > 0$ such that

$$F(x, \xi) \ge \theta |\xi|^p - M_{12}, \tag{47}$$

for almost every $x \in \Omega$ and every $\xi \in \mathbb{R}$.

By using the right-hand side of $(\widehat{C}_M)$ and (47), we have for every $u \in \mathbb{R} \oplus E$

$$\varphi(u) = \Phi(u) - \int_\Omega F(x, u(x))dx \le \frac{m_1}{p}\|u\|^p - \theta\|u\|^p_{L^p(\Omega)} + M_{13}, \tag{48}$$

for some $M_{13} > 0$. Now, since the space $\mathbb{R} \oplus E$ is finite-dimensional, all the norms are equivalent and we have

$$\varphi(u) \le \left(\frac{m_1}{p} - \theta M_{14}\right)\|u\|^p + M_{13}, \tag{49}$$

for some $M_{14} > 0$. By relation (49) and bearing in mind that θ was arbitrary, we conclude that $\varphi(u) \to -\infty$ as $\|u\| \to +\infty$, $u \in \mathbb{R} \oplus E$. $\square$

4.3. *Proof of Theorem* 4.3 *concluded*

We apply Theorem 3.1 for $Y := \mathbb{R}$,

$$V := \left\{u \in W^{1,p}(\Omega) : \int_\Omega u(x)dx = 0\right\},$$

and

$$\varphi(u) := \frac{1}{p}\widehat{M}\left(\int_\Omega |\nabla u(x)|^p dx\right) - \int_\Omega \left(\int_0^{u(x)} f(x,t)dt\right) dx, \quad \forall\, u \in W^{1,p}(\Omega).$$

The definition of φ shows that it maps bounded sets into bounded sets. Moreover, Lemma 4.5 and Propositions 4.6 and 4.7 allow the application of Theorem 3.1. Thus, we can find a weak solution $u \in W^{1,p}(\Omega) \setminus \{0\}$ of problem $(N^p_{M,f})$. The proof is now complete.

4.4. *Example*

We conclude this paper with an illustration of Theorem 4.3 for an elliptic nonlocal Neumann problem involving the Laplace operator. We first remark that for $p = 2$, the function

$$M(t) := 1 + \frac{\cos t}{1+t^2}$$

satisfies conditions (C^1_M) and (C^2_M).

Example 4.8. Consider the nonlinear problem

$$\begin{cases} -M\left(\displaystyle\int_\Omega |\nabla u(x)|^2 dx\right) \Delta u = f(u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where f is given in Remark 4.2 and

$$M\left(\int_\Omega |\nabla u(x)|^2 dx\right) := 1 + \frac{\cos\left(\displaystyle\int_\Omega |\nabla u(x)|^2 dx\right)}{1 + \left(\displaystyle\int_\Omega |\nabla u(x)|^2 dx\right)^2}.$$

Thus, by Theorem 4.3, the above problem admits at least one nontrivial weak solution.

Remark 4.9. We notice that there are several multiplicity results for nonlinear Neumann problems driven by p-Laplacian type operators. For instance, Motreanu and Papageorgiou [25] studied a nonlinear Neumann problem driven by a nonhomogeneous quasilinear degenerate elliptic differential operator. In [25], the reaction term is a Carathéodory function that exhibits subcritical growth in the second variable. The authors, using variational methods based on the mountain pass and deformation theorems, together with truncation and minimization techniques, showed that the problem has three nontrivial smooth solutions, two of which have constant sign (one positive, the other negative).

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