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Ground states of weighted 4D biharmonic equations with exponential growth

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In this paper, we are concerned with the existence of a ground state solution for a logarithmic weighted biharmonic equation under Dirichlet boundary conditions in the unit ball B of $\mathbb{R}^4$. The reaction term of the equation is assumed to have exponential growth, in view of Adams' type inequalities. It is proved that there is a ground state solution using min-max techniques and the Nehari method. The associated energy functional loses compactness at a certain level. An appropriate asymptotic condition allows us to bypass the non-compactness levels of the functional.

KEYWORDS

Adams inequality, compactness level, mountain pass method, Nehari manifold, nonlinearity of exponential growth

MSC CLASSIFICATION

31B30, 35J40, 58E05

1 | INTRODUCTION

In this work, we consider the following fourth-order weighted equations:

$$\begin{cases} \Delta(w(x)\Delta u) = f(x,u) & \text{in} \quad B \\ u = \frac{\partial u}{\partial n} = 0 & \text{on} \quad \partial B, \end{cases} \tag{1}$$

where $B = B(0,1)$ is the unit open ball in $\mathbb{R}^4$, the weight $w(x)$ is given by

$$w(x) = \left(\log \frac{e}{|x|}\right)^{\beta}, \ \beta \in (0,1)\cdot \tag{2}$$

The nonlinearity $f(x,t)$ is a radial function with respect to x and behaves like $\exp(\alpha|t|^{\frac{2}{1-\beta}})$ as $t \to +\infty$.

In the literature, the notion of critical exponential growth, in the Sobolev space $W_0^{1,N}(\Omega)$, $N \geq 2$, $\Omega \subset \mathbb{R}^N$, is given by the well-known Trudinger–Moser inequality [1, 2]

$$\sup_{\int_\Omega |\nabla u|^N \leq 1} \int_\Omega e^{\alpha|u|^{\frac{N}{N-1}}} dx < +\infty \ \text{ if and only if } \ \alpha \leq \alpha_N,$$

where $\alpha_N = \omega_{N-1}^{\frac{1}{N-1}}$ with ω_{N-1} is the area of the unit sphere S^{N-1} in $\mathbb{R}^N$. The latter result allowed the study of second-order problems with exponential growth nonlinearities in non-weighted Sobolev spaces. For example, we cite the following problems in dimension $N \geq 2$:

$$-\Delta_N u = -\text{div}(|\nabla u|^{N-2}\nabla u) = f(x,u) \ \text{ in } \ \Omega \subset \mathbb{R}^N$$

which have been studied considerably [3–6].

Later, this notion of critical exponential growth was extended to include higher order.

For bounded domains $\Omega \subset \mathbb{R}^4$, in the literature [7, 8], the authors extended the Trudinger–Moser inequalities [1, 2] to the higher-order space $W_0^{2,2}(\Omega)$ and obtained the so-called Adams' inequalities,

$$\sup_{u \in S} \int_\Omega (e^{\alpha u^2)} - 1) dx < +\infty \ \Leftrightarrow \ \alpha \leq 32\pi^2$$

where

$$S = \left\{ u \in W_0^{2,2}(\Omega) \mid \left(\int_\Omega |\Delta u|^2 dx \right)^{\frac{1}{2}} \leq 1 \right\}.$$

These results allowed to investigate fourth-order problems with subcritical or critical exponential nonlinearity involving continuous potential [9, 10].

Recently, a result related to Adams' inequalities has been extended to weighted Sobolev spaces. More precisely, Wang and Zhu [11] proved the following result:

Theorem 1 (Wang and Zhu [11]). *Let $\beta \in (0,1)$ and let w given by (2), then*

$$\sup_{\substack{u \in W_{0,rad}^{2,2}(B,w) \\ \int_B w(x)|\Delta u|^2 dx \leq 1}} \int_B e^{\alpha|u|^{\frac{2}{1-\beta}}} dx < +\infty \ \Leftrightarrow \ \alpha \leq \alpha_\beta = 4[8\pi^2(1-\beta)]^{\frac{1}{1-\beta}}, \tag{3}$$

where $W_{0,rad}^{2,2}(B,w)$ denotes the weighted Sobolev space of radial functions given by

$$W_{0,rad}^{2,2}(B,w) = \text{closure}\left\{ u \in C_{0,rad}^{\infty}(B) \mid \int_B w(x)|\Delta u|^2 dx < \infty \right\},$$

endowed with the norm $\|u\| = \left(\int_B w(x)|\Delta u|^2 dx \right)^{\frac{1}{2}}$.

This result is an extension of the following Trudinger–Moser inequalities in Sobolev spaces with logarithmic weights due to Calanchi and Ruff [12].

Theorem 2 (Calanchi and Ruff [12]).

(i) *Let* $\beta \in [0,1)$ *and let* ρ *given by* $\rho(x) = \left(\log \frac{1}{|x|}\right)^{\beta(N-1)}$, *then*

$$\int_B e^{|u|^\tau} dx < +\infty, \ \forall\, u \in W^{1,N}_{0,rad}(B,\rho), \ \text{if and only if } \tau \le \tau_{N,\beta} = \frac{N}{(N-1)(1-\beta)} = \frac{N'}{1-\beta}$$

and

$$\sup_{\substack{u\in W^{1,N}_{0,rad}(B,\rho)\\ \int_B |\nabla u|^N w(x)dx\le 1}} \int_B e^{\alpha |u|^{\tau_{N,\beta}}} dx < +\infty \ \Leftrightarrow\ \alpha \le \alpha_{N,\beta} = N\left[\omega_{N-1}^{\frac{1}{N-1}}(1-\beta)\right]^{\frac{1}{1-\beta}}$$

where ω_{N-1} *is the area of the unit sphere* S^{N-1} *in* $\mathbb{R}^N$ *and* N' *is the Hölder conjugate of* N.

(ii) *Let* ρ *given by* $\rho(x) = \left(\log \frac{e}{|x|}\right)^{N-1}$, *then*

$$\int_B exp\left\{e^{|u|^{\frac{N}{N-1}}}\right\} dx < +\infty, \ \forall\, u \in W^{1,N}_{0,rad}(B,\rho)$$

and

$$\sup_{\substack{u\in W^{1,N}_{0,rad}(B,\rho)\\ \|u\|_\rho\le 1}} \int_B exp\left\{\beta e^{\omega_{N-1}^{\frac{1}{N-1}} |u|^{\frac{N}{N-1}}}\right\} dx < +\infty \ \Leftrightarrow\ \beta \le N,$$

where $W^{1,N}_{0,rad}(B,\rho) = \text{closure}\left\{u \in C^\infty_{0,rad}(B) \mid \int_B |\nabla u|^N \rho(x)dx < \infty\right\}$, is equipped with the norm $\|u\|_\rho = \left(\int_B |\nabla u|^N w(x)\ dx\right)^{\frac{1}{N}}$.

These results have allowed a study of weighted elliptic problems of the second-order in dimension $N \ge 2$. For instance, recently, in the case $V = 0$ or $V \ne 0$, previous studies [13–15] have proved the existence of a nontrivial solution for the following boundary value problem:

$$\begin{cases} -\text{div}(\sigma(x)|\nabla u(x)|^{N-2}\nabla u(x)) + V(x)|u|^{N-2}u = f(x,u) & \text{in} \quad B \\ u = 0 & \text{on} \quad \partial B, \end{cases}$$

where B is the unit ball in $\mathbb{R}^N$, $N \ge 2$, the nonlinearity $f(x,u)$ is continuous in $B \times \mathbb{R}$ and has critical growth in the sense of Theorem 2. The authors proved that there is a non-trivial solution to this problem using mountain pass Theorem. Also, Abid et al. [16] investigated the weighted second-order elliptic problem of Kirchhoff type, which is defined as follows:

$$\begin{cases} -g\left(\int_B \sigma(x)|\nabla u|^N dx\right) \text{div}\left(\sigma(x)|\nabla u|^{N-2}\nabla u\right) = f(x,u) & \text{in} \quad B, \\ u > 0 & \text{in} \quad B, \\ u = 0 & \text{on} \quad \partial B, \end{cases}$$

where $N \ge 2$, $\rho(x) = (\log(e/|x|))^{N-1}$, and the function $f(x,t)$ is continuous in $B \times \mathbb{R}$ and behaves like $\exp\left\{e^{\alpha t^{\frac{N}{N-1}}}\right\}$ as time $t \to +\infty$, for some $\alpha > 0$. The Kirchhoff function g satisfies certain conditions. The authors proved that this problem has a positive ground state solution using minimax techniques combined with Trudinger–Moser inequality. Radulescu and Vetro [17] studied on a bounded domain $\Omega \subseteq \mathbb{R}^n$ a fourth-order problem involving a sign-changing Kirchhoff $p(x)$-Laplace biharmonic operator as follows:

$$\begin{cases} \left(a - b\int_\Omega \frac{1}{|p(x)|}|\Delta u|^{p(x)} dx\right) \Delta^2_{p(x)} u = f(x,u,\nabla u,\Delta u) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Notably, the presence of a reaction term dependent on the gradient of solutions disrupts the problem's variational structure. To surmount this challenge, the authors employ topological techniques from operator theory, along with the Galerkin method and fixed-point arguments, to establish the existence of solutions. In addition to their work, we also cite the research of Bohner et al. [18]. In this study, the authors investigated the existence of at least three solutions for a class of double eigenvalue discrete anisotropic Kirchhoff-type problems by employing critical point theory and variational methods. We also cite the research of Gupta and Dwivedi [19]. In their work, the authors established the existence of a ground state solution to the Kirchhoff problem:

$$\begin{cases} -k\left(\int_\Omega |\nabla u|^N dx\right)\Delta_N u = \frac{f(x,u)}{|x|^a} + \lambda g(x) \; in\, \Omega, \\ u = 0 \; on\, \partial\Omega, \end{cases}$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with smooth boundary and where the nonlinearity function f satisfies critical exponential growth at infinity, albeit without satisfying the Ambrosetti–Rabinowitz condition. Consequently, they establish the existence of a ground state weak solution by applying the mountain pass theorem and the Nehari manifold technique. Recently, Dridi et al. [20], investigate the existence of least energy nodal solutions for the following nonlocal weighted Schrödinger–Kirchhoff problem:

$$\begin{cases} \mathcal{L}_{(\rho,\xi)}(u) = f(x,u) & \text{in } B \\ u = 0 & \text{on } \partial B, \end{cases}$$

where B is the unit ball of $\mathbb{R}^N$, $N > 2$. The nonlinearity $f(x,t)$ is continuous in $B \times \mathbb{R}$ and has exponential critical growth in the sense of Theorem 2 and the operator $\mathcal{L}_{(\omega,\xi)}$ is defined as

$$\mathcal{L}_{(\rho,\xi)}(u) := m\left(\int_B (\rho(x)|\nabla u|^N + \xi(x)|u|^N)dx\right)\left[-\text{div}(\rho(x)|\nabla u|^{N-2}\nabla u) + \xi(x)|u|^{N-2}u\right],$$

where m is a continuous positive function on $(0,+\infty)$ satisfying some mild conditions. The weight $\rho(x)$ is of logarithm type and the potential $\xi : \bar{B} \to \mathbb{R}$ is a positive continuous function and bounded away from zero in B. The existence result is obtained by the constrained minimization in Nehari set, the quantitative deformation Lemma and degree theory results.

Now, let $\Omega \subset \mathbb{R}^4$, be a bounded domain and $w \in L^1(\Omega)$ be a nonnegative function. We introduce the Sobolev space

$$W_0^{2,2}(\Omega, w) = \text{closure}\left\{u \in C_0^\infty(\Omega) \mid \int_\Omega w(x)|\Delta u|^2 dx < \infty\right\}.$$

Since the nonlinearity is radial, the natural space for a variational treatment of the biharmonic problem (1) is the subspace

$$\mathcal{X} = W_{0,rad}^{2,2}(\Omega, w) = \text{closure}\left\{u \in C_{0,rad}^\infty(B) \mid \int_B w(x)|\Delta u|^2 dx < \infty\right\},$$

endowed with the norm

$$\|u\| = \left(\int_B w(x)|\Delta u|^2 dx\right)^{\frac{1}{2}}.$$

We note that this norm is issued from the inner product

$$\langle u, v\rangle = \int_B w(x)\Delta u\, \Delta v\, dx.$$

Let $\gamma := \frac{2}{1-\beta}$. In view of inequality (3), we say that f has critical growth at infinity if there exists some $\alpha_0 > 0$,

$$\lim_{s\to+\infty} \frac{|f(x,s)|}{e^{\alpha s^\gamma}} = 0, \ \forall \alpha \ \text{such that} \ \alpha > \alpha_0 \ \text{and} \ \lim_{s\to+\infty} \frac{|f(x,s)|}{e^{\alpha s^\gamma}} = +\infty, \ \forall \alpha < \alpha_0. \tag{4}$$

Inspired by the last work cited above, we study the existence of ground state solutions when the nonlinear terms have critical exponential growth in the sense of Adams' inequalities [11]. Our approach is variational. Let us now state our result.

We suppose that $f(x,t)$ has critical growth at infinity and satisfies the following hypotheses:

(A_1) The nonlinearity $f : \bar{B} \times \mathbb{R} \to \mathbb{R}$ is continuous, radial in x and $f(x,t) = 0$ for all $t \leq 0$.

(A_2) There exist $t_0 > 0$ and $M_0 > 0$ such that for all $|t| > t_0$ and for all $x \in B$ we have

$$0 < F(x,t) \leq M_0 |f(x,t)|,$$

where

$$F(x,t) = \int_0^t f(x,s)ds.$$

(A_3) $\lim_{t\to 0} \frac{f(x,t)}{t} = 0$.

(A_4) For each $x \in B$, $t \mapsto \frac{f(x,t)}{t}$ is increasing for all $t > 0$.

(A_5) In the case that (3) holds, there exists $\gamma_0 > 0$ such that

$$\lim_{t\to\infty} \frac{f(x,t)t}{e^{\alpha_0 t^\gamma}} \geq \gamma_0 \quad \text{uniformly in } x, \ \text{with} \ \gamma_0 > 256\left(\frac{4}{\alpha_0}\right)^{1-\beta}(1-\beta)\cdot$$

The condition (A_2) implies that for any $\varepsilon > 0$, there exists a real $t_\varepsilon > 0$ such that

$$F(x,t) \leq \varepsilon t f(x,t), \ \forall |t| > t_\varepsilon, \ \text{uniformly in } x \in B. \tag{5}$$

We give an example of f. Let $f(t) = F'(t)$, with $F(t) = \frac{t^4}{4} + t^4 e^{\alpha_0 t^\gamma}$. A simple calculation shows that f verifies the conditions (A_1), (A_2), (A_3), (A_4) and (A_5).

It will be said that u is a solution to problem (1), if u is a weak solution in the following sense.

Definition 1. A function u is called a solution to (1) if $u \in \mathcal{X}$ and

$$\int_B w(x)\Delta u \Delta\varphi\, dx = \int_B f(x,u)\varphi\, dx, \quad \text{for all } \varphi \in \mathcal{X}$$

The energy functional associated to (1) is defined by $\mathcal{E} : \mathcal{X} \to \mathbb{R}$

$$\mathcal{E}(u) = \frac{1}{2}\|u\|^2 - \int_B F(x,u)dx, \tag{6}$$

where

$$F(x,t) = \int_0^t f(x,s)ds.$$

Definition 2. A solution u is a ground state solution to problem (1), if u is a solution and

$$\mathcal{E}(u) = r, \ \text{with} \ r = \inf_{u\in\mathcal{S}} \mathcal{E}(u) \ \text{where} \ \mathcal{S} = \{u \in \mathcal{X} : \mathcal{E}'(u) = 0, u \neq 0\},$$

and

$$\mathcal{E}'(u)\varphi := \langle \mathcal{E}'(u), \varphi \rangle = \int_B \omega(x) \Delta u \Delta \varphi \, dx - \int_B f(x,u) \varphi \, dx, \ \varphi \in \mathcal{X}.$$

It is clear that finding weak solutions to the problem (1) is equivalent to finding non-zero critical points of the functional $\mathcal{E}$ over $\mathcal{X}$.

The major difficulty is the loss of compactness for the energy $\mathcal{E}$. To circumvent it, we use appropriate Adams' functions and prove a concentration compactness result.

Our result is as follows:

Theorem 3. *Assume that $f(x,t)$ verifies (4) for some α_0. If in addition f satisfies the conditions (A_1), (A_2), (A_3) , (A_4) and (A_5), then problem (1) has a ground state solution.*

To the best of our knowledge, the present papers results have not been covered yet in the literature.

This paper is organized as follows. In Section 2, we present some necessary preliminary knowledge about functional space. In Section 3, we give some useful lemmas for the compactness analysis and we prove a concentration compactness result of Lions type. In Section 4, we prove that the energy $\mathcal{E}$ has a mountain pass geometry. Section 5 is devoted to estimate the minimax level of energy. In this estimation, the asymptotic condition (A_5) will play a decisive role. Finally, we conclude with the proofs of the main result in Section 6.

Throughout this paper, the constant C may change from one line to another and we sometimes index the constants in order to show how they change.

2 | WEIGHTED LEBESGUE AND SOBOLEV SPACES SETTING

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain in $\mathbb{R}^N$ and let $w \in L^1(\Omega)$ be a nonnegative function. To deal with weighted operator, we need to introduce some functional spaces $L^p(\Omega, w)$, $W^{m,p}(\Omega, w)$, $W_0^{m,p}(\Omega, w)$ and some of their properties that will be used later. Let $S(\Omega)$ be the set of all measurable real-valued functions defined on Ω and two measurable functions are considered as the same element if they are equal almost everywhere.

Following Drabek et al. and Kufner [21, 22], the weighted Lebesgue space $L^p(\Omega, w)$ is defined as follows:

$$L^p(\Omega, w) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable; } \int_\Omega w(x)|u|^p \, dx < \infty \right\}$$

for any real number $1 \leq p < \infty$.

This is a normed vector space equipped with the norm

$$\|u\|_{p,w} = \left(\int_\Omega w(x)|u|^p \, dx \right)^{\frac{1}{p}}.$$

For $w(x) = 1$, one finds the standard Lebesgue space $L^p(\Omega)$ endowed with the norm $\|u\|_p = \left(\int_\Omega |u|^p \, dx \right)^{\frac{1}{p}}$.

For $m \geq 2$, let w be a given family of weight functions w_τ, $|\tau| \leq m$,

$$w = \{w_\tau(x) \, x \in \Omega, \ |\tau| \leq m\}.$$

In Drabek et al. [21], the weighted Sobolev space corresponds to the following definition:

$$W^{m,p}(\Omega, w) = \{u \in L^p(\Omega) \text{ such that } D^\tau u \in L^p(\Omega) \text{ for all } |\tau| \leq m-1, \, D^\tau u \in L^p(\Omega, w) \text{ for all } |\tau| = m\}$$

equipped with the norm:

$$\|u\|_{W^{m,p}(\Omega,w)} = \left(\sum_{|\tau|\le m-1} \int_{\Omega} |D^{\tau}u|^p dx + \sum_{|\tau|=m} \int_{\Omega} w(x)|D^{\tau}u|^p dx \right)^{\frac{1}{p}}.$$

If we also assume that $w(x) \in L^1_{loc}(\Omega)$, then $C_0^{\infty}(\Omega)$ is a subset of $W^{m,p}(\Omega, w)$ and one can introduce the space

$$W_0^{m,p}(\Omega, w)$$

as the closure of $C_0^{\infty}(\Omega)$ in $W^{m,p}(\Omega, w)$. In addition, the following embedding is compact:

$$W^{m,p}(\Omega, w) \hookrightarrow W^{m-1,p}(\Omega).$$

Also, $(L^p(\Omega, w), \|\cdot\|_{p,w})$ and $(W^{m,p}(\Omega, w), \|\cdot\|_{W^{m,p}(\Omega,w)})$ are separable, reflexive Banach spaces provided that $w(x)^{\frac{-1}{p-1}} \in L^1_{loc}(\Omega)$.

Then the space $\mathcal{X}$ is a Banach and reflexive space. The space $\mathcal{X}$ is endowed with the norm

$$\|u\| = \left(\int_B w(x)|\Delta u|^2 dx \right)^{\frac{1}{2}}$$

which is equivalent to the following norm (see Lemma 1):

$$\|u\|_{W^{2,2}_{0,rad}(B,w)} = \left(\int_B u^2 dx + \int_B |\nabla u|^2 dx + \int_B w(x)|\Delta u|^2 dx \right)^{\frac{1}{2}}.$$

We also have the continuous embedding $\mathcal{X} \hookrightarrow L^q(B)$ for all $q \ge 1$. Moreover, $\mathcal{X}$ is compactly embedded in $L^q(B)$ for all $q \ge 1$ (see Lemma 1).

3 | PRELIMINARIES FOR THE COMPACTNESS ANALYSIS

In this section, we will establish several technical lemmas that we can use later. We begin with the radial lemma.

Lemma 1. *Let u be a radially symmetric function in $C_0^2(B)$. Then, we have*

(*i*) *Wang and Zhu [11]*

$$|u(x)| \le \frac{1}{2\sqrt{2\pi}} \frac{\left| |\log\left(\frac{e}{|x|}\right)|^{1-\beta} - 1\right|^{\frac{1}{2}}}{\sqrt{1-\beta}} \int_B w(x)|\Delta u|^2 dx \le \frac{1}{2\sqrt{2\pi}} \frac{\left| |\log\left(\frac{e}{|x|}\right)|^{1-\beta} - 1\right|^{\frac{1}{2}}}{\sqrt{1-\beta}} \|u\|.$$

(*ii*) $\int_B e^{|u|^{\gamma}} dx < +\infty, \ \forall u \in \mathcal{X}.$

(*iii*) *The norms* $\|.\|$ *and* $\|u\|_{W^{2,2}_{0,rad}(B,w)} = \left(\int_B u^2 dx + \int_B |\nabla u|^2 dx + \int_B w(x)|\Delta u|^2 dx \right)^{\frac{1}{2}}$ *are equivalents.*

(*iv*) *The following embedding is continuous:*

$$\mathcal{X} \hookrightarrow L^q(B) \ \text{for all} \ q \ge 2.$$

(*v*) $\mathcal{X}$ *is compactly embedded in* $L^q(B)$ *for all* $q \ge 1$.

Proof. (*i*) See Wang and Zhu [11].

(*ii*) From (*i*) and using the identity $\log(\frac{e}{|x|}) - |\log(|x|)| = 1\ \forall x \in B$ and the fact that $\sqrt{t-1} \le \sqrt{t}, \forall t \ge 1$, we get

$$|u(x)|^{\gamma} \le \frac{1}{\alpha_\beta}\left| |\log\left(\frac{e}{|x|}\right)|^{1-\beta} - 1\right|^{\frac{1}{1-\beta}} \|u\|^{\gamma} \le \frac{1}{\alpha_\beta}\left(1 + |\log(|x|)|\right) \|u\|^{\gamma}.$$

Hence, using the fact that the function $r \mapsto r^3 e^{\frac{\|u\|^{\gamma}(1+\log r)}{\alpha_\beta}}$ is increasing, we get

$$\int_B e^{|u|^{\gamma}} dx \le 2\pi^2 \int_0^1 r^3 e^{\frac{\|u\|^{\gamma}(1+\log r)}{\alpha_\beta}} dr \le 2\pi^2 e^{\frac{\|u\|^{\gamma}}{\alpha_\beta}} < +\infty,\ \forall\, u \in \mathcal{X}.$$

Then (*ii*) follows by density.

(*iii*) By Poincaré inequality, for all $u \in W^{1,2}_{0,rad}(B)$

$$\int_B u^2 dx \le C \int_B |\nabla u|^2 dx.$$

Using the Green formula, we get

$$\int_B |\nabla u|^2 dx = \int_B \nabla u \nabla u dx = -\int_B u\Delta u dx + \underbrace{\int_{\partial B} u\frac{\partial u}{\partial n}}_{=0} \le \left|\int_B u\Delta u dx\right|.$$

By Young inequality, we get for all $\varepsilon > 0$

$$\left|\int_B u\Delta u dx\right| \le \frac{1}{2\varepsilon}\int_B |\Delta u|^2 dx + \frac{\varepsilon}{2}\int_B u^2 dx \le \frac{1}{2\varepsilon}\int_B w(x)|\Delta u|^2 dx + \frac{\varepsilon}{2}\int_B u^2 dx.$$

Hence,

$$\left(1 - \frac{\varepsilon}{2}C^2\right)\int_B |\nabla u|^2 dx \le \frac{1}{2\varepsilon}\int_B w(x)|\Delta u|^2 dx,$$

so

$$\int_B u^2 dx + \int_B |\nabla u|^2 dx + \int_B w(x)|\Delta u|^2 dx \le C\int_B w(x)|\Delta u|^2 dx \le C\|u\|^2.$$

Then (*iii*) follows.

(*iv*) Since $w(x) \ge 1$, then following embeddings are continuous:

$$\mathcal{X} \hookrightarrow W^{2,2}_{0,rad}(B,w) \hookrightarrow W^{2,2}_{0,rad}(B) \hookrightarrow L^q(B)\ \forall q \ge 2$$

and from (*i*), we have that $\mathcal{X} \hookrightarrow L^1(B)$ is continuous.

(*iv*) Since $W^{2,2}_0(B,w) \hookrightarrow W^{1,2}(B)$ is compact, then (*iv*) follows. This concludes the lemma. □

In the next, we give the following useful lemma.

Lemma 2 (Figueiredo et al. [4]). *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $f : \bar{\Omega} \times \mathbb{R}$ a continuous function. Let $\{u_n\}_n$ be a sequence in $L^1(\Omega)$ converging to u in $L^1(\Omega)$. Assume that $f(x, u_n)$ and $f(x, u)$ are also in $L^1(\Omega)$. If*

$$\int_\Omega |f(x, u_n)u_n| dx \leq C,$$

where C is a positive constant, then

$$f(x, u_n) \to f(x, u) \text{ in } L^1(\Omega).$$

Next, we prove a concentration compactness result of Lions type.

Theorem 4. *Let $(u_k)_k$ be a sequence in $\mathcal{X}$. Suppose that*
$\|u_k\| = 1$, $u_k \rightharpoonup u$ weakly in $\mathcal{X}$, $u_k(x) \to u(x)$ a.e $x \in B$, and $u \not\equiv 0$. Then

$$\sup_k \int_B e^{p\,\alpha_\beta |u_k|^\gamma} dx < +\infty, \quad \textit{where} \quad \alpha_\beta = 4[8\pi^2(1-\beta)]^{\frac{1}{1-\beta}},$$

for all $1 < p < U(u)$ where $U(u)$ is given by

$$U(u) := \begin{cases} \frac{1}{(1-\|u\|^2)^{\frac{\gamma}{2}}} & \textit{if} \quad \|u\| < 1 \\ +\infty & \textit{if} \quad \|u\| = 1 \cdot \end{cases}$$

Proof. For $a, b \in \mathbb{R}$, $q > 1$. If q' its conjugate, that is, $\frac{1}{q} + \frac{1}{q'} = 1$ we have, by Young inequality, that

$$e^{a+b} \leq \frac{1}{q} e^{qa} + \frac{1}{q'} e^{q'b}.$$

Also, we have

$$(1+a)^q \leq (1+\epsilon)a^q + \left(1 - \frac{1}{(1+\epsilon)^{\frac{1}{q-1}}}\right)^{1-q}, \ \forall a \geq 0, \ \forall \epsilon > 0 \ \forall q > 1. \tag{7}$$

So we get

$$\begin{aligned} |u_k|^\gamma &= |u_k - u + u|^\gamma \\ &\leq (|u_k - u| + |u|)^\gamma \\ &\leq (1+\epsilon)|u_k - u|^\gamma + \left(1 - \frac{1}{(1+\epsilon)^{\frac{1}{\gamma-1}}}\right)^{1-\gamma} |u|^\gamma, \end{aligned}$$

which implies that

$$\int_B e^{p\,\alpha_\beta |u_k|^\gamma} dx \leq \frac{1}{q} \int_B e^{pq\,\alpha_\beta (1+\epsilon)|u_k - u|^\gamma} dx + \frac{1}{q'} \int_B e^{pq'\,\alpha_\beta \left(1 - \frac{1}{(1+\epsilon)^{\frac{1}{\gamma-1}}}\right)^{1-\gamma} |u|^\gamma} dx,$$

for any $p > 1$. From Lemma 1 (*ii*), the last integral is finite.

To complete the evidence, we have to prove that for every p such that $1 < p < U(u)$,

$$\sup_k \int_B e^{pq\,\alpha_\beta (1+\epsilon)|u_k - u|^\gamma} dx < +\infty, \tag{8}$$

for some $\epsilon > 0$ and $q > 1$.

In the following, we suppose that $\|u\| < 1$ and in the case of $\|u\| = 1$, the proof is similar.
When

$$\|u\| < 1$$

and

$$p < \frac{1}{(1 - \|u\|^2)^{\frac{\gamma}{2}}},$$

there exists $\nu > 0$ such that

$$p(1 - \|u\|^2)^{\frac{\gamma}{2}}(1 + \nu) < 1.$$

On the other hand, we have

$$\|u_k - u\|^2 = \|u_k\|^2 - \|u\|^2 + o(1) \ \text{ where } \ o(1) \to 0 \ \text{ as } \ k \to +\infty. \tag{9}$$

Then,

$$\|u_k - u\|^2 = 1 - \|u\|^2 + o(1),$$

so

$$\lim_{k\to+\infty} \|u_k - u\|^\gamma = (1 - \|u\|^2)^{\frac{\gamma}{2}}.$$

Therefore, for every $\varepsilon > 0$, there exists $k_\varepsilon \geq 1$ such that

$$\|u_k - u\|^\gamma \leq (1 + \varepsilon)(1 - \|u\|^2)^{\frac{\gamma}{2}}, \ \forall\, k \geq k_\varepsilon.$$

If we take $q = 1 + \varepsilon$ with $\varepsilon = \sqrt[3]{1 + \nu} - 1$, then $\forall k \geq k_\varepsilon$, we have

$$pq(1 + \varepsilon)\|u_k - u\|^\gamma \leq 1.$$

Consequently,

$$\int_B e^{pq\,\alpha_\beta(1+\varepsilon)|u_k-u|^\gamma} dx \leq \int_B e^{(1+\varepsilon)pq\,\alpha_\beta\left(\frac{|u_k-u|}{\|u_k-u\|}\right)^\gamma \|u_k-u\|^\gamma} dx \leq \int_B e^{\alpha_\beta\left(\frac{|u_k-u|}{\|u_k-u\|}\right)^\gamma} dx \leq \sup_{\|u\|\leq 1} \int_B e^{\alpha_\beta|u|^\gamma} dx < +\infty.$$

Now, (8) follows from (3). This completes the proof of lemma 4. □

4 | THE MOUNTAIN PASS STRUCTURE OF THE ENERGY

According to (A_3), for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x,t)| \leq \varepsilon|t|, \ \forall\, 0 < |t| \leq \delta, \ \ \text{uniformly in } \ x \in B. \tag{10}$$

Moreover, since f is critical at infinity, for every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\forall t \geq C_\varepsilon \ |f(x,t)| \leq \varepsilon \exp(a|t|^\gamma) \ \text{ with } \ a > \alpha_0 \ \text{ uniformly in } \ x \in B. \tag{11}$$

In particular, we obtain for $q > 2$,

$$|f(x,t)t| \leq \frac{\varepsilon}{C_\varepsilon^{q-1}}|t|^{q-1}\exp(a|t|^\gamma) \ \text{ with } \ a > \alpha_0 \ \text{ uniformly in } \ x \in B. \tag{12}$$

Hence, using (10), (11), (12) and the continuity of f, for every $\varepsilon > 0$, for every $q > 2$, there exist positive constants C and c such that

$$|f(x,t)| \leq \varepsilon t + C|t|^{q-1}e^{a|t|^{\gamma}}, \ \forall (x,t) \in B \times \mathbb{R}. \tag{13}$$

It follows from (13), (5) and the continuity of F, that for all $\varepsilon > 0$, there exists $C > 0$ such that

$$F(x,t) \leq \varepsilon|t|^2 + C|t|^q e^{a\,t^{\gamma}}, \quad \text{for all } \ (x,t) \in B \times \mathbb{R}. \tag{14}$$

So, by (3) and (14), the functional $\mathcal{E}$ given by (6), is well defined. Moreover, by standard arguments, $\mathcal{E} \in C^1(\mathcal{X}, \mathbb{R})$.

In order to establish the existence of ground state solution to problem (1), we will prove the existence of a nonzero critical point of the functional $\mathcal{E}$ by using the theorem introduced by Ambrosetti and Rabinowitz [23] (Mountain Pass Theorem) without the Palais–Smale condition.

Theorem 5 (Ambrosetti and Rabinowitz [23]). *Let E be a Banach space and $J : E \to \mathbb{R}$ a C^1 functional satisfying $J(0) = 0$. Suppose that there exist $\rho, \ \bar{\beta}_0 > 0$ and $e \in E$ with $\|e\| > \rho$ such that*

$$\inf_{\|u\|=\rho} J(u) \geq \beta_0 \ \text{ and } \ J(e) \leq 0.$$

Then there is a sequence $(u_n) \subset E$ such

$$J(u_n) \to \bar{c} \ \text{ and } \ J'(u_n) \to 0,$$

where

$$\bar{c} := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)) \geq \bar{\beta}_0$$

and

$$\Gamma := \{\gamma \in C([0,1], E) \ \text{ such that } \ \gamma(0) = 0 \ \text{ and } \ \gamma(1) = e\}.$$

The number $\bar{c}$ is called mountain pass level or minimax level of the functional J.

In the next lemmas, we prove that the functional $\mathcal{E}$ has the mountain pass geometry of the theorem 5.

Lemma 3. *Suppose that f has critical growth at $+\infty$. In addition if (A_1) and (A_3) hold, then there exist $\rho, \beta_0 > 0$ such that $\mathcal{E}(u) \geq \beta_0$ for all $u \in \mathcal{X}$ with $\|u\| = \rho$.*

Proof. From (13), for all $\varepsilon > 0$, there exists $C > 0$ such that

$$F(x,t) \leq \varepsilon|t|^2 + C|t|^q e^{a\,t^{\gamma}}, \quad \text{for all } \ t \in \mathbb{R}.$$

Then, using the last inequality, we get

$$\mathcal{E}(u) \geq \frac{1}{2}\|u\|^2 - \varepsilon \int_B |u|^2 dx - C \int_B |u|^q e^{a\,u^{\gamma}}\, dx.$$

From the Hölder inequality, we obtain

$$\mathcal{E}(u) \geq \frac{1}{2}\|u\|^2 - \varepsilon \int_B |u|^2 dx - C\left(\int_B e^{2a|u|^{\gamma}} dx\right)^{\frac{1}{2}} \|u\|_{2q}^q. \tag{15}$$

From Theorem 2, if we choose $u \in \mathcal{X}$ such that

$$2a\|u\|^{\gamma} \leq \alpha_{\beta_0}, \tag{16}$$

we get

$$\int_B e^{2a|u|^\gamma}dx = \int_B e^{2a\|u\|^\gamma\left(\frac{|u|}{\|u\|}\right)^\gamma}\Bigg) dx < +\infty.$$

On the other hand, from Sobolev embedding Lemma 1, there exist constants $C_1 > 0$ and $C_2 > 0$ such that $\|u\|_{2q} \leq C_1\|u\|$ and $\|u\|_2^2 \leq C_1\|u\|^2$. So

$$\mathcal{E}(u) \geq \frac{1}{2}\|u\|^2 - \varepsilon C_1\|u\|^2 - C\|u\|^q = \|u\|^2\left(\frac{1}{2} - \varepsilon C_1 - C\|u\|^{q-2}\right),$$

for all $u \in \mathcal{X}$ satisfying (16). Since $q > 2$, we can choose $\rho = \|u\| \leq \left(\frac{\alpha_\beta}{2a}\right)^{\frac{1}{\gamma}}$ and for ε such that $\frac{1}{2C_1} > \varepsilon$, there exists $\beta_0 = \rho^2\left(\frac{1}{2} - \varepsilon C_1 - C\rho^{q-2}\right) > 0$ with $\mathcal{E}(u) \geq \beta_0 > 0$. □

By the following lemma, we prove the second geometric property for the functional $\mathcal{E}$.

Lemma 4. *Suppose that (A_1) and (A_2) hold. Then there exists $e \in \mathcal{X}$ with $\mathcal{E}(e) < 0$ and $\|e\| > \rho$.*

Proof. It follows from the condition (A_2) that

$$f(x,t) = \frac{\partial}{\partial t}F(x,t) \geq \frac{1}{M}F(x,t),$$

for all $t \geq t_0$. So

$$F(x,t) \geq Ce^{\frac{t}{M}},\ \forall\, t \geq t_0. \tag{17}$$

In particular, for $p > 2$, there exists $C_2 > 0$ such that

$$F(x,t) \geq C_3|t|^p - C_3,\ \forall t \in \mathbb{R},\ x \in B.$$

Next, one arbitrarily picks $\bar{u} \in \mathcal{E}$ such that $\|\bar{u}\| = 1$. Thus, from (17), for all $t \geq 1$,

$$\mathcal{E}(t\bar{u}) \leq \frac{1}{2}t^2\|\bar{u}\|^2 - C_3\|\bar{u}\|_p^p\, t^p - \pi^2 C_3.$$

Therefore,

$$\lim_{t\to+\infty} \mathcal{E}(t\bar{u}) = -\infty.$$

We take $e = \bar{t}\bar{u}$, for some $\bar{t} > 0$ large enough. So Lemma 4 follows. □

5 | THE MINIMAX ESTIMATE OF THE ENERGY

Since $\mathcal{E}$ has a mountain pass geometry, this allows us to consider the minimax level given by the mountain pass theorem of Ambrosetti and Rabinowitz. Thus, let

$$\mu := \inf_{\gamma\in\Lambda}\max_{t\in[0,1]} \mathcal{E}(\gamma(t)) > 0$$

where

$$\Lambda := \{\gamma \in C([0,1],\mathcal{X})\ \text{ such that }\ \gamma(0) = 0\ \text{ and }\ \mathcal{E}(\gamma(1)) < 0\}.$$

Our aim is to obtain a precise upper estimate for the mountain pass level. The idea is to construct a sequence of functions $(v_n) \in \mathcal{X}$, and estimate $\max\{\mathcal{E}(tv_n) : t \geq 0\}$. For this purpose, let consider the following Adam's function defined for all

$n \geq 3$ by

$$\psi_n(x) = \begin{cases} \left(4\frac{\log\left(e\sqrt[4]{n}\right)}{\alpha_\beta}\right)^{\frac{1}{\gamma}} - \frac{|x|^{2(1-\beta)}}{2\left(\frac{\alpha_\beta}{4n}\right)^{\frac{1}{\gamma}}\left(\log\left(e\sqrt[4]{n}\right)\right)^{\frac{\gamma-1}{\gamma}}} + \frac{1}{2\left(\frac{\alpha_\beta}{4}\right)^{\frac{1}{\gamma}}\left(\log\left(e\sqrt[4]{n}\right)\right)^{\frac{\gamma-1}{\gamma}}} & \text{if } 0 \leq |x| \leq \frac{1}{\sqrt[4]{n}} \\ \frac{\left(\log\left(\frac{e}{|x|}\right)\right)^{1-\beta}}{\left(\frac{\alpha_\beta}{4}\log\left(e\sqrt[4]{n}\right)\right)^{\frac{1}{\gamma}}} & \text{if } \frac{1}{\sqrt[4]{n}} \leq |x| \leq \frac{1}{2} \\ \zeta_n & \text{if } \frac{1}{2} \leq |x| \leq 1 \end{cases} \tag{18}$$

where $\zeta_n \in C^\infty_{0,rad}(B)$ is such that $\zeta_n\left(\frac{1}{2}\right) = \frac{1}{\left(\frac{\alpha_\beta}{16}\log(e^4 n)\right)^{\frac{1}{\gamma}}}(\log 2e)^{1-\beta}$, $\frac{\partial \zeta_n}{\partial r}\left(\frac{1}{2}\right) = \frac{-2(1-\beta)}{\left(\frac{\alpha_\beta}{4}\log\left(e\sqrt[4]{n}\right)\right)^{\frac{1}{\gamma}}}\left(\log(2e)\right)^{-\beta}$ $\zeta_n(1) = \frac{\partial \zeta_n}{\partial r}(1) = 0$ and ξ_n, $\nabla\xi_n$, $\Delta\xi_n$ are all $o\left(\frac{1}{\left[\log\left(e\sqrt[4]{n}\right)\right]^{\frac{1}{\gamma}}}\right)$. Here, $\frac{\partial \zeta_n}{\partial r}$ denotes the first derivative of ζ_n in the radial variable r.

Let $v_n(x) = \frac{\psi_n}{\|\psi_n\|}$. We have, $v_n \in \mathcal{X}$, $\|v_n\|^2 = 1$.

We compute $\Delta\psi_n(x)$, we get

$$\Delta\psi_n(x) = \begin{cases} \frac{-(1-\beta)(4-2\beta)|x|^{-2\beta}}{\left(\frac{\alpha_\beta}{4n}\right)^{\frac{1}{\gamma}}\left(\log\left(e\sqrt[4]{n}\right)\right)^{\frac{\gamma-1}{\gamma}}} & \text{if } 0 \leq |x| \leq \frac{1}{\sqrt[4]{n}} \\ \frac{-(1-\beta)}{|x|^2}\frac{\left(\log\left(\frac{e}{|x|}\right)\right)^{-\beta}\left(2+\beta\left(\log\frac{e}{|x|}\right)^{-1}\right)}{\left(\frac{\alpha_\beta}{4}\log\left(e\sqrt[4]{n}\right)\right)^{\frac{1}{\gamma}}} & \text{if } \frac{1}{\sqrt[4]{n}} \leq |x| \leq \frac{1}{2} \\ \Delta\zeta_n & \text{if } \frac{1}{2} \leq |x| \leq 1 \end{cases}$$

So

$$\|\Delta\psi_n\|^2_{2,w} = \underbrace{2\pi^2\int_0^{\frac{1}{\sqrt[4]{n}}} r^3|\Delta\psi_n(x)|^2\left(\log\frac{e}{r}\right)^\beta dr}_{I_1} + \underbrace{2\pi^2\int_{\frac{1}{\sqrt[4]{n}}}^{\frac{1}{2}} r^3|\Delta\psi_n(x)|^2\left(\log\frac{e}{r}\right)^\beta dr}_{I_2} + \underbrace{2\pi^2\int_{\frac{1}{2}}^{1} r^3|\Delta\psi_n(x)|^2\left(\log\frac{e}{r}\right)^\beta dr}_{I_3}$$

By using integration by parts, we obtain

$$\begin{aligned} I_1 &= 2\pi^2\frac{(1-\beta)^2(4-2\beta)^2}{\left(\frac{\alpha_\beta}{4n}\right)^{\frac{2}{\gamma}}\left(\log\left(e\sqrt[4]{n}\right)\right)^{\frac{2(\gamma-1)}{\gamma}}}\int_0^{\frac{1}{\sqrt[4]{n}}} r^{3-4\beta}\left(\log\frac{e}{r}\right)^\beta dr \\ &= 2\pi^2\frac{(1-\beta)^2(4-2\beta)^2}{\left(\frac{\alpha_\beta}{4n}\right)^{\frac{2}{\gamma}}\left(\log\left(e\sqrt[4]{n}\right)\right)^{\frac{2(\gamma-1)}{\gamma}}}\left[\frac{r^{4-4\beta}}{4-4\beta}\left(\log\frac{e}{r}\right)^\beta\right]_0^{\frac{1}{\sqrt[4]{n}}} \\ &\quad + 2\pi^2\frac{\beta(1-\beta)^2(4-2\beta)^2}{\left(\frac{\alpha_\beta}{4n}\right)^{\frac{2}{\gamma}}\left(\log\left(e\sqrt[4]{n}\right)\right)^{\frac{2(\gamma-1)}{\gamma}}}\int_0^{\frac{1}{\sqrt[4]{n}}} \frac{r^{4-4\beta}}{4-4\beta}\left(\log\frac{e}{r}\right)^{\beta-1} dr \\ &= o\left(\frac{1}{\log e\sqrt[4]{n}}\right). \end{aligned}$$

Also,

$$I_2 = 2\pi^2 \frac{(1-\beta)^2}{\left(\frac{\alpha_\beta}{4}\right)^{\frac{2}{\gamma}} \left(\log\left(e\sqrt[4]{n}\right)\right)^{\frac{2}{\gamma}}} \int_{\frac{1}{\sqrt[4]{n}}}^{\frac{1}{\frac{1}{2}}} \frac{1}{r}\left(\log\frac{e}{r}\right)^{-\beta}\left(2+\beta\left(\log\frac{e}{r}\right)^{-1}\right)^2 dr$$

$$= -2\pi^2 \frac{(1-\beta)^2}{\left(\frac{\alpha_\beta}{4}\right)^{\frac{2}{\gamma}} \left(\log\left(e\sqrt[4]{n}\right)\right)^{\frac{2}{\gamma}}} \left[\frac{\beta^2}{-1-\beta}\left(\log\frac{e}{r}\right)^{-\beta-1} + 4\left(\log\frac{e}{r}\right)^{-\beta} + \frac{4}{1-\beta}\left(\log\frac{e}{r}\right)^{1-\beta}\right]_{\frac{1}{\sqrt[4]{n}}}^{\frac{1}{2}}$$

$$= 1 + o\left(\frac{1}{\left(\log e\sqrt[4]{n}\right)^{\frac{2}{\gamma}}}\right).$$

and $I_3 = o\left(\frac{1}{\left(\log e\sqrt[4]{n}\right)^{\frac{2}{\gamma}}}\right)$. Then $\|\Delta\psi_n\|_{2,w}^2 = 1 + o\left(\frac{1}{\left(\log e\sqrt[4]{n}\right)^{\frac{2}{\gamma}}}\right)$.

5.1 | Estimate of the energy $\mathcal{E}$

We are now going to prove the desired estimate.

Lemma 5. *Assume that (A_1) and (A_2) holds, then*

$$\mu < \frac{1}{2}\left(\frac{\alpha_\beta}{\alpha_0}\right)^{\frac{2}{\gamma}}.$$

Proof. We have $v_n \geq 0$ and $\|v_n\| = 1$. Then from Lemma 4 $\mathcal{E}(tv_n) \to -\infty$ as $t \to +\infty$. As a consequence,

$$\mu \leq \max_{t\geq 0} \mathcal{E}(tv_n).$$

We argue by contradiction and suppose that for all $n \geq 1$,

$$\max_{t\geq 0} \mathcal{E}(tv_n) \geq \frac{1}{2}\left(\frac{\alpha_\beta}{\alpha_0}\right)^{\frac{2}{\gamma}}.$$

Since $\mathcal{E}$ possesses the mountain pass geometry, for any $n \geq 1$, there exists $t_n > 0$ such that

$$\max_{t\geq 0} \mathcal{E}(tv_n) = \mathcal{E}(t_n v_n) \geq \frac{1}{2}\left(\frac{\alpha_\beta}{\alpha_0}\right)^{\frac{2}{\gamma}}$$

Using the fact that $F(x,t) \geq 0$ for all $(x,t) \in B \times \mathbb{R}$ we get,

$$t_n^2 \geq \left(\frac{\alpha_\beta}{\alpha_0}\right)^{\frac{2}{\gamma}}. \tag{19}$$

On the other hand,

$$\frac{d}{dt}\mathcal{E}(tv_n)\Big|_{t=t_n} = t_n - \int_B f(x, t_n v_n) v_n dx = 0,$$

that is

$$t_n^2 = \int_B f(x, t_n v_n) t_n v_n dx. \tag{20}$$

Now, we claim that the sequence (t_n) is bounded in $(0, +\infty)$.

Indeed, it follows from (A_5) that for all $\varepsilon > 0$, there exists $t_\varepsilon > 0$ such that

$$f(x,t)t \geq (\gamma_0 - \varepsilon) e^{\alpha_0 t^\gamma} \; \forall |t| \geq t_\varepsilon, \quad \text{uniformly in } \; x \in B. \tag{21}$$

$$t_n^2 = \int_B f(x, t_n v_n) t_n v_n dx \geq \int_{0 \leq |x| \leq \frac{1}{\sqrt[4]{n}}} f(x, t_n v_n) t_n v_n dx \cdot$$

Since

$$\frac{t_n}{\|\psi_n\|} \left(\frac{\log e\sqrt[4]{n}}{\alpha_\beta} \right)^{\frac{1}{\gamma}} \to \infty \;\text{ as }\; n \to +\infty,$$

then it follows from (21) that for all $\varepsilon > 0$, there exists n_0 such that for all $n \geq n_0$

$$t_n^2 \geq (\gamma_0 - \varepsilon) \int_{0 \leq |x| \leq \frac{1}{\sqrt[4]{n}}} e^{\alpha_0 t_n^\gamma v_n^\gamma} dx$$

From (20) and (21), for n large enough, we get

$$t_n^2 \geq (\gamma_0 - \varepsilon) \int_{0 \leq |x| \leq \frac{1}{\sqrt[4]{n}}} e^{\alpha_0 t_n^\gamma v_n^\gamma} dx \geq 2\pi^2 (\gamma_0 - \varepsilon) \int_0^{\frac{1}{\sqrt[4]{n}}} r^3 e^{\alpha_0 t_n^\gamma \left(4 \frac{\log\left(e\sqrt[4]{n}\right)}{\|\psi_n\|^\gamma \alpha_\beta} \right)} dr \cdot = 2\pi^2 (\gamma_0 - \varepsilon) \, e^{\alpha_0 t_n^\gamma \left(\left(\frac{\log\left(e\sqrt[4]{n}\right)}{\|\psi_n\|^\gamma \alpha_\beta} \right) - \log 4n} . \tag{22}$$

There holds

$$1 \geq 2\pi^2 (\gamma_0 - \varepsilon) \, e^{\alpha_0 t_n^\gamma \left(\frac{\log\left(e\sqrt[4]{n}\right)}{\|\psi_n\|^\gamma \alpha_\beta} \right) - \log 4n - 2\log t_n} .$$

It follows that (t_n) is a bounded sequence. We must note that if

$$\lim_{n \to +\infty} t_n^\gamma > \left(\frac{\alpha_\beta}{\alpha_0} \right), \tag{23}$$

then we get a contradiction with the boundedness of (t_n). Indeed if (23) is accurate, then there exists some $\delta > 0$ such that for n large enough,

$$t_n^\gamma \geq \delta + \left(\frac{\alpha_\beta}{\alpha_0} \right) \cdot$$

Then the right hand of (23) tends to infinity which contradicts the boundedness of (t_n). Consequently (23) can not hold, and we get

$$\lim_{n \to +\infty} t_n^2 = \left(\frac{\alpha_\beta}{\alpha_0} \right)^{\frac{2}{\gamma}} \cdot \tag{24}$$

We claim that (24) leads to a contradiction with (A_5). Indeed, let us introduce the sets:

$$A_n = \{ x \in B | t_n v_n \geq t_\varepsilon \} \;\text{ and }\; C_n = B \setminus A_n,$$

where t_ε is given in (21). We have

$$
\begin{aligned}
t_n^2 &= \int\limits_B f(x, t_n v_n) t_n v_n dx = \int\limits_{\mathcal{A}_n} f(x, t_n v_n) t_n v_n dx + \int\limits_{C_n} f(x, t_n v_n) t_n v_n \\
&\geq (\gamma_0 - \varepsilon) \int\limits_{\mathcal{A}_n} e^{\alpha_0 t_n^\gamma v_n^\gamma} dx + \int\limits_{C_n} f(x, t_n v_n) t_n v_n dx \\
&= (\gamma_0 - \varepsilon) \int\limits_B e^{\alpha_0 t_n^\gamma v_n^\gamma} dx - (\gamma_0 - \varepsilon) \int\limits_{C_n} e^{\alpha_0 t_n^\gamma v_n^\gamma} dx \\
&\quad + \int\limits_{C_n} f(x, t_n v_n) t_n v_n dx.
\end{aligned}
$$

Since $v_n \to 0$ a.e in B, $\chi_{C_n} \to 1$ a.e in B, therefore, using the dominated convergence theorem, we get

$$
\int\limits_{C_n} f(x, t_n v_n) t_n v_n dx \to 0 \text{ and } \int\limits_{C_n} e^{\alpha_0 t_n^\gamma v_n^\gamma} dx \to \frac{\pi^2}{2}\cdot
$$

On the other hand,

$$
\int\limits_B e^{\alpha_0 t_n^\gamma v_n^\gamma} dx \geq \int\limits_{\frac{1}{\sqrt[4]{n}} \leq |x| \leq \frac{1}{2}} e^{\alpha_0 t_n^\gamma v_n^\gamma} dx + \int\limits_{C_n} e^{\alpha_0 t_n^\gamma v_n^\gamma} dx\cdot
$$

Then

$$
\lim_{n \to +\infty} t_n^2 = \left(\frac{\alpha_\beta}{\alpha_0}\right)^{\frac{2}{\gamma}} \geq (\gamma_0 - \varepsilon) \lim_{n \to +\infty} \int\limits_{\frac{1}{\sqrt[4]{n}} \leq |x| \leq \frac{1}{2}} e^{\alpha_0 t_n^\gamma v_n^\gamma} dx
$$

Using the fact that

$$
t_n^2 \geq \left(\frac{\alpha_\beta}{\alpha_0}\right)^{\frac{2}{\gamma}},
$$

we get

$$
\lim_{n \to +\infty} t_n^2 \geq \lim_{n \to +\infty} (\gamma_0 - \varepsilon) \int\limits_B e^{\alpha_0 t_n^\gamma v_n^\gamma} dx \geq \lim_{n \to +\infty} (\gamma_0 - \varepsilon) 2\pi^2 \int_{\frac{1}{\sqrt[4]{n}}}^{\frac{1}{2}} r^3 e^{\frac{4\left(\log \frac{e}{r}\right)^2}{\log\left(e\sqrt[4]{n}\right)\|\psi_n\|^\gamma}} dr.
$$

We make the change of variable

$$
s = \frac{4 \log \frac{e}{r}}{\log\left(e\sqrt[4]{n}\right) \|\psi_n\|^\gamma},
$$

to get

$$
\begin{aligned}
\lim_{n \to +\infty} t_n^2 &\geq \lim_{n \to +\infty} (\gamma_0 - \varepsilon) \int\limits_B e^{\alpha_0 t_n^\gamma v_n^\gamma} dx \\
&\geq \lim_{n \to +\infty} 2\pi^2 (\gamma_0 - \varepsilon) \frac{\|\psi_n\|^\gamma \log\left(e\sqrt[4]{n}\right)}{4} e^4 \int_{\frac{4 \log 2e}{\|\psi_n\|^\gamma \log\left(e\sqrt[4]{n}\right)}}^{\frac{4}{\|\psi_n\|^\gamma}} e^{\frac{\|\psi_n\|^\gamma \log\left(e\sqrt[4]{n}\right)}{4}\left(s^2 - 4s\right)} ds \\
&\geq \lim_{n \to +\infty} 2\pi^2 (\gamma_0 - \varepsilon) \frac{\|\psi_n\|^\gamma \log\left(e\sqrt[4]{n}\right)}{4} e^4 \int_{\frac{4 \log 2e}{\|\psi_n\|^\gamma \log\left(e\sqrt[4]{n}\right)}}^{\frac{4}{\|\psi_n\|^\gamma}} e^{-\frac{\|\psi_n\|^\gamma \log\left(e\sqrt[4]{n}\right)}{4} 4s} ds \\
&= \lim_{n \to +\infty} (\gamma_0 - \varepsilon) \frac{\pi^2}{2} e^4 \left(-e^{-4 \log e\sqrt[4]{n}} + e^{-4 \log(2e)}\right) \\
&= (\gamma_0 - \varepsilon) \frac{\pi^2 e^{4(1 - \log 2e)}}{2} = (\gamma_0 - \varepsilon) \frac{\pi^2}{32}\cdot
\end{aligned}
$$

This leads to

$$\left(\frac{\alpha_\beta}{\alpha_0}\right)^{\frac{2}{\gamma}} \geq (\gamma_0-\varepsilon)\frac{\pi^2}{32} \quad \text{for all} \quad \varepsilon>0.$$

Since ε is arbitrary, we obtain

$$\gamma_0 \leq 256\left(\frac{4}{\alpha_0}\right)^{1-\beta}(1-\beta)\cdot$$

This contradicts (A_5) and the lemma is proved. □

6 | PROOF OF MAIN RESULTS

Now, we consider the Nehari manifold associated to the functional $\mathcal{E}$, namely,

$$\mathcal{N}=\{u\in\mathcal{X} : \langle\mathcal{E}'(u),u\rangle=0, u\neq 0\},$$

and the number $c=\inf_{u\in\mathcal{N}}\mathcal{E}(u)$. We have the following key lemmas.

Lemma 6. *Assume that the condition (A_1) and (A_3) hold, then for each $x\in B$,*

$$t\mapsto tf(x,t)-2F(x,t) \quad \textit{is increasing for} \quad t>0.$$

In particular, $tf(x,t)-2F(x,t)\geq 0$ for all $(x,t)\in B\times[0,+\infty)$.

Proof. Assume that $0<t<s$. For each $x\in B$, we have

$$\begin{aligned} tf(x,t)-2F(x,t) &= \frac{f(x,t)}{t}t^2-2F(x,s)+2\int_t^s f(x,v)dv \\ &< \frac{f(x,t)}{s}t^2-2F(x,s)+\frac{f(x,s)}{s}(s^2-t^2) \\ &= sf(x,s)-2F(x,s)\cdot \end{aligned}$$

□

Lemma 7. *If (A_1) and (A_3) are satisfied then $\mu\leq c$.*

Proof. Let $\bar{u}\in\mathcal{N}$ and consider the function $\psi : (0,+\infty)\to\mathbb{R}$ defined by $\psi(t)=\mathcal{E}(t\bar{u})$.
ψ is differentiable and we have

$$\psi'(t)=\langle\mathcal{E}'(t\bar{u}),\bar{u}\rangle=t\|\bar{u}\|^2-\int_B f(x,t\bar{u})\bar{u}dx, \quad \text{for all} \quad t>0.$$

Since $\bar{u}\in\mathcal{N}$, we have $\langle\mathcal{E}'(\bar{u}),\bar{u}\rangle=0$ and therefore $\|\bar{u}\|^2=\int_B f(x,\bar{u})\bar{u}dx$. Hence,

$$\psi'(t)=t\int_B\left(\frac{f(x,\bar{u})}{\bar{u}}-\frac{f(x,t\bar{u})}{t\bar{u}}\right)\bar{u}\,dx\cdot$$

We have that $\psi'(1)=0$. We also have by (A_3) that $\psi'(t)>0$ for all $0<t<1$, $\psi'(t)\leq 0$ for all $t>1$. It follows that

$$\mathcal{E}(\bar{u})=\max_{t\geq 0}\mathcal{E}(t\bar{u}).$$

We define the function $\lambda : [0,1] \to \mathcal{X}$ such that $\lambda(t) = t\bar{t}\bar{u}$, with $\mathcal{E}(\bar{t}\bar{u}) < 0$. We have $\lambda \in \Lambda$, and hence,

$$\mu \leq \max_{t\in[0,1]} \mathcal{E}(\lambda(t)) \leq \max_{t\geq 0} \mathcal{E}(t\bar{u}) = \mathcal{E}(\bar{u}).$$

Since $\bar{u} \in \mathcal{N}$ is arbitrary then $\mu \leq c$.

Proof of Theorem 1.2.

Since $\mathcal{E}$ possesses the mountain pass geometry, there exists $u_n \in \mathcal{X}$ such that

$$\mathcal{E}(u_n) = \frac{1}{2}\|u_n\|^2 - \int_B F(x, u_n)dx \to \mu,\ n \to +\infty \tag{25}$$

and

$$|\langle \mathcal{E}'(u_n), \varphi\rangle| = \left| \int_B w(x)\Delta u_n.\Delta\varphi dx - \int_B f(x, u_n)\varphi dx \right| \leq \varepsilon_n\|\varphi\|, \tag{26}$$

for all $\varphi \in \mathcal{X}$, where $\varepsilon_n \to 0$, when $n \to +\infty$.

In order to obtain a ground state solution for problem (1), it is enough to show that there is $u \in \mathcal{N}$ such that $\mathcal{E}(u) = \mu\ (\mu \leq c \leq r)$.

From (25), for n large enough, there exists a constant $C > 0$ such that

$$\frac{1}{2}\|u_n\|^2 \leq C + \int_B F(x, u_n)dx\cdot$$

From (5), for all $\varepsilon > 0$, there exists $t_\varepsilon > 0$ such that

$$F(x,t) \leq \varepsilon t f(x,t), \quad \text{for all } |t| > t_\varepsilon \text{ and uniformly in } x \in B.$$

It follows that

$$\frac{1}{2}\|u_n\|^2 \leq C + \int_{|u_n|\leq t_\varepsilon} F(x, u_n)dx + \varepsilon \int_B f(x, u_n)u_n dx\cdot$$

From (26), we get

$$\frac{1}{2}\|u_n\|^2 \leq C_3 + \varepsilon\varepsilon_n\|u_n\| + \varepsilon\|u_n\|^2,$$

for some constant $C_3 > 0$.

We deduce that the sequence (u_n) is bounded in $\mathcal{X}$. As consequence, there exists $u \in \mathcal{X}$ such that up to subsequence, $u_n \rightharpoonup u$ weakly in $\mathcal{X}$, $u_n \to u$ strongly in $L^q(B)$, for all $q \geq 1$ and $u_n(x) \to u(x)$ $a.e$ in B.

Furthermore, we have from (25) and (26), that

$$0 < \int_B |f(x, u_n)u_n|dx \leq C, \tag{27}$$

and

$$0 < \int_B F(x, u_n) \leq C. \tag{28}$$

By Lemma 2, we have

$$f(x, u_n) \to f(x, u) \text{ in } L^1(B)\ as\ n \to +\infty. \tag{29}$$

It follows from (A_2) and the generalized Lebesgue dominated convergence theorem that

$$F(x,u_n) \to F(x,u) \;\text{ in }\; L^1(B)\; as\; n \to +\infty. \tag{30}$$

Also, by the definition of weak convergence, we get $\langle u_n, \varphi\rangle \to \langle u, \varphi\rangle$. Then, passing to the limit in (26) and using (29),we obtain that u is a weak solution of the problem (1), that is,

$$\int_B w(x)\Delta u\,\Delta\varphi\,dx = \int_B f(x,u)\varphi\,dx, \quad \text{for all} \;\; \varphi \in \mathcal{X}. \tag{31}$$

By (30) and (25), we obtain

$$\lim_{n\to+\infty} \|u_n\|^2 = 2\left(\mu + \int_B F(x,u)dx\right). \tag{32}$$

Next, we are going to make some claims.

Claim 1. $u \neq 0$.

Indeed, we argue by contradiction and suppose that $u \equiv 0$. Therefore, $\int_B F(x,u_n)dx \to 0$, and consequently, we get

$$\frac{1}{2}\|u_n\|^2 \to \mu < \frac{1}{2}\left(\frac{\alpha_\beta}{\alpha_0}\right)^{\frac{2}{\gamma}}.$$

So there exist $n_0 \in \mathbb{N}$ and $\eta \in (0,1)$ such that $\alpha_0\|u_n\|^\gamma = (1-\eta)\alpha_\beta$, for all $n \geq n_0$.

By (26), we also have

$$\left|\|u_n\|^2 - \int_B f(x,u_n)u_n dx\right| \leq C\varepsilon_n.$$

First, we claim that there exists $q > 1$ such that

$$\int_B |f(x,u_n)|^q dx \leq C. \tag{33}$$

In fact, since f has critical growth, for every $\varepsilon > 0$ and $q > 1$, there exists $t_\varepsilon > 0$ and $C > 0$ such that for all $|t| \geq t_\varepsilon$, we have

$$|f(x,t)|^q \leq Ce^{\alpha_0(\varepsilon+1)t^\gamma}. \tag{34}$$

Consequently,

$$\begin{aligned}\int_B |f(x,u_n)|^q dx &= \int_{\{|u_n|\leq t_\varepsilon\}} |f(x,u_n)|^q dx + \int_{\{|u_n|>t_\varepsilon\}} |f(x,u_n)|^q dx \\ &\leq 2\pi^2 \max_{B\times[-t_\varepsilon,t_\varepsilon]} |f(x,t)|^q + C\int_B e^{\alpha_0(\varepsilon+1)|u_n|^\gamma} dx.\end{aligned}$$

Since, there exist $n_0 \in \mathbb{N}$ and $\eta \in (0,1)$ such that $\alpha_0\|u_n\|^\gamma = (1-\eta)\alpha_\beta$, for all $n \geq n_0$, then

$$\alpha_0(1+\varepsilon)\left(\frac{|u_n|}{\|u_n\|}\right)^\gamma \|u_n\|^\gamma \leq (1+\varepsilon)(1-\eta)\alpha_\beta.$$

We choose $\varepsilon > 0$ small enough to get

$$\alpha_0(1+\varepsilon)\|u_n\|^\gamma \leq \alpha_\beta.$$

Therefore, the second integral is uniformly bounded in view of (3), and the claim is proved.

Now, using (33), we get

$$\|u_n\|^2 \le C\varepsilon_n + \left(\int_B |f(x,u_n)|^q\right)^{\frac{1}{q}} dx \left(\int_B |u_n|^{q'}\right)^{\frac{1}{q'}}$$

where q' is the conjugate of q. Since (u_n) converge to $u = 0$ in $L^{q'}(B)$

$$\lim_{n\to+\infty} \|u_n\|^2 = 0.$$

Therefore, $\mathcal{E}(u_n) \to 0$ which is in contradiction with $\mu > 0$.

Claim 2. $\|u\|^2 \ge \int_B f(x,u)u dx$.

We argue by contradiction and we suppose that $\|u\|^2 < \int_B f(x,u)udx$. Hence, $\langle \mathcal{E}'(u), u\rangle < 0$. The function $\psi : t \to \psi(t) = \langle \mathcal{E}'(tu), u\rangle$ is positive for t small enough. In fact, from (13), for every $\varepsilon > 0$, for every $q > 2$, there exist positive constants C and a such that

$$|f(x,t)| \le \varepsilon|t| + C|t|^{q-1}e^{a|t|^\gamma}, \quad \forall (x,t) \in B \times \mathbb{R}.$$

Then, using the Hölder inequality, we obtain

$$\psi(t) = |t|\|u\|^2 - \int_B f(x,tu)udx \ge |t|\|u\|^2 - \varepsilon|t|\int_B u^2dx - C|t|^{q-1}\left(\int_B e^{2a|t|^\gamma|u|^\gamma}dx\right)^{\frac{1}{2}}\left(\int_B |u|^{2q}dx\right)^{\frac{1}{2}}.$$

In view of (3), the integral $\int_B e^{2a|t|^\gamma|u|^\gamma}dx \le \int_B e^{2a|t|^\gamma \frac{u^\gamma}{\|u\|^\gamma}\|u\|^\gamma}\Big) dx \le C$, provided $|t| \le \frac{1}{\|u\|}\left(\frac{\alpha_\beta}{2a}\right)^{\frac{1}{\gamma}}$. Using the radial Lemma 1 we get $\|u\|_{2q}^2 \le C'\|u\|^q$. Then,

$$\psi(t) \ge |t|\|u\|^2 - C_1\varepsilon|t|\|u\|^2 - C_2|t|^{q-1}\|u\|^q = \|u\|^2|t|\left[(1-C_1\varepsilon) - C_2|t|^{q-2}\|u\|^{q-2}\right].$$

We choose $\varepsilon > 0$, such that $1 - C_1\varepsilon > 0$ and since $q > 2$, for small t, we get $\psi : t \to \psi(t) = \langle \mathcal{E}'(tu), u\rangle > 0$. So there exists $\eta \in (0,1)$ such that $\psi(\eta u) = 0$. Therefore $\eta u \in \mathcal{N}$. Using Lemma 6 and the semicontinuity of norm and Fatou's lemma, we get

$$\begin{aligned}
\mu \le c \le \mathcal{E}(\eta u) &= \mathcal{E}(\eta u) - \frac{1}{2}\langle \mathcal{E}'(\eta u), \eta u\rangle \\
&= \frac{1}{2}\int_B (f(x,\eta u)\eta u - 2F(x,\eta u))\, dx \\
&< \frac{1}{2}\int_B (f(x,u)u - 2F(x,u)) \\
&\le \liminf_{n\to+\infty}\left[\frac{1}{2}\|u_n\|^2 - \frac{1}{2}\|u_n\|^2\right] \\
&\quad + \liminf_{n\to+\infty}\left[\frac{1}{2}\int_B (f(x,u_n)u_n - 2F(x,u_n))\, dx\right] \\
&\le \lim_{n\to+\infty}\left[\mathcal{E}(u_n) - \frac{1}{2}\langle \mathcal{E}'(u_n), u_n\rangle\right] = \mu,
\end{aligned}$$

which is absurd and the claim is well established.

On the other hand, by claim 2, (A_3) and Lemma 6, we obtain

$$\mathcal{E}(u) \ge \frac{1}{2}\int_B [f(x,u) - 2F(x,u)]dx \ge 0. \tag{35}$$

Claim 3. $\mathcal{E}(u) = \mu$. Indeed, since (u_n) is bounded, up to a subsequence, $\|u_n\| \to \zeta > 0$. Now, using the semicontinuity of the norm and (25) we get,

$$\mathcal{E}(u) \leq \frac{1}{2}\liminf_{n\to\infty} \|u_n\|^2 - \int_B F(x,u)dx = \mu.$$

Suppose that

$$\mathcal{E}(u) < \mu, \tag{36}$$

then

$$\|u\|^2 < \zeta^2. \tag{37}$$

In addition,

$$\zeta^2 = \lim_{n\to+\infty} \|u_n\|^2 = 2\left(\mu + \int_B F(x,u)dx\right). \tag{38}$$

Set

$$v_n = \frac{u_n}{\|u_n\|}$$

and

$$v = \frac{u}{\zeta}.$$

We have $\|v_n\| = 1$, $v_n \rightharpoonup v$ in $\mathcal{X}$, $v \not\equiv 0$ and $\|v\| < 1$. So, by Theorem 4, we get

$$\sup_n \int_B e^{p\alpha_\beta |v_n|^\gamma} dx < \infty$$

provided $1 < p < (1 - \|v\|^2)^{-\frac{\gamma}{2}}$.

From (38), Lemma 6 and the following equality:

$$2\mu - 2\mathcal{E}(u) = \zeta^2 - \|u\|^2,$$

we get

$$\zeta^2 \leq 2\mu + \|u\|^2 < \left(\frac{\alpha_\beta}{\alpha_0}\right)^{\frac{2}{\gamma}} + \|u\|^2. \tag{39}$$

Since

$$\zeta^2 = \frac{\zeta^2 - \|u\|^2}{1 - \|v\|^2},$$

we deduce from (30) that

$$\zeta^2 < \frac{\left(\frac{\alpha_\beta}{\alpha_0}\right)^{\frac{2}{\gamma}}}{1 - \|v\|^2}.$$

Then there exists $\delta \in \left(0, \frac{1}{2}\right)$ such that $\zeta^\gamma = (1 - 2\delta)\frac{\frac{\alpha_\beta}{\alpha_0}}{(1-\|v\|^2)^{\frac{\gamma}{2}}}$.

On one hand, we have this estimate $\int_B |f(x,u_n)|^q dx < C$. Indeed, for $\varepsilon > 0$,

$$\begin{aligned}\int_B |f(x,u_n)|^q dx &= \int_{\{|u_n|\le t_\varepsilon\}} |f(x,u_n)|^q dx + \int_{\{|u_n|>t_\varepsilon\}} |f(x,u_n)|^q dx \\ &\le 2\pi^2 \max_{B\times[-t_\varepsilon,t_\varepsilon]} |f(x,t)|^q + C\int_B e^{\alpha_0(1+\varepsilon)|u_n|^\gamma} dx \\ &\le C_\varepsilon + C\int_B e^{\alpha_0(1+\varepsilon)\|u_n\|^\gamma |v_n|^\gamma}\Bigg) dx \le C,\end{aligned}$$

provided $\alpha_0(1+\varepsilon)\|u_n\|^\gamma \le p\alpha_\beta$, for p such that $1 < p < (1-\|v\|^2)^{-\frac{\gamma}{2}}$.

On the other hand, since

$$\lim_{n\to+\infty} \|u_n\|^\gamma = \zeta^\gamma,$$

then for n large enough, we get

$$\alpha_0(1+\varepsilon)\|u_n\|^\gamma \le \alpha_0(1+\varepsilon)\zeta^\gamma \le (1+\varepsilon)(1-\delta)\frac{\alpha_\beta}{(1-\|v\|^2)^{\frac{\gamma}{2}}}\cdot$$

We choose $\varepsilon > 0$ small enough such that $(1+\varepsilon)(1-\delta) < 1$, which means

$$\alpha_0(1+\varepsilon)\|u_n\|^\gamma < \frac{\alpha_\beta}{(1-\|v\|^2)^{\frac{\gamma}{2}}}\cdot$$

It follows that the sequence $(f(x,u_n))$ is bounded in $L^q(B)$, $q > 1$. Using the Hölder inequality and the Sobolev embedding theorem, we deduce that

$$\begin{aligned}\left|\int_B f(x,u_n)(u_n-u)dx\right| &\le \left(\int_B |f(x,u_n)|^q dx\right)^{\frac{1}{q}}\left(\int_B |u_n-u|^{q'}\right)^{\frac{1}{q'}} dx \\ &\le C\left(\int_B |u_n-u|^{q'}\right)^{\frac{1}{q'}} dx \to 0 \, as\, n\to+\infty,\end{aligned}$$

where $\frac{1}{q}+\frac{1}{q'} = 1$.

Since $\langle \mathcal{E}'(u_n), u_n-u\rangle = o_n(1)$, it follows that

$$\int_B w(x)\Delta u_n(\Delta u_n - \Delta u) \to 0.$$

On the other side,

$$\int_B w(x)\Delta u_n(\Delta u_n - \Delta u) = \|u_n\|^2 - \langle u_n, u\rangle\cdot$$

Passing to the limit in the last equality, we get

$$\zeta^2 - \|u\|^2 = 0.$$

Therefore $\|u\| = \zeta$ and $\|u_n\| \to \|u\|$. This is in contradiction with (36). It follows that $\mathcal{E}(u) = \mu$ and Claim 3 is proved.

Finally, from Claim 3 and (31), we deduce that u is a ground state solution to problem (1).

The proof is now complete. □

AUTHOR CONTRIBUTIONS

Sami Baraket: Conceptualization; methodology; visualization; writing—review and editing; supervision; funding acquisition; project administration; resources; writing—original draft. **Brahim Dridi**: Conceptualization; writing—original draft, writing—review and editing, methodology. **Rached Jaidane**: Conceptualization; methodology; validation; writing—original draft. **Vicenţiu D. Rădulescu**: Conceptualization; methodology; validation; supervision; funding acquisition; project administration; writing—review and editing.

CONFLICT OF INTEREST STATEMENT

This work does not have any conflicts of interest.

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