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Nodal solutions for the nonlinear Robin problem in Orlicz spaces

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ABSTRACT

In this paper we consider a non-linear Robin problem driven by the Orlicz g-Laplacian operator. Using variational technique combined with a suitable truncation and Morse theory (critical groups), we prove two multiplicity theorems with sign information for all the solutions. In the first theorem, we establish the existence of at least two non-trivial solutions with fixed sign. In the second, we prove the existence of at least three non-trivial solutions with sign information (one positive, one negative, and the other change sign) and order. The result of the nodal solution is new for the non-linear g-Laplacian problems with the Robin boundary condition.

1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 3)$ with C^2-boundary $\partial\Omega$. We consider the following Robin problem:

$$\begin{cases} -\mathrm{div}(a(|\nabla u(x)|)\nabla u(x)) + a(|u(x)|)u(x) = \lambda f(x, u(x)), & x \in \Omega \\ a(|\nabla u(x)|)\dfrac{\partial u(x)}{d\nu} + b(x)|u(x)|^{p-2}u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{P}$$

where ν is the unit exterior vector on $\partial\Omega$, $\lambda > 0$, $p > 0$, and $b \in C^{1,\theta}(\partial\Omega)$ for some $\theta \in (0,1)$, $\inf_{x\in\partial\Omega} b(x) > 0$ and the function $a(|t|)t$ is an increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$. In the right side of problem (P) there is a Carathéodory function $f : \Omega\times\mathbb{R} \longrightarrow \mathbb{R}$, that is $x \longmapsto f(x,t)$ is measurable for all $t \in \mathbb{R}$ and $t \longmapsto f(x,t)$ continuous for a.e. $x \in \Omega$.

The main feature of this paper is to deal with the existence of smooth nodal (i.e. sign-changing) solutions for the Robin problem of type (P) without assuming the well-known Ambrosetti–Rabinowitz ((AR) for short) or monotonicity condition on f. To the best of our knowledge, this is the first paper proving the existence of sign-changing solutions for problem (P). The tools used are a combination of cut-off techniques (truncation), together with variational methods based on the critical point and critical groups theories. More precisely we assume two classes of hypotheses on f. For the first class, we prove the existence of $\lambda_* > 0$ such that,

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for all $0 < \lambda \le \lambda_*$, problem (P) admits positive and negative solutions. For the second we show that, for all $\lambda > 0$, problem (P) has at least three non-trivial smooth solutions, all with sign information: positive, negative, and nodal solutions.

The study of variational problems in the classical Sobolev and Orlicz-Sobolev spaces is an interesting topic of research due to its significant role in many fields of mathematics, such as approximation theory, partial differential equations, calculus of variations, non-linear potential theory, the theory of quasi-conformal mappings, non-Newtonian fluids, differential geometry, geometric function theory, probability theory, and image processing (see [1–7]). In particular, when incorporating Robin boundary conditions in image processing tasks, the context is often related to how the boundary of the image domain interacts with the processing algorithm. Robin boundary conditions represent a mix of Dirichlet and Neumann conditions and can be particularly useful in modeling various physical phenomena or constraints at the boundaries of the image domain, see [8].

We point out that if we let $a(t) = 1$, problem (P) turns into the well-known Laplace equation. The existence of nodal solutions fro these kinds of classical problems have been studied by many authors because of their various applications to different disciplines see [9–15] and references therein. Moreover, there are a lot of papers dealing with the existence of smooth nodal solutions for problems driven by the p-Laplacian, $p(x)$-Laplacian or the Orlicz g-Laplacian operator, see [16–23] and references therein. In [16], using truncation techniques together with critical groups theory, Aizicovici et al. proved the existence of five non-trivial smooth solutions, two positive, two negative, and one nodal, for the following Neumann equation

$$\begin{cases} -\operatorname{div}(|\nabla u(x)|^{p-2}\nabla u(x)) + \beta|u(x)|^{p-2}u(x) = f(x,u(x)), & x \in \Omega \\ \dfrac{\partial u(x)}{d\nu} = 0, & x \in \partial\Omega, \quad 2 \le p < \infty, \end{cases}$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with a C^2 boundary $\partial\Omega$, $\beta > 0$, $f(x,t)$ is a Carathéodory function.

As a generalization of the p-Laplacian operator, Papageorgiou et al. [21], produced three non-trivial smooth solutions with sign information (one is a nodal solution) for the following non-linear non-homogeneous Robin equation

$$\begin{cases} -\operatorname{div}(a(|\nabla u(x)|)) + \xi(x)|u(x)|^{p-2}u(x) = f(x,u(x)), & x \in \Omega \\ a(|\nabla u|)\dfrac{\partial u(x)}{d\nu} + \beta(x)|u(x)|^{p-2}u(x) = 0, & x \in \partial\Omega, \end{cases}$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with a C^2 boundary $\partial\Omega$, $f(x,t)$ is a Carathéodory function, $\xi(.) \in L^\infty(\Omega)$, a is a continuous, strictly monotone map, $\beta(.) \in C^{1,\alpha}(\partial\Omega)$, for some $\alpha \in (0,1)$, $\beta \ge 0$ and ν is the outward unit normal on $\partial\Omega$.

For the $p(x)$-Laplacian operator, Gasiński and Papageorgiou [20] established the existence of at least three non-trivial smooth solutions: two with constant sign (one positive, the other negative) and the third with an unknown sign, for a non-linear Neumann problems driven by the $p(x)$-Laplacian operator

$$\begin{cases} -\operatorname{div}(|\nabla u(x)|^{p(x)-2}\nabla u(x)) = f(x,u(x)), & x \in \Omega \\ \dfrac{\partial u(x)}{d\nu} = 0, & x \in \partial\Omega, \end{cases}$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with a C^2 boundary $\partial\Omega$, $f(x,t)$ is a Carathéodory function and ν is the outward unit normal on $\partial\Omega$. In [23], using the arguments employed in [16], Papageorgiou and Winkert studied the existence of a nodal solution for the above problem. Precisely, the authors proved the existence of a sign-changing solution for an anisotropic Robin problem driven by the $p(x)$-Laplacian with a little modification in the assumptions on f. In 2013, Zhong and Fang (see [17]) treated the existence of smooth nodal solutions for the following Dirichlet problem driven by the Orlicz g-Laplacian operator

$$\begin{cases} -\operatorname{div}(a(|\nabla u(x)|)\nabla u(x)) = f(x,u(x)), & x \in \Omega \\ u(x) = 0, & x \in \partial\Omega, \end{cases}$$

where $\Omega \in \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $a(|t|)t \in C(\mathbb{R})$ and $f(x,t) \in C(\Omega \times \mathbb{R})$. In order to prove the existence of nodal solutions, the authors gave some new regularity results which are crucial in the application of the truncation and sub-supersolution methods.

In [18], by using the nodal Nehari manifold method, Figueiredo and Santos established the existence of a sign-changing solution for Kirchhoff equations driven by the Orlicz operator. Here, we would like to mention that the last cited paper about the Orlicz problem considered only the equations with Dirichlet boundary value condition.

To the best of our knowledge, there are no results concerning the existence of sign-changing solutions for the Orlicz equations with Robin boundary value condition. Hence, a natural question is whether or not there exist nodal solutions of problem (P).

The main aim of our work is the study of the existence of smooth nodal solutions for non-linear problems driven by the Orlicz g-Laplacian operator of type (P). Such problems, present challenging mathematical difficulties (non-homogeneity of the g-Laplacian operator). To overcome these difficulties, we produce new technical lemmas.

Specifically, we consider two classes of assumptions on the non-linear term f.

- **The first class:**

(f_1) $f(x,0) = 0$ and there exist an odd increasing homomorphism $h \in C^1(\mathbb{R},\mathbb{R})$, and a positive function $\hat{a}(t) \in L^\infty(\Omega)$ such that

$$|f(x,t)| \le \hat{a}(x)(1 + h(|t|)), \quad \forall\, t \in \mathbb{R}, \ \forall x \in \overline{\Omega}$$

and

$$G \prec\prec H \prec\prec G_*,$$

$$1 < g^+ < h^- := \inf_{t>0} \frac{h(t)t}{H(t)} \le h^+ := \sup_{t>0} \frac{h(t)t}{H(t)} \le \frac{g_*^-}{g^-},$$

$$1 < h^- - 1 := \inf_{t>0} \frac{h'(t)t}{h(t)} \le h^+ - 1 := \sup_{t>0} \frac{h'(t)t}{h(t)}$$

where

$$H(t) := \int_0^t h(s)\, ds$$

is an N-function.

(f_2) $\lim_{t\to\pm\infty} \frac{F(x,t)}{|t|^{g^+}} = +\infty$, uniformly in $x \in \Omega$, where $F(x,t) = \int_0^t f(x,s)ds$.

(f_3) $f(x,t) = o(|t|^{g^- -1})$ as $|t| \to 0$ uniformly in $x \in \Omega$.

(f_4) $\tilde{F}(x,t) = \frac{1}{g^+} f(x,t)t - F(x,t) > 0$, for $|t|$ large and there exist constants $\sigma > \frac{N}{g^-}$, $\tilde{c} > 0$ and $r_0 > 0$, such that

$$|f(x,t)|^\sigma \le \tilde{c}|t|^{(g^- -1)\sigma} \tilde{F}(x,t), \quad \forall\ (x,t) \in \Omega \times \mathbb{R}, \quad |t| \ge r_0.$$

• **The second class:** We suppose that f satisfies $(f_1)-(f_2)$ and the following conditions

(f_3') There exist an odd increasing homomorphism $q \in C^1(\mathbb{R},\mathbb{R})$, and a positive constants $c_0 \ge 0$, $\delta \ge 0$ such that

$$c_0 q(t)t \le f(x,t)t \le q^+ F(x,t), \text{ for almost all } x \in \Omega \text{ and for all } 0 < |t| \le \delta$$

and

$$Q \prec\prec G,$$

$$1 < q^- := \inf_{t>0} \frac{q(t)t}{Q(t)} \le q^+ := \sup_{t>0} \frac{q(t)t}{Q(t)} < p < g^-,$$

$$1 < q^- - 1 := \inf_{t>0} \frac{q'(t)t}{q(t)} \le q^+ - 1 := \sup_{t>0} \frac{q'(t)t}{q(t)}$$

where

$$Q(t) = \int_0^t q(s)ds$$

is an N-function.

(f_4') There exist $\eta_- < 0$ and $\eta_+ > 0$ such that

$$f(x,\eta_+) < 0 < f(x,\eta_-), \quad \text{for a.a. } x \in \Omega.$$

Remark 1.1. (i) In the first class of assumptions, condition (f_4) will be important in the proof of the boundedness of the Cerami sequence (see Proposition 3.2). To the best of our knowledge, a similar condition to (f_4) was firstly introduced in [24] for some scalar Schrödinger equation. Moreover, the assumption (f_4) weaker then the well-known Ambrosetti–Rabinowitz condition.

(ii) The following function satisfies $(f_1)-(f_4)$ and it does not satisfies the (AR) condition (see [25, p. 1277])

• $f(x,t) = |t|^{\beta-2}t\ln(1+|t|)$, $\beta \in 2\mathbb{N}$ and $2 < \beta < N \le \beta + 1$, while $a(t) = |t|^{\beta-2}$, $g(t) = |t|^{\beta-2}t$ and $g^- = g^+ = \beta$. (iii) For the second class of assumptions on f, the following function satisfies hypotheses $(f_1)-(f_2)$ and $(f_3')-(f_4')$, but fails to fulfill the (AR) condition.

$$\bullet f(x,t) = \begin{cases} |t|^{\alpha-2}t - 2|t|^{p-2}t & \text{if } |t| \le 1 \\ |t|^{g^+-2}t\ln(|t|) - |t|^{\alpha-2}t & \text{if } 1 < |t|, \end{cases}$$

where $0 < \alpha + 1 < p < g^-$ and $2 < g^- \le g^+ < N \le g^- + 1$.

(iv)The above assumptions related to the Robin boundary condition (on $\partial\Omega$, b, and p) are intricately connected to the regularity results obtained in [26]. These results will be instrumental in establishing the main results.

Next, we give the assumption on the Young function g. So, let

$$g(t) := \begin{cases} a(|t|)t, & \text{if } t \ne 0, \\ 0, & \text{if } t = 0, \end{cases}$$

be an odd increasing homeomorphism from $\mathbb{R}$ onto itself. Let

$$G(t) := \int_0^t g(s)ds \text{ and } \tilde{G}(t) := \int_0^t g^{-1}(s)ds.$$

In order to construct an Orlicz-Sobolev space setting for problem (P), we impose the following condition on G, a and g:

(g_1) : $a(t) \in C^1(0,+\infty)$, $a(t) > 0$ and $a(t)$ is an increasing function for $t > 0$.
(g_2) : $1 < p < g^- := \inf_{t>0} \frac{g(t)t}{G(t)} \le g^+ := \sup_{t>0} \frac{g(t)t}{G(t)} < N$.
(g_3) : $0 < g^- - 1 = a^- := \inf_{t>0} \frac{g'(t)t}{g(t)} \le g^+ - 1 = a^+ := \sup_{t>0} \frac{g'(t)t}{g(t)}$.
(g_4) : $t \mapsto G(\sqrt{t})$ is convex on $[0,+\infty)$, $\int_1^{+\infty} \frac{G^{-1}(t)}{t^{\frac{N+1}{N}}} dt = \infty$ and $\int_0^1 \frac{G^{-1}(t)}{t^{\frac{N+1}{N}}} dt < \infty$.

Remark 1.2. Here are some examples of N-functions:

- For the non-linear elasticity: $G(t) = (1 + t^2)^\alpha - 1$.
- For the plasticity: $G(t) = t^\alpha (\log(1 + t))^\beta$, $\alpha \ge 1$, $\beta > 0$.

Now, we can set our results. The aim of this paper is summarized in these theorems.

Theorem 1.3. *Assume that f, g and G satisfy $(f_1) - (f_4)$ and $(g_1) - (g_4)$. Then, there exists $\lambda_* > 0$ such that for all $0 < \lambda \le \lambda_*$, problem (P) admits a positive smooth solution $u_0 \in W^{1,G}(\Omega) \cap int(C^1(\overline{\Omega})_+)$ and a negative smooth solution $v_0 \in W^{1,G}(\Omega) \cap (-int(C^1(\overline{\Omega})_+))$ in the sense of Definition 2.13.*

Theorem 1.4. *Assume that f, g and G satisfy $(f_1) - (f_2)$, $(f_3') - (f_4')$ and $(g_1) - (g_4)$. Then, for all $\lambda > 0$, problem (P) admits a positive smooth solution $u_0 \in W^{1,G}(\Omega) \cap int(C^1(\overline{\Omega})_+)$ and a negative smooth solution $v_0 \in W^{1,G}(\Omega) \cap (-int(C^1(\overline{\Omega})_+))$ in the sense of Definition 2.13.*

Theorem 1.5. *Assume that f, g and G satisfy $(f_1) - (f_2)$, $(f_3') - (f_4')$ and $(g_1) - (g_4)$. Then, for all $\lambda > 0$, problem (P) admits a nodal solution.*

Our plan for the proof of the existence of the nodal solution is divided into four steps. In step one, we prove that the sets of positive and negative solutions are non-empty. In the next step, we show that the set of positive solutions has a smallest element u_* and the set of negative solutions has a greatest element v_*. In the third step, we prove the existence of another solution y_0 for the problem (P) lies between u_* and v_*. Evidently, $y_0 = 0$ or y_0 is a nodal solution for our problem. In the final step, we compute the critical groups at the origin and at y_0 to prove that y_0 cannot be zero.

The paper is organized as follows. In Section 2, we recall the basic properties of the Orlicz Sobolev spaces and the Orlicz Laplacian operator. Moreover, we mention some tools/definitions we need later (Cerami-condition, critical groups). In Section 3, for each class of assumptions on f, we prove the existence of at least positive and negative solutions (Theorems 1.3 and 1.4). Finally, we establish the existence of a nodal solution to our problem which lies between the extremal constant sign solutions (Theorem 1.5).

2. Preliminaries

In this section, we provide the mathematical background and framework for our problem (P).

2.1. Mathematical background: Orlicz and Orlicz-Sobolev spaces, critical groups

In this subsection, we recall some general properties about Orlicz spaces, Orlicz-Sobolev spaces, critical groups and some tools\definitions needed in the sequel (see [27–29]).

We start by recalling the definition of the well-known N-functions. Let g be a real-valued function defined on $\mathbb{R}$ and having the following properties:

(g_0) (1) $g(0) = 0$, $g(t) > 0$ if $t > 0$ and $\lim_{t\to+\infty} g(t) = +\infty$.
(2) g is non-decreasing and odd function.
(3) g is right continuous.

The real-valued function G defined on $\mathbb{R}$ by

$$G(t) = \int_0^t g(s)\, ds$$

is called an N-function. G is even, positive, continuous and convex function, Moreover $G(0) = 0$, $\frac{G(t)}{t} \to 0$ as $t \to 0$ and $\frac{G(t)}{t} \to +\infty$ as $t \to +\infty$.

The complementary N-function of G is defined by

$$\tilde{G}(t) = \int_0^t \tilde{g}(s)\, ds,$$

where $\tilde{g} : \mathbb{R} \to \mathbb{R}$ is given by $\tilde{g}(t) = \sup\{s : \ g(s) \le t\}$. If g is continuous on $\mathbb{R}$, then $\tilde{g}(t) = g^{-1}(t)$ for all $t \in \mathbb{R}$. Moreover, we have

$$st \le G(s) + \tilde{G}(t), \tag{2.1}$$

which is known as the Young inequality. Equality in (2.1) holds if and only if either $t = g(s)$ or $s = \tilde{g}(t)$.

We say that G satisfies the $\triangle_2$-condition, if there exists $C > 0$, such that

$$G(2t) \le CG(t), \text{ for all } t > 0. \tag{2.2}$$

An equivalent condition to (2.2) is: there exist g^- and g^+ such that

$$1 < g^- := \inf_{t>0} \frac{g(t)t}{G(t)} \le g^+ := \sup_{t>0} \frac{g(t)t}{G(t)} < +\infty. \tag{2.3}$$

If A and B are two N-functions, we say that A grow essentially more slowly than B ($A << B$ in symbols), if and only if for every positive constant k, we have

$$\lim_{t\to+\infty} \frac{A(kt)}{B(t)} = 0. \tag{2.4}$$

Another important function related to function G, is the Sobolev conjugate function G_* defined by

$$G_*^{-1}(t) = \int_0^t \frac{G^{-1}(s)}{s^{\frac{N+1}{N}}}\, ds, \ t > 0.$$

If G satisfies the $\triangle_2$-condition, then G_* satisfies the $\triangle_2$-condition. Namely, there exist $g_*^- = \frac{Ng^-}{N-g^-}$ and $g_*^+ = \frac{Ng^+}{N-g^+}$ such that

$$g^+ < g_*^- := \inf_{t>0} \frac{g_*(t)t}{G_*(t)} \le g_*^+ := \sup_{t>0} \frac{g_*(t)t}{G_*(t)} < +\infty. \tag{2.5}$$

Let G be an N-function satisfies the $\triangle_2$-condition. Then we can define the Orlicz space $L^G(\Omega)$ as the vectorial space of measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\rho(u) = \int_\Omega G(|u(x)|)\, dx < \infty.$$

$L^G(\Omega)$ is a Banach space under the Luxemburg norm

$$\|u\|_{(G)} = \inf \left\{ \lambda > 0 \ : \ \rho(\frac{u}{\lambda}) \le 1 \right\}.$$

For Orlicz spaces, the Hölder inequality reads as follows

$$\int_\Omega uv dx \le \|u\|_{(G)} \|v\|_{(\tilde{G})}, \ \text{ for all } u \in L^G(\Omega) \ \text{ and } u \in L^{\tilde{G}}(\Omega).$$

Next, we introduce the Orlicz-Sobolev space. We denote by $W^{1,G}(\Omega)$ the Orlicz-Sobolev space defined by

$$W^{1,G}(\Omega) := \left\{ u \in L^G(\Omega) : \ \frac{\partial u}{\partial x_i} \in L^G(\Omega), \ i = 1, \dots, N \right\}.$$

$W^{1,G}(\Omega)$ is a Banach space with respect to the norm

$$\|u\|_G = \|u\|_{(G)} + \|\nabla u\|_{(G)}.$$

Another equivalent norm is

$$\|u\| = \inf \left\{ \lambda > 0 \ : \ \mathcal{K}(\frac{u}{\lambda}) \le 1 \right\},$$

where

$$\mathcal{K}(u) = \int_\Omega G(|\nabla u(x)|) dx + \int_\Omega G(|u(x)|)\, dx. \tag{2.6}$$

In the sequel, we give a general results related to the N-function and the Orlicz, Orlicz-Sobolev spaces.

Lemma 2.1 (*see [30]*). *Let G be an N-function satisfying (2.3) such that $G(t) = \int_0^t g(s) ds$ and we denote by $\tilde{G}$ its complementary function. Then*

$$\tilde{G}(g(t)) \le (g^+ + 1)G(t) \ \text{ and } \ \tilde{G}\left(\frac{G(t)}{t}\right) \le G(t),$$

for all $t \ge 0$, where g^+ is defined in (2.3).

Lemma 2.2 (*see [27,29]*).. *Let G be an N-function and $\tilde{G}$ its complementary N-function. If G and $\tilde{G}$ satisfy (2.3), then, $L^G(\Omega)$, $W^{1,G}(\Omega)$ are separable and reflexive Banach spaces.*

Lemma 2.3 (*see [28]*). *Let G be an N-function satisfying (2.3) such that $G(t) = \int_0^t g(s) ds$. Then*

(1) if $0 < t < 1$, then $G(z)t^{g^+} \le G(tz) \le G(z)t^{g^-}$, for $z \in \mathbb{R}$;
(2) if $1 < t$, then $G(z)t^{g^-} \le G(tz) \le G(z)t^{g^+}$, for $z \in \mathbb{R}$;

(3) if $0 < t < 1$, then $g(z)t^{g^+-1} \leq g(tz) \leq g(z)t^{g^--1}$, for $z \in \mathbb{R}$;
(4) if $1 < t$, then $g(z)t^{g^--1} \leq g(tz) \leq g(z)t^{g^+-1}$, for $z \in \mathbb{R}$;
(5) if $\|u\|_{(G)} < 1$, then $\|u\|_{(G)}^{g^+} \leq \rho(u) \leq \|u\|_{(G)}^{g^-}$;
(6) if $\|u\|_{(G)} \geq 1$, then $\|u\|_{(G)}^{g^-} \leq \rho(u) \leq \|u\|_{(G)}^{g^+}$;
(7) if $\|u\| < 1$, then $\|u\|^{g^+} \leq \mathcal{K}(u) \leq \|u\|^{g^-}$;
(8) if $\|u\| \geq 1$, then $\|u\|^{g^-} \leq \mathcal{K}(u) \leq \|u\|^{g^+}$.

Theorem 2.4 (*see [27,29]*).. *Let G and H be N-functions, such that H grow essentially more slowly than G_* (where G_* is the Sobolev conjugate function of G).*

(1) If $\int_1^{+\infty} \frac{G^{-1}(t)}{t^{\frac{N+1}{N}}}dt = \infty$ and $\int_0^1 \frac{G^{-1}(t)}{t^{\frac{N+1}{N}}}dt < \infty$, then the embedding $W^{1,G}(\Omega) \hookrightarrow L^H(\Omega)$ is compact and the embedding $W^{1,G}(\Omega) \hookrightarrow L^{G_*}(\Omega)$ is continuous.
(2) If $\int_1^{+\infty} \frac{G^{-1}(t)}{t^{\frac{N+1}{N}}dt} < \infty$, then the embedding $W^{1,G}(\Omega) \hookrightarrow L^H(\Omega)$ is compact and the embedding $W^{1,G}(\Omega) \hookrightarrow L^\infty(\Omega)$ is continuous.

Theorem 2.5 (*see [26, Theorem 2.7, p. 5]*). *Let G be an N-function satisfies (2.3). Then, the Orlicz-Sobolev space $W^{1,G}(\Omega)$ is continuously and compactly embedded in the classical Lebesgue spaces $L^r(\Omega)$ and $L^r(\partial\Omega)$ for all $1 \leq r < g_*^-$, where g_*^- is defined in (2.5).*

Another mathematical tool that we will use in the sequel is the Morse theory and in particular critical groups. So, let us recall some basic definitions from the theory.

Let $\mathbb{X}$ be a Banach space and $Y_2 \subseteq Y_1 \subseteq \mathbb{X}$. For every integer $k \geq 0$ we denote by $H_k(Y_1, Y_2)$ the kth relative singular homology group for the pair (Y_1, Y_2) with integer coefficient. We recall that $H_k(Y_1, Y_2) = 0$ for all integer $k < 0$.

Given $J \in C^1(\mathbb{X})$ and $c \in \mathbb{R}$, we introduce the following sets:

$$J^c = \{x \in \mathbb{X},\ J(x) \leq c\}, \quad \text{and} \quad K_J = \left\{x \in \mathbb{X},\ J'(x) = 0\right\}.$$

The critical groups of J at an isolated critical point $x_0 \in \mathbb{X}$ with $c = J(x_0)$ are defined by

$$C_k(J, x_0) = H_k(J^c \cap U, (J^c \cap U)\backslash\{x_0\}) \ \text{ for all } k \geq 0,$$

where U is a neighborhood of x_0 such that $K_J \cap J^c \cap U = \{x_0\}$. The excision property of singular homology theory implies that the above definition of critical groups is independent of the choice of the neighborhood U.

Given $u \in W^{1,G}(\Omega)$, we set $u^\pm = \max\{\pm u, 0\}$ being the positive and negative part of u, respectively. We know that $u = u^+ - u^-$, $|u| = u^+ + u^-$ and $u^\pm \in W^{1,G}(\Omega)$. If $u, v : \Omega \to \mathbb{R}$ are measurable functions and $u(x) \leq v(x)$ for a. a. $x \in \Omega$, then we introduce the following order interval in $W^{1,G}(\Omega)$

$$[u, v] = \left\{y \in W^{1,G}(\Omega) : u(x) \leq y(x) \leq v(x) \ \text{ for a.a. } x \in \Omega\right\}.$$

Moreover, we need the Banach space $C^1(\overline{\Omega})$. This is an ordered Banach space with positive order cone

$$C^1(\overline{\Omega})_+ = \left\{u \in C^1(\overline{\Omega}),\ u(x) \geq 0 \text{ for all } x \in \overline{\Omega}\right\}.$$

This cone has a nonempty interior given by

$$\mathrm{int}(C^1(\overline{\Omega})_+) = \left\{u \in C^1(\overline{\Omega})_+,\ u(x) > 0 \text{ for all } x \in \overline{\Omega}\right\}.$$

Definition 2.6. Let $J \in C^1(W^{1,G}(\Omega))$. We say that J satisfies the "Cerami condition", C-condition for short, if every sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,G}(\Omega)$ such that $\{J(u_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded and

$$(1 + |u_n|)J'(u_n) \longrightarrow 0 \ \text{ in } (W^{1,G}(\Omega))^* \text{ as } \ n \to +\infty,$$

admits a strongly convergent subsequence. Where $(W^{1,G}(\Omega))^*$ is the topological dual of $W^{1,G}(\Omega)$.

Definition 2.7 (*see [16]*).

(1) A nonempty set S is said "downward directed", if $u_1, u_2 \in S$, then we can find $u \in S$ such that $u \leq u_1$ and $u \leq u_2$.
(2) A nonempty set S is said "upward directed", if $u_1, u_2 \in S$, then we can find $u \in S$ such that $u_1 \leq u$ and $u_2 \leq u$.

2.2. Further properties of orlicz-Sobolev spaces and the non-linear term f

In this subsection, we assume conditions $(g_1) - (g_4)$ and explore new properties of the Orlicz-Sobolev spaces. We also establish the variational framework for problem (P).

Evidently, G and $\tilde{G}$ are complementary N-functions. Under the condition (g_2), the function $G(t)$ satisfies the $\triangle_2$-condition. We assume also that the complementary N-function $\tilde{G}$ satisfies the $\triangle_2$-condition.

In the following, we give the proofs of some lemmas and theorem which will be used in the proofs of our results.

Lemma 2.8. *Suppose that* $(g_1)-(g_4)$ *are satisfied. Then, for all* $a, b \in \mathbb{R}$, *we have*

$$g(a+b)b - g(a)b \geq 0 \quad \text{and} \quad (g(a)-g(b))(b-a) \leq 0.$$

Proof. The right inequality is simple. Indeed, since g is increasing on $\mathbb{R}$, then

$$sgn\,(g(a)-g(b)) = -sgn(b-a), \quad \text{for all } a, b \in \mathbb{R},$$

which gives us

$$(g(a)-g(b))(b-a) \leq 0, \quad \text{for all } a, b \in \mathbb{R}.$$

For the left inequality, we make four cases (in all the cases have in mind the fact that g is increasing on $\mathbb{R}$). Case 1: If $a \geq 0$ and $b \geq 0$, then $a+b \geq a$ and

$$\underbrace{(g(a+b)-g(a))}_{\geq 0}\underbrace{\quad b \quad}_{\geq 0} \geq 0.$$

Case 2: If $a \leq 0$ and $b \leq 0$, then $a+b \leq a$ and

$$\underbrace{(g(a+b)-g(a))}_{\leq 0}\underbrace{\quad b \quad}_{\leq 0} \geq 0.$$

Case 3: If $a \geq 0$ and $b \leq 0$, then $a+b \leq a$ and

$$\underbrace{(g(a+b)-g(a))}_{\leq 0}\underbrace{\quad b \quad}_{\leq 0} \geq 0.$$

Case 4: If $a \leq 0$ and $b \geq 0$, then $a+b \geq a$ and

$$\underbrace{(g(a+b)-g(a))}_{\geq 0}\underbrace{\quad b \quad}_{\geq 0} \geq 0.$$

From Case 1–Case 4, we get

$$g(a+b)b - g(a)b \geq 0, \quad \text{for all} \quad a, b \in \mathbb{R}.$$

This ends the proof. □

Lemma 2.9 ([26, Lemma 2.9]). *Let* G *be an* N*-function satisfying* $(g_1)-(g_3)$ *such that* $G(t) = \int_0^t g(s)\, ds = \int_0^t a(|s|)s\, ds$. *Then for every* $\xi, \eta \in \mathbb{R}^N$, *we have*

$$\langle a(|\eta|)\eta - a(|\xi|)\xi, \eta - \xi\rangle_{\mathbb{R}^N} \geq 0$$

where $\langle . \rangle_{\mathbb{R}^N}$ *is the inner product on* $\mathbb{R}^N$.

Lemma 2.10 ([31, Lemma 2.1]). *Let* G *be an* N*-function satisfying* $(g_1)-(g_4)$. *Then for every* $\eta, \xi \in \mathbb{R}^N$,

$$\frac{G(|\eta|)+G(|\xi|)}{2} \geq G\left(|\frac{\eta+\xi}{2}|\right) + G\left(|\frac{\eta-\xi}{2}|\right).$$

Lemma 2.11 ([32, Lemma 3.4]). *Let* G *be an* N*-function satisfying* $(g_1)-(g_4)$ *such that* $G(t) = \int_0^t g(s)ds = \int_0^t a(|s|)sds$. *Then for every* $\eta, \xi \in \mathbb{R}^N \setminus \{0\}$, *we have*

$$(a(|\eta|)\eta - a(|\xi|)\xi)\cdot(\eta-\xi) \geq 4G\left(\frac{|\eta-\xi|}{2}\right).$$

Proof. Let $\eta, \xi \in \mathbb{R}^N \setminus \{0\}$. Since G is convex, we have

$$G(|\eta|) \leq G\left(|\frac{\eta+\xi}{2}|\right) + g(|\eta|)\frac{\eta}{|\eta|}\cdot\frac{\eta-\xi}{2}$$

and

$$G(|\xi|) \leq G\left(|\frac{\eta+\xi}{2}|\right) + g(|\xi|)\frac{\xi}{|\xi|}\cdot\frac{\xi-\eta}{2}.$$

Adding the above two relations, we find that

$$\frac{1}{2}\left(g(|\eta|)\frac{\eta}{|\eta|} - g(|\xi|)\frac{\xi}{|\xi|}\right)\cdot(\eta-\xi) \geq G(|\eta|)+G(|\xi|) - 2G\left(|\frac{\eta+\xi}{2}|\right) \tag{2.7}$$

for all $\eta, \xi \in \mathbb{R}^N \setminus \{0\}$.

On the other hand, we deduce by Lemma 2.10, that

$$G(|\eta|)+G(|\xi|)\geq 2G\left(|\frac{\eta+\xi}{2}|\right)+2G\left(|\frac{\eta-\xi}{2}|\right) \tag{2.8}$$

for all $\eta,\xi\in\mathbb{R}^N$.

From (2.7) and (2.8), we get

$$\left(g(|\eta|)\frac{\eta}{|\eta|}-g(|\xi|)\frac{\xi}{|\xi|}\right)\cdot(\eta-\xi)\geq 4G\left(\frac{|\eta-\xi|}{2}\right) \tag{2.9}$$

for all $\eta,\xi\in\mathbb{R}^N\setminus\{0\}$.

Using (2.9) and the fact that $g(t)=a(|t|)t$, for all $t\in\mathbb{R}\setminus\{0\}$, we deduce our desired result

$$(a(|\eta|)\eta-a(|\xi|)\xi)\cdot(\eta-\xi)\geq 4G\left(\frac{|\eta-\xi|}{2}\right)$$

for all $\eta,\xi\in\mathbb{R}^N\setminus\{0\}$. □

In what follows, we give some definitions and lemmas related to the variational setting of problem (P).

Lemma 2.12 (*see [17,28]*). *Under the assumptions* $(g_1)-(g_2)$. *If* $u_n\rightharpoonup u$ *in* $W^{1,G}(\Omega)$ *and*

$$\limsup_{n\to+\infty}\int_\Omega a(|\nabla u_n|)\nabla u_n.\nabla(u_n-u)+a(|u_n|)u_n(u_n-u)dx\leq 0,$$

then

$$u_n\to u\ \text{ in }\ W^{1,G}(\Omega).$$

Definition 2.13. We say that $u\in W^{1,G}(\Omega)$ is a weak solution of problem (P) if

$$\int_\Omega [a(|\nabla u|)\nabla u.\nabla v+a(|u|)uv]\,dx+\int_{\partial\Omega} b(x)|u|^{p-2}uv d\gamma=\lambda\int_\Omega f(x,u)v dx,\ \forall v\in W^{1,G}(\Omega) \tag{2.10}$$

where $d\gamma$ is the measure on the boundary $\partial\Omega$.

The energy functional corresponding to problem (P) is defined as $J: W^{1,G}(\Omega)\to\mathbb{R}$

$$J(u)=\mathcal{K}(u)+\frac{1}{p}\int_{\partial\Omega} b(x)|u|^p d\gamma-\lambda\int_\Omega F(x,u)dx, \tag{2.11}$$

where $\mathcal{K}$ is defined in (2.6).

Definition 2.14.

(1) A function $u\in W^{1,G}(\Omega)$ is an "upper solution" for problem (P) if

$$\int_\Omega a(|\nabla u|)\nabla u.\nabla v dx+\int_\Omega a(|u|)uv dx+\int_{\partial\Omega} b(x)|u|^{p-2}uv d\gamma\geq\lambda\int_\Omega f(x,u)v dx, \tag{2.12}$$

for all $v\in W^{1,G}(\Omega)_+$.

(2) A function $u\in W^{1,G}(\Omega)$ is a " lower solution" for problem (P) if

$$\int_\Omega a(|\nabla u|)\nabla u.\nabla v dx+\int_\Omega a(|u|)uv dx+\int_{\partial\Omega} b(x)|u|^{p-2}uv d\gamma\leq\lambda\int_\Omega f(x,u)v dx, \tag{2.13}$$

for all $v\in W^{1,G}(\Omega)_+$.

Definition 2.15.

(1) We say that $u_0\in W^{1,G}(\Omega)$ is a local $C^1(\overline{\Omega})$-minimizer of J, if we can find $r_0>0$ such that

$$J(u_0)\leq J(u_0+v),\ \text{for all } v\in C^1(\overline{\Omega}) \text{ with } \|v\|_{C^1(\overline{\Omega})}\leq r_0.$$

(2) We say that $u_0\in W^{1,G}(\Omega)$ is a local $W^{1,G}(\Omega)$-minimizer of J, if we can find $r_1>0$ such that

$$J(u_0)\leq J(u_0+v),\ \text{for all } v\in W^{1,G}(\Omega) \text{ with } \|v\|\leq r_1.$$

Now, we give some technical Lemmas related to the assumptions on the non-linear term f.

Lemma 2.16. *Suppose that* f *satisfies* $(f_2)-(f_4)$. *Then, for any* $\varepsilon>0$, *there exist* $C_\varepsilon>0$ *and* $r\in(g^-,g_*^-)$ *such that*

$$|f(x,t)|\leq\varepsilon|t|^{g^--1}+C_\varepsilon|t|^{r-1},\quad |F(x,t)|\leq\varepsilon\frac{1}{g^-}|t|^{g^-}+\frac{1}{r}C_\varepsilon|t|^r, \tag{2.14}$$

for each $x\in\Omega$ *and* $t\in\mathbb{R}$.

Proof. For $\varepsilon > 0$, we can use (f_3) to obtain $\delta > 0$, such that

$$f(x,t) \le \varepsilon |t|^{g^- - 1}, \quad \forall x \in \Omega, \ |t| \le \delta. \tag{2.15}$$

By (f_2), there exists $r_1 > 0$ such that

$$F(x,t) \ge |t|^{g^+}, \quad \text{for all } x \in \Omega \text{ and } |t| \ge r_1,$$

which gives us

$$\tilde{F}(x,t) = \frac{1}{g^+} f(x,t)t - F(x,t) \le \frac{1}{g^+} f(x,t)t, \quad \text{for all } x \in \Omega \text{ and } |t| \ge r_1. \tag{2.16}$$

From (2.16) and (f_4), we get

$$|f(x,t)|^\sigma \le \tilde{c}|t|^{(g^- -1)\sigma} \tilde{F}(x,t) \le \frac{\tilde{c}}{g^+}|t|^{(g^- -1)\sigma + 1}|f(x,t)|,$$

for all $x \in \Omega$ and $|t| \ge r_2 = \max\{r_0, r_1\}$, which is equivalent to

$$|f(x,t)|^{\sigma - 1} \le \frac{\tilde{c}}{g^+}|t|^{(g^- -1)\sigma + 1} \tag{2.17}$$

for all $x \in \Omega$ and $|t| \ge r_2$. It follows, by (2.17), that

$$|f(x,t)| \le \left(\frac{\tilde{c}}{g^+}\right)^{\frac{1}{\sigma-1}} |t|^{\frac{(g^- -1)\sigma+1}{\sigma-1}} = \left(\frac{\tilde{c}}{g^+}\right)^{\frac{1}{\sigma-1}} |t|^{r-1}, \tag{2.18}$$

for all $x \in \Omega$ and $|t| \ge r_2$, where $g^- < r = \frac{g^- \sigma}{\sigma - 1} < g^-_*$.

Next, since we have $\frac{f(x,t)}{|t|^{r-1}}$ is continuous, then we can find $C_\varepsilon > 0$, such that

$$|f(x,t)| \le C_\varepsilon |t|^{r-1}, \tag{2.19}$$

for all $x \in \Omega$ and $\delta \le |t| \le r_2$.

Putting together (2.15), (2.18) and (2.19), we get

$$|f(x,t)| \le \varepsilon |t|^{g^- - 1} + C_\varepsilon |t|^{r-1}, \quad \text{for all } x \in \Omega \text{ and } t \in \mathbb{R}.$$

Then

$$|F(x,t)| \le \varepsilon \frac{1}{g^-}|t|^{g^-} + C_\varepsilon \frac{1}{r}|t|^r, \quad \text{for all } x \in \Omega \text{ and } t \in \mathbb{R}.$$

Thus the proof. □

Lemma 2.17. *Under the assumptions (f_1), (f_3') and (g_3), we have*

(1)

$$t \longmapsto \frac{g(t)}{t^{q^+ - 1}} \ \textit{is increasing on} \ (0, +\infty), \tag{2.20}$$

(2)

$$t \longmapsto \frac{q(t)}{t^{q^+ - 1}} \ \textit{is non-increasing on} \ (0, +\infty), \tag{2.21}$$

(3)

$$t \longmapsto \frac{h(t)}{t^{q^+ - 1}} \ \textit{is increasing on} \ (0, +\infty), \tag{2.22}$$

(4)

$$t \longmapsto G(t^{\frac{1}{q^+}}) \ \textit{is convex on} \ (0, +\infty). \tag{2.23}$$

Proof. (1) The proof of (1) follows directly from the following fact

$$\begin{aligned} g'(t)t^{q^+ - 1} - (q^+ - 1)t^{q^+ - 2}g(t) &\ge (g^- - 1)g(t)t^{q^+ - 2} - (q^+ - 1)t^{q^+ - 2}g(t) \\ &= [(g^- - 1) - (q^+ - 1)]g(t)t^{q^+ - 2} > 0, \end{aligned}$$

for all $t > 0$. In the above inequality, we used assumption (g_3).

(2) From (f_3'), we find

$$q'(t)t^{q^+ - 1} - (q^+ - 1)t^{q^+ - 2}q(t) \le (q^+ - 1)q(t)t^{q^+ - 2} - (q^+ - 1)t^{q^+ - 2}q(t) = 0, \quad \text{for all } t > 0.$$

This proves (2).

(3) By (f_1), we have

$$h'(t)t^{q^+-1} - (q^+-1)t^{q^+-2}h(t) \geq (h^- - 1)h(t)t^{q^+-2} - (q^+-1)t^{q^+-2}h(t) = [(h^- - 1) - (q^+-1)]h(t)t^{q^+-2} > 0,$$

for all $t > 0$. Which conclude the proof of (3).

(4) Using (f_3') and (g_3), we obtain that

$$\begin{aligned}\frac{1}{(q^+)^2}g'(t^{\frac{1}{q^+}})t^{\frac{2-2q^+}{q^+}} + \frac{1-q^+}{(q^+)^2}g(t^{\frac{1}{q^+}})t^{\frac{1-2q^+}{q^+}} &= \frac{1}{(q^+)^2}g'(t^{\frac{1}{q^+}})t^{\frac{2-2q^+}{q^+}} - \frac{q^+-1}{(q^+)^2}g(t^{\frac{1}{q^+}})t^{\frac{1-2q^+}{q^+}}\\ &\geq \frac{g^- - 1}{(q^+)^2}g(t^{\frac{1}{q^+}})t^{\frac{2-2q^+}{q^+}-\frac{1}{q^+}} - \frac{q^+-1}{(q^+)^2}g(t^{\frac{1}{q^+}})t^{\frac{1-2q^+}{q^+}}\\ &= g(t^{\frac{1}{q^+}})t^{\frac{1-2q^+}{q^+}}\left[\frac{g^- - 1}{(q^+)^2} - \frac{q^+-1}{(q^+)^2}\right] > 0 \ \ (\text{since } q^+ < g^-).\end{aligned}$$

Thus ends the proof. □

3. Fixed sign solutions

In this Section, under each class of assumptions on f, we prove the existence of at least two weak solutions with constant sign (fixed sign) to the problem (P). Namely, we give the proofs of Theorems 1.3 and 1.4.

Before that we give the following result.

Proposition 3.1. *Assume that the assumptions (f_1) and $(g_1) - (g_2)$ hold. Let $u \in W^{1,G}(\Omega) \cap (C^1(\overline{\Omega})_+)$ be a non-negative ($v \in W^{1,G}(\Omega) \cap (-C^1(\overline{\Omega})_+)$ be a non-positive) weak solution for the problem (P). Then $u \in int(C^1(\overline{\Omega})_+)$ is a positive ($v \in -int(C^1(\overline{\Omega})_+)$ is a negative) weak solution for the problem (P).*

Proof. Let $u \in W^{1,G}(\Omega) \cap (C^1(\overline{\Omega})_+)$ be a non-negative weak solution for the problem (P). Then, we can fix $M > \max\{\|\nabla u\|_\infty, 1\}$ and

$$\tilde{a}(t) = \begin{cases} a(t), & t \leq M \\ \frac{t^{g^- -2}}{M^{g^- -2}}a(M), & t > M. \end{cases} \tag{3.24}$$

Using the assumption (g_2) and Lemma 2.3, we obtain

$$\begin{aligned}\tilde{a}(|\eta|)|\eta|^2 &= a(|\eta|)|\eta|^2 = g(|\eta|)|\eta| \geq g^- G(|\eta|)\\ &\geq G(1)\min\{|\eta|^{g^-}, |\eta|^{g^+}\}\\ &\geq G(1)(|\eta|^{g^-} - 1), \quad \text{for } |\eta| \leq M.\end{aligned} \tag{3.25}$$

Hence, by Lemma 2.3, we get

$$\tilde{a}(|\eta|)|\eta|^2 = \frac{|\eta|^{g^- -2}}{M^{g^- -2}}a(M)|\eta|^2 = \frac{a(M)}{M^{g^- -2}}|\eta|^{g^-}, \quad \text{for } |\eta| > M. \tag{3.26}$$

From (3.25) and (3.26), we can find $\alpha_1, \alpha_2 > 0$ such that

$$\tilde{a}(|\eta|)|\eta|^2 \geq \alpha_1|\eta|^{g^-} - \alpha_2, \quad \forall\, \eta \in \mathbb{R}^N. \tag{3.27}$$

Next, we define the function $\tilde{A} : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ by $\tilde{A}(x,\eta) = \frac{1}{\alpha_1}\tilde{a}(|\eta|)\eta$, that is

$$\tilde{A}(x,\eta) = \begin{cases} \frac{1}{\alpha_1}a(|\eta|)\eta, & |\eta| \leq M \\ \frac{1}{\alpha_1}\frac{|\eta|^{g^- -2}}{M^{g^- -2}}a(M)\eta, & |\eta| > M. \end{cases} \tag{3.28}$$

Again, using assumption (g_2) and Lemma 2.3, we obtain

$$\begin{aligned}|\tilde{A}(x,\eta)| &= \frac{1}{\alpha_1}a(|\eta|)|\eta| = \frac{1}{\alpha_1}g(|\eta|) \leq \frac{g(1)}{\alpha_1}\max\{|\eta|^{g^- -1}, |\eta|^{g^+ -1}\}\\ &\leq \frac{g(1)}{\alpha_1}\left(|\eta|^{g^- -1} + M^{g^+ -1}\right), \quad \text{for } |\eta| \leq M.\end{aligned} \tag{3.29}$$

Also, by (3.28), we have

$$|\tilde{A}(x,\eta)| = \frac{1}{\alpha_1}\frac{a(M)}{M^{g^- -1}}|\eta|^{g^- -1}, \quad \text{for } |\eta| > M. \tag{3.30}$$

Putting together (3.29) and (3.30), we infer that

$$|\tilde{A}(x,\eta)| \leq c_1|\eta|^{g^- -1} + c_2, \quad \text{for all } \eta \in \mathbb{R}^N \tag{3.31}$$

for some $c_1, c_2 > 0$.

From (3.27), it follows that

$$\tilde{A}(x,\eta)\cdot\eta = \frac{1}{\alpha_1}\tilde{a}(|\eta|)|\eta|^2 \geq |\eta|^{g^-} - \frac{\alpha_2}{\alpha_1}, \quad \text{for all } \eta \in \mathbb{R}^N. \tag{3.32}$$

Setting $B(x,u) := \frac{1}{\alpha_1}a(|u(x)|)u(x) - f(x,u(x))$, we infer that u is a weak solution for the following quasilinear problem

$$-\mathrm{div}(\tilde{A}(x,\nabla u)) + \tilde{B}(x,u) = 0 \text{ in } \Omega.$$

Since $\tilde{A}$ satisfies the inequalities (3.31) and (3.32), then, from [33, Theorem 1.1, p. 724], we deduce that $u \in \mathrm{int}(C^1(\overline{\Omega})_+)$ is a positive weak solution for the problem (P).

On the other side, if we let $v \in W^{1,G}(\Omega)\cap(-C^1(\overline{\Omega})_+)$ be a non-positive weak solution for the problem (P) such that $u \in -C^1(\overline{\Omega})_+$. Then, we can see that $w = -v \in W^{1,G}(\Omega) \cap C^1(\overline{\Omega})_+$ is a non-negative weak solution for the following problem

$$\begin{cases} -\mathrm{div}(a(|\nabla u(x)|)\nabla u(x)) + a(|u(x)|)u(x) = \lambda g(x,u(x)), & x \in \Omega \\ a(|\nabla u(x)|)\frac{\partial u(x)}{d\nu} + b(x)|u(x)|^{p-2}u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{P1}$$

where $g(x,t) = -f(x,-t)$ for all $x \in \Omega$ and $t \in \mathbb{R}$.

Using the same argument as above, we prove that $w \in \mathrm{int}(C^1(\overline{\Omega})_+)$ is a positive weak solution for the problem (P1). So, we deduce that $v \in -\mathrm{int}(C^1(\overline{\Omega})_+)$ is a negative weak solution for the problem (P). Thus, the proof is complete. □

3.1. *Proof of Theorem 1.3*

First, let us introduce the Carathéodory functions $f_+ : \Omega\times\mathbb{R} \to \mathbb{R}$ defined by

$$f_+(x,t) = \begin{cases} \lambda f(x,t) & \text{if } t \geq 0 \\ 0 & \text{if } t < 0, \end{cases} \tag{3.33}$$

and $f_- : \Omega\times\mathbb{R} \to \mathbb{R}$ defined by

$$f_-(x,t) = \begin{cases} \lambda f(x,t) & \text{if } t \leq 0 \\ 0 & \text{if } t > 0. \end{cases} \tag{3.34}$$

We set $F_\pm(x,s) = \int_0^s f_\pm(x,t)dt$ and consider the C^1-functionals $J_\pm : W^{1,G}(\Omega) \to \mathbb{R}$ defined by

$$J_\pm(u) = \mathcal{K}(u) + \frac{1}{p}\int_{\partial\Omega} b(x)|u|^p d\gamma - \int_\Omega F_\pm(x,u)dx, \text{ for all } u \in W^{1,G}(\Omega). \tag{3.35}$$

Let $\tilde{F}_\pm(x,t) = \frac{1}{g^+}f_\pm(x,t)t - F_\pm(x,t)$, for all $x \in \Omega$ and $t \in \mathbb{R}$.

Proposition 3.2. *Assume that the assumptions of Theorem 1.3 hold. If $\{u_n\}_{n\in\mathbb{N}} \subset W^{1,G}(\Omega)$ is a $(C)_{c_\pm}$-sequence for $J_\pm$, that is*

$$J_\pm(u_n) \longrightarrow c_\pm \;\; \textit{and} \;\; (1+\|u_n\|)\|J'_\pm(u_n)\|_{(W^{1,G}(\Omega))^*} \longrightarrow 0, \;\; \textit{as} \;\; n \longrightarrow +\infty.$$

Then $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W^{1,G}(\Omega)$.

Proof. We give the proof for the functional J_+; the proof for J_- is similar.

Let $\{u_n\}_{n\in\mathbb{N}} \subset W^{1,G}(\Omega)$ be a $(C)_{c_+}$-sequence for J_+, that is

$$J_+(u_n) \longrightarrow c_+ \;\; \text{and} \;\; (1+\|u_n\|)\|J'_+(u_n)\|_{(W^{1,G}(\Omega))^*} \longrightarrow 0, \;\; \text{as} \;\; n \longrightarrow +\infty.$$

Then

$$J_+(u_n) \longrightarrow c_+ \;\; \text{and} \;\; \langle J'_+(u_n), u_n\rangle \longrightarrow 0, \;\; \text{as} \;\; n \longrightarrow +\infty. \tag{3.36}$$

By (3.36) and assumption (g_2), for n sufficiently large, there exists $C > 0$ such that

$$\begin{aligned} C &\geq J_+(u_n) - \frac{1}{g^+}\langle J'_+(u_n), u_n\rangle \\ &\geq \int_\Omega \left[\frac{1}{g^+}f_+(x,u_n)u_n - F_+(x,u_n)\right] dx \\ &= \int_\Omega \tilde{F}_+(x,u_n)dx. \end{aligned} \tag{3.37}$$

Arguing by contradiction, we assume that $\|u_n\| \longrightarrow +\infty$, then $\|u_n\| \geq 1$ for n large enough. Let $v_n = \frac{u_n}{\|u_n\|} \in W^{1,G}(\Omega)$ with $\|v_n\| = 1$ and up to subsequence we can assume that

$$v_n \rightharpoonup v \text{ in } W^{1,G}(\Omega) \text{ and } \; v_n(x) \to v(x) \text{ a.e. } x \in \Omega.$$

Note that, exploiting Lemma 2.3 and assumption (g_2), we find that

$$\begin{aligned}
\langle J'_+(u_n), u_n\rangle &= \int_\Omega a(|\nabla u_n|)\nabla u_n.\nabla u_n dx + \int_\Omega a(|u_n|)u_n.u_n dx \\
&+ \int_{\partial\Omega} b(x)|u_n|^p dx - \int_\Omega f_+(x,u_n)u_n dx \\
&\geq \int_\Omega a(|\nabla u_n|)\nabla u_n.\nabla u_n dx + \int_\Omega a(|u_n|)u_n.u_n dx \\
&- \int_\Omega f_+(x,u_n)u_n dx \\
&\geq g^- \mathcal{K}(u_n) - \int_\Omega f_+(x,u_n)u_n dx \\
&\geq \|u_n\|^{g^-} - \int_\Omega f_+(x,u_n)u_n dx, \quad (\text{since } \|u_n\| \geq 1)
\end{aligned} \tag{3.38}$$

for n large enough, thus

$$\frac{\langle J'_+(u_n), u_n\rangle}{\|u_n\|^{g^-}} \geq 1 - \int_\Omega \frac{f_+(x,u_n)}{\|u_n\|^{g^-}} u_n dx. \tag{3.39}$$

It follows, from (3.36) and (3.39), that

$$\limsup_{n\to+\infty} \int_\Omega \frac{f_+(x,u_n)}{\|u_n\|^{g^-}} u_n dx \geq 1. \tag{3.40}$$

Set for $r \geq 0$

$$\mathfrak{F}(r) := \inf\left\{\tilde{F}_+(x,s) : \; x \in \Omega \text{ and } s \in \mathbb{R} \text{ with } s \geq r\right\}.$$

By (f_4), we have

$$\mathfrak{F}(r) > 0, \text{ for all } r \text{ large, and } \mathfrak{F}(r) \to +\infty \text{ as } r \to +\infty.$$

For $0 \leq a < b \leq +\infty$ let

$$A_n(a,b) := \left\{x \in \Omega : \; a \leq |u_n(x)| < b\right\}$$

and

$$c_a^b := \inf\left\{\frac{\tilde{F}_+(x,s)}{|s|^{g^-}} : \; x \in \Omega \text{ and } s \in \mathbb{R}\setminus\{0\} \text{ with } a \leq |s| < b\right\}.$$

Note that

$$\tilde{F}_+(x,u_n) \geq c_a^b |u_n|^{g^-}, \quad \text{for all } x \in A_n(a,b). \tag{3.41}$$

It follows from (3.37) that

$$\begin{aligned}
C &\geq \int_\Omega \tilde{F}_+(x,u_n)dx \\
&= \int_{A_n(0,a)} \tilde{F}_+(x,u_n)dx + \int_{A_n(a,b)} \tilde{F}_+(x,u_n)dx + \int_{A_n(b,+\infty)} \tilde{F}_+(x,u_n)dx \\
&\geq \int_{A_n(0,a)} \tilde{F}_+(x,u_n)dx + c_a^b \int_{A_n(a,b)} |u_n|^{g^-} dx + \mathfrak{F}(b)|A_n(b,+\infty)|
\end{aligned} \tag{3.42}$$

for n large enough.

Using Theorem 2.5, we get $\gamma_3 > 0$ such that $\|v_n\|_r \leq \gamma_3\|v_n\| = \gamma_3$ with $1 \leq r < g_*^-$.

Let $0 < \varepsilon < \frac{1}{3}$. By assumption (f_3), there exists $a_\varepsilon > 0$ such that

$$|f_+(x,s)| \leq \frac{\varepsilon}{3\gamma_3}|s|^{g^- -1} \text{ for each } |s| \leq a_\varepsilon. \tag{3.43}$$

From (3.43) and Theorem 2.5, we obtain

$$\begin{aligned}
\int_{A_n(0,a_\varepsilon)} \frac{f_+(x,u_n)}{\|u_n\|^{g^-}} u_n dx &\leq \frac{\varepsilon}{3\gamma_3} \int_{A_n(0,a_\varepsilon)} \frac{|u_n|^{g^- -1}}{\|u_n\|^{g^-}} u_n dx \\
&\leq \frac{\varepsilon}{3\gamma_3} \int_{A_n(0,a_\varepsilon)} |v_n|^{g^-} dx
\end{aligned}$$

$$\le \frac{\varepsilon}{3\gamma_3}\gamma_3\|v_n\|^{g^-}$$
$$= \frac{\varepsilon}{3}, \quad \text{for all } n \in \mathbb{N}. \tag{3.44}$$

Now, exploiting (3.42) and assumption (f_4), we see that

$$C' \ge \int_{A_n(b,+\infty)} \tilde{F}_+(x,u_n)dx \ge \mathfrak{F}(b)|A_n(b,+\infty)|,$$

where $C' > 0$. It follows, using the fact $\mathfrak{F}(b) \to +\infty$ as $b \to +\infty$, that

$$|A_n(b,+\infty)| \to 0, \quad \text{as } b \to +\infty, \quad \text{uniformly in } n. \tag{3.45}$$

Set $\sigma' = \frac{\sigma}{\sigma-1}$ (where σ is defined in (f_4)). Since $\sigma > \frac{N}{g^-}$, one sees that $g^-\sigma' \in (g^-, g_*^-)$.

Let $\tau \in (g^-\sigma', g_*^-)$. Using Theorem 2.5, the Hölder inequality and (3.45), for b large, we find

$$\left(\int_{A_n(b,+\infty)} |v_n|^{g^-\sigma'}dx\right)^{\frac{1}{\sigma'}} \le |A_n(b,+\infty)|^{\frac{\tau-g^-\sigma'}{\tau\sigma'}} \left(\int_{A_n(b,+\infty)} |v_n|^{g^-\sigma'\frac{\tau}{g^-\sigma'}}dx\right)^{\frac{g^-}{\tau}}$$
$$\le |A_n(b,+\infty)|^{\frac{\tau-g^-\sigma'}{\tau\sigma'}} \left(\int_{A_n(b,+\infty)} |v_n|^{\tau}dx\right)^{\frac{g^-}{\tau}}$$
$$\le |A_n(b,+\infty)|^{\frac{\tau-g^-\sigma'}{\tau\sigma'}} \gamma\|v_n\|^{g^-}$$
$$= |A_n(b,+\infty)|^{\frac{\tau-g^-\sigma'}{\tau\sigma'}} \gamma$$
$$\le \frac{\varepsilon}{3}, \quad \text{uniformly in } n. \tag{3.46}$$

By (f_4), Hölder inequality, (3.37) and (3.46), we can choose $b_\varepsilon \ge r_0$ large so that

$$\int_{A_n(b_\varepsilon,+\infty)} \frac{f_+(x,u_n)}{\|u_n\|^{g^-}}u_n dx \le \int_{A_n(b_\varepsilon,+\infty)} \frac{f_+(x,u_n)}{|u_n|^{g^--1}}|v_n|^{g^-}dx$$
$$\le \left(\int_{A_n(b_\varepsilon,+\infty)} \left|\frac{f_+(x,u_n)}{|u_n|^{g^--1}}\right|^{\sigma}dx\right)^{\frac{1}{\sigma}} \left(\int_{A_n(b_\varepsilon,+\infty)} |v_n|^{g^-\sigma'}dx\right)^{\frac{1}{\sigma'}}$$
$$\le \left(\tilde{c}\int_{A_n(b_\varepsilon,+\infty)} \tilde{F}_+(x,u_n)dx\right)^{\frac{1}{\sigma}} \left(\int_{A_n(b_\varepsilon,+\infty)} |v_n|^{g^-\sigma'}dx\right)^{\frac{1}{\sigma'}}$$
$$\le \frac{\varepsilon}{3}, \quad \text{uniformly in } n. \tag{3.47}$$

Next, from (3.42), we have

$$\int_{A_n(a,b)} |v_n|^{g^-}dx = \frac{1}{\|u_n\|^{g^-}}\int_{A_n(a,b)} |u_n|^{g^-}dx$$
$$\le \frac{C''}{c_a^b\|u_n\|^{g^-}} \longrightarrow 0 \quad \text{as } n \longrightarrow +\infty, \tag{3.48}$$

where $C'' > 0$. Since $\frac{f_+(x,s)}{|s|^{g^--1}}$ is continuous on $a \le |s| \le b$, then, there exists $c > 0$ depend on a and b and independent from n, such that

$$|f_+(x,u_n)| \le c|u_n|^{g^--1}, \quad \text{for all } x \in A_n(a,b). \tag{3.49}$$

Using (3.48) and (3.49), we can choose n_0 large enough such that

$$\int_{A_n(a_\varepsilon,b_\varepsilon)} \frac{f_+(x,u_n)}{\|u_n\|^{g^-}}u_n dx \le \int_{A_n(a_\varepsilon,b_\varepsilon)} \frac{f_+(x,u_n)}{|u_n|^{g^--1}}|v_n|^{g^-}dx$$
$$\le c\int_{A_n(a_\varepsilon,b_\varepsilon)} |v_n|^{g^-}dx$$
$$\le c\frac{C+1}{c_{a_\varepsilon}^{b_\varepsilon}\|u_n\|^{g^-}}$$
$$\le \frac{\varepsilon}{3}, \quad \text{for all } n \ge n_0. \tag{3.50}$$

Putting together (3.44), (3.47) and (3.50), we find that

$$\int_\Omega \frac{f_+(x,u_n)}{\|u_n\|^{g^-}}u_n dx \le \varepsilon, \quad \text{for all } n \ge n_0.$$

Which is contradict with (3.40). Therefore, $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W^{1,G}(\Omega)$. □

Proposition 3.3. *Assume that* $(f_1)-(f_4)$ *and* $(g_1)-(g_4)$ *hold. Then,* $J_\pm$ *satisfies the* C*-condition at level* $c_\pm$.

Proof. Let $\{u_n\} \subset W^{1,G}(\Omega)$ be a $(C)_{c_\pm}$-sequence for $J_\pm$, that is

$$J_\pm(u_n) \longrightarrow c_\pm \quad \text{and} \quad (1+\|u_n\|)\|J'_\pm(u_n)\|_{(W^{1,G}(\Omega))^*} \longrightarrow 0, \quad \text{as} \quad n \longrightarrow +\infty. \tag{3.51}$$

By Proposition 3.2, we see that $\{u_n\}$ is bounded. Then, up to a subsequence, there exists $u \in W^{1,G}(\Omega)$ such that u_n converges to u weakly in $W^{1,G}(\Omega)$, strongly in $L^r(\Omega)$, $1 \le r < g_*^-$, and a.e. in Ω.

From (3.51), we have

$$\lim_{n\to+\infty} \langle J'_\pm(u_n), u_n - u\rangle = 0. \tag{3.52}$$

Since the embedding $W^{1,G}(\Omega) \hookrightarrow L^r(\partial\Omega)$ is compact for $1 \le r < g_*^-$ (see Theorem 2.5), then, by Hölder inequality, we get

$$\int_{\partial\Omega} b(x)|u_n|^{p-2}u_n(u_n-u)dx \le \|b\|_{L^\infty(\partial\Omega)}\|u_n\|_p^{p-1}\|u_n-u\|_p \longrightarrow 0, \text{ as } n \longrightarrow +\infty. \tag{3.53}$$

Using, (f_1), Hölder inequality, Lemma 2.1 and the fact that the embedding $W^{1,G}(\Omega) \hookrightarrow L^H(\Omega)$ and $W^{1,G}(\Omega) \hookrightarrow L^1(\Omega)$ are compact (see Theorems 2.4 and 2.5), we find that

$$\begin{aligned}\int_\Omega f_\pm(x,u_n)(u_n-u)dx &\le \|\hat{a}\|_{L^\infty(\Omega)}\left(\int_\Omega |u_n-u|dx + \int_\Omega h(|u_n|)|u_n-u|dx\right)\\ &\le \|\hat{a}\|_{L^\infty(\Omega)}\left(\int_\Omega |u_n-u|dx + \|h(|u_n|)\|_{(\tilde{H})}\|u_n-u\|_{(H)}\right)\\ &\le \|\hat{a}\|_{L^\infty(\Omega)}\left(\int_\Omega |u_n-u|dx + \tilde{C}\|u_n-u\|_{(H)}\right)\\ &\longrightarrow 0, \text{ as } n \to +\infty\end{aligned} \tag{3.54}$$

where $\tilde{C} > 0$.

From (3.52), (3.53) and (3.54), we obtain

$$\limsup_{n\to+\infty} \int_\Omega a(|\nabla u_n|)\nabla u_n.\nabla(u_n-u) + a(|u_n|)u_n(u_n-u)dx \le 0.$$

It follows, by Lemma 2.12, that

$$u_n \to u \text{ in } W^{1,G}(\Omega).$$

Thus the proof. □

The next result deals with Mountain Pass Geometry of $J_\pm$.

Proposition 3.4. *Under the assumptions of Theorem 1.3, there exists* $\lambda_* > 0$ *such that the functionals* $J_\pm$ *satisfy the following conditions for all* $\lambda \le \lambda_*$:

(1) there exist $\rho, \alpha > 0$ *such that*

$$J_\pm(u) \ge \alpha, \quad \textit{for all } u \in W^{1,G}(\Omega) \textit{ with } \|u\| = \rho;$$

(2) there exists $e \in B_\rho^c(0)$ *verifying* $J_\pm(e) < 0$.

Proof. (1) Let $u \in W^{1,G}(\Omega)$ such that $\|u\| \le 1$. Using Lemmas 2.3, 2.16 (for $\varepsilon = 1$) and Theorem 2.5, we infer that

$$\begin{aligned}J_\pm(u) &\ge \|u\|^{g^+} - \int_\Omega F_\pm(x,u)dx\\ &\ge \|u\|^{g^+} - \lambda\left(\int_\Omega |u|^{g^-}dx + C_1\int_\Omega |u|^r dx\right)\\ &\ge \|u\|^{g^+} - \lambda\left(C_{g^-}\|u\|^{g^-} + C_rC_1\|u\|^r\right)\\ &\ge \|u\|^{g^+} - \lambda\left(C_{g^-} + C_rC_1\right)\|u\|^{g^-} \quad (\text{since } \ g^- < r).\end{aligned} \tag{3.55}$$

Choosing $\lambda_* = \frac{1}{2(C_{g^-}+C_rC_1)}$, then,

$$2\lambda(C_{g^-} + C_rC_1) < 1, \quad \text{for all } 0 < \lambda < \lambda^*.$$

Let, for $\lambda \in (0,\lambda^*)$, $(2\lambda\alpha)^{\frac{1}{g^+-g^-}} < \rho < 1$ such that

$$J_\pm(u) \ge \alpha = \frac{\rho^{g^+}}{2}, \quad \text{for } \|u\| = \rho.$$

(2) From (f_2), for any $A > 0$, there exists $R_A > 0$, such that

$$A|t|^{g^+} \le F_\pm(x,t), \text{ for all } x \in \Omega \text{ and all } |t| \ge R_A. \tag{3.56}$$

Let $u \in W^{1,G}(\Omega)$ such that $\|u\| \ge 1$. Using (3.56) and Lemmas 2.3, 2.16, for $|t| \ge R_A$, we obtain

$$\begin{aligned} J_\pm(tu) &\le \|tu\|^{g^+} + \frac{1}{p}\int_{\partial\Omega} b(x)|tu|^p d\gamma - A\int_\Omega |tu|^{g^+} \\ &\le |t|^{g^+}\|u\|^{g^+} + \frac{|t|^p}{p}\|b\|_\infty \int_{\partial\Omega} |u|^p d\gamma - A|t|^{g^+}\int_\Omega |u|^{g^+} \\ &\le |t|^{g^+}\|u\|^{g^+} + \frac{|t|^p}{p}\|b\|_\infty \int_{\partial\Omega} |u|^p d\gamma - A|t|^{g^+}\|u\|_{g^+}^{g^+} \\ &\le |t|^{g^+}\|u\|^{g^+} + \frac{|t|^p}{p}\|b\|_\infty C_p\|u\|^p - A|t|^{g^+}\|u\|_{g^+}^{g^+} \\ &\le |t|^{g^+}\left(\|u\|^{g^+} + \frac{|t|^{p-g^+}}{p}\|b\|_\infty C_p\|u\|^{g^+} - A\|u\|_{g^+}^{g^+}\right) \end{aligned} \tag{3.57}$$

In (3.57), when $A \ge \frac{\|u\|^{g^+}}{\|u\|_{g^+}^{g^+}} \ge 0$, we deduce that

$$J_\pm(tu) \to -\infty, \text{ as } |t| \to +\infty.$$

Therefore, there exists $\bar{t}$ large enough such that

$$e = \bar{t}u \in B_\rho^c(0) \text{ and } J_\pm(e) < 0.$$

Thus the proof. □

Proof of Theorem 1.3 concluded.

From Propositions 3.3 and 3.4, we can apply the Mountain Pass Theorem in [34]. Therefore, there exist $u_0, v_0 \in W^{1,G}(\Omega)$ such that $J_+(u_0) = c_+$, $J_+'(u_0) = 0$ and $J_-(v_0) = c_-$, $J_-'(v_0) = 0$.

Since u_0 and v_0 are critical points respectively for J_+ and J_-, then

$$\langle J_+'(u_0), v\rangle = 0, \quad \text{for all } v \in W^{1,G}(\Omega) \tag{3.58}$$

and

$$\langle J_-'(v_0), v\rangle = 0, \quad \text{for all } v \in W^{1,G}(\Omega). \tag{3.59}$$

In (3.58), we act with $v = u_0^-$, we get

$$\int_\Omega \left[a(|\nabla u_0|)\nabla u_0.\nabla u_0^- + a(|u_0|)u_0u_0^-\right]dx + \int_{\partial\Omega} b(x)|u_0|^{p-2}u_0u_0^- d\gamma = \int_\Omega f_+(x,u_0)u_0^- dx. \tag{3.60}$$

From (3.33), (3.60) and assumption (g_2), one has

$$\mathcal{K}(v_0^-) \le \int_\Omega \left[a(|\nabla u_0^-|)|\nabla u_0^-|^2 + a(|u_0^-|)(u_0^-)^2\right]dx \le 0$$

which give us, by Lemma 2.3, that $u_0^- = 0$. Thus, u_0 is a non-negative function.

In (3.59), we act with $v = v_0^+$, we get

$$\int_\Omega \left[a(|\nabla v_0|)\nabla v_0.\nabla v_0^+ + a(|v_0|)v_0v_0^+\right]dx + \int_{\partial\Omega} b(x)|v_0|^{p-2}v_0v_0^+ d\gamma = \int_\Omega f_-(x,v_0)v_0^+ dx. \tag{3.61}$$

From (3.34), (3.61) and assumption (g_2), one has

$$\mathcal{K}(v_0^+) \le \int_\Omega \left[a(|\nabla v_0^+|)|\nabla v_0^+|^2 + a(|v_0^+|)(v_0^+)^2\right]dx \le 0$$

which give us, by Lemma 2.3, that $v_0^+ = 0$. Thus, v_0 is a non-positive function.

Next, by the truncation on f (see (3.33) and (3.34)), we infer that u_0 is a non-negative and v_0 is a non-positive weak solutions for the problem (P). Hence, from [26, Theorems 2.13 and 2.14, p. 7], one sees that u_0 and v_0 are bounded and

$$u_0 \in C^1(\overline{\Omega})_+ \text{ and } v_0 \in -C^1(\overline{\Omega})_+. \tag{3.62}$$

Exploiting (3.62) and Proposition 3.1, we deduce that $u_0 \in W^{1,G}(\Omega)\cap\text{int}(C^1(\overline{\Omega})_+)$ and $v_0 \in W^{1,G}(\Omega)\cap(-\text{int}(C^1(\overline{\Omega})_+))$ are, respectively, a positive and a negative weak solutions for the problem (P). This ends the proof. □

3.2. Proof of Theorem 1.4

Let us introduce the Carathéodory functions $\bar{f}_+ : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$\bar{f}_+(x,t) = \begin{cases} \lambda f(x,t^+) & \text{if } t \leq \eta_+ \\ \lambda f(x,\eta_+) & \text{if } t > \eta_+, \end{cases} \tag{3.63}$$

and $\bar{f}_- : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$\bar{f}_-(x,t) = \begin{cases} \lambda f(x,\eta_-) & \text{if } t < \eta_- \\ \lambda f(x,t^-) & \text{if } t \geq \eta_-, \end{cases} \tag{3.64}$$

where η_+ and η_- are defined in (f_4').

We set $\bar{F}_\pm(x,s) = \int_0^s \bar{f}_\pm(x,t)dt$ and consider the C^1-functionals $\bar{J}_\pm : W^{1,G}(\Omega) \to \mathbb{R}$ defined by

$$\bar{J}_\pm(u) = \mathcal{K}(u) + \frac{1}{p}\int_{\partial\Omega} b(x)|u|^p d\gamma - \int_\Omega \bar{F}_\pm(x,u)dx, \quad \text{for all } u \in W^{1,G}(\Omega). \tag{3.65}$$

Proof of Theorem 1.4 concluded. We start by the existence of positive solution. Using (3.63), (3.65) and Lemma 2.3, we find that

$$\bar{J}_+(u) \geq \min\left\{\|u\|^{g^-}, \|u\|^{g^+}\right\} - C_1|\Omega| - C_2\|u\|, \quad \text{for all } u \in W^{1,G}(\Omega) \tag{3.66}$$

where C_1 and C_2 are two positive constants. From (3.66) it is clear that $\bar{J}_+$ is coercive.

Let $\{u_n\}_{n\in\mathbb{N}} \subset W^{1,G}(\Omega)$, such that $u_n \rightharpoonup u$ in $W^{1,G}(\Omega)$. We have

$$\liminf_{n\to+\infty} \bar{J}_+(u_n) \geq \liminf_{n\to+\infty} \mathcal{K}(u_n) + \liminf_{n\to+\infty} \frac{1}{p}\int_{\partial\Omega} b(x)|u_n|^p d\gamma - \limsup_{n\to+\infty} \int_\Omega \bar{F}_+(x,u_n)dx. \tag{3.67}$$

Exploiting Fatou's lemma and the compactness embedding theorem (see Theorem 2.5) in (3.67), we obtain

$$\liminf_{n\to+\infty} \bar{J}_+(u_n) \geq \bar{J}_+(u).$$

Therefore, $\bar{J}_+$ is sequentially weakly lower semi-continuous. Then, by the Weierstrass–Tonelli theorem we can find $u_0 \in W^{1,G}(\Omega)$ such that

$$\bar{J}_+(u_0) = \min\left\{\bar{J}_+(u) : \quad u \in \ W^{1,G}(\Omega)\right\}. \tag{3.68}$$

Let $u \in \operatorname{int}(C^1(\Omega)_+)$ and choose $t \in (0,1)$ smalls enough such that

$$0 < tu(x) \leq \min\{\delta, \eta_+\}, \quad \text{for all } \ x \in \overline{\Omega}.$$

Since $\bar{F}_+(x,tu) = \lambda F(x,tu)$, using Lemma 2.3 and assumption (f_3'), we get

$$\begin{aligned}\bar{J}_+(tu) &= \int_\Omega [G(|\nabla tu|) + G(|tu|)]\,dx + \frac{1}{p}\int_{\partial\Omega} b(x)|tu|^p d\gamma - \int_\Omega \bar{F}_+(x,tu)dx \\ &\leq |t|^{g^-}\int_\Omega [G(|\nabla u|) + G(|u|)]\,dx + \frac{1}{p}|t|^p\int_{\partial\Omega} b(x)|u|^p d\gamma - \lambda\int_\Omega F(x,tu)dx \\ &\leq |t|^{g^-}\mathcal{K}(u) + \frac{1}{p}|t|^p\int_{\partial\Omega} b(x)|u|^p d\gamma - c_0\frac{g^-}{q^+}\lambda|t|^{q^+}\int_\Omega Q(|u|)dx.\end{aligned}$$

Since $q^+ < p < g^-$, we can choose $t \in (0,1)$ sufficiently small such that $\bar{J}_+(tu) < 0$. Hence, by (3.68), we get $\bar{J}_+(u_0) \leq \bar{J}_+(tu) < 0 = \bar{J}_+(0)$. Therefore, $u_0 \neq 0$.

Recall that u_0 is a global minimizer of $\bar{J}_+$, then

$$\langle \bar{J}_+'(u_0), u\rangle = 0 \ \text{ for all } u \in W^{1,G}(\Omega). \tag{3.69}$$

We act with $u = u_0^-$ in (3.69), we obtain

$$\int_\Omega a(|\nabla u_0|)\nabla u_0.\nabla u_0^- dx + \int_\Omega a(|u_0|)u_0u_0^- dx + \int_{\partial\Omega} b(x)|u_0|^{p-2}u_0u_0^- d\gamma = \int_\Omega \bar{f}_+(x,u_0)u_0^- dx,$$

it follows, by assumption (g_2) and the truncation (3.63), that

$$\mathcal{K}(u_0^-) \leq \int_\Omega a(|\nabla u_0^-|)|\nabla u_0^-|^2 dx + \int_\Omega a(|u_0^-|)(u_0^-)^2 dx \leq 0.$$

Thus $u_0^- = 0$. Then $u_0 \neq 0$ and $u_0 \geq 0$.

Again, in (3.69), we act with $u = (u_0 - \eta_+)^+$ and using (f_4'), we infer that

$$\int_\Omega a(|\nabla u_0|)\nabla u_0.\nabla(u_0-\eta_+)^+ dx + \int_\Omega a(|u_0|)u_0(u_0-\eta_+)^+ dx$$

$$\leq \int_\Omega \bar{f}_+(x,u_0)(u_0-\eta_+)^+dx$$
$$= \lambda \int_\Omega f(x,\eta_+)(u_0-\eta_+)^+dx$$
$$\leq 0. \tag{3.70}$$

Exploiting (3.70) and Lemmas 2.3, 2.11, we get

$$\begin{aligned}0 &\geq \int_{u_0\geq\eta_+} a(|\nabla u_0|)\nabla u_0.\nabla(u_0-\eta_+)dx + \int_{u_0\geq\eta_+} a(|u_0|)u_0(u_0-\eta_+)dx \\ &\geq 4\int_{u_0\geq\eta_+} G\left(\frac{|\nabla(u_0-\eta_+)|}{2}\right) + 4\int_{u_0\geq\eta_+} G\left(\frac{|u_0-\eta_+|}{2}\right) \\ &= 4\int_\Omega G\left(\frac{|\nabla(u_0-\eta_+)^+|}{2}\right) + 4\int_\Omega G\left(\frac{|(u_0-\eta_+)^+|}{2}\right) \\ &\geq 4\min\left\{\|\frac{(u_0-\eta_+)^+}{2}\|^{g^-}, \|\frac{(u_0-\eta_+)^+}{2}\|^{g^+}\right\}.\end{aligned} \tag{3.71}$$

Therefore, we infer that $(u_0-\eta_+)^+ = 0$. Namely, $u_0 \in [0,\eta_+]$. Next, From the truncation (3.63), we conclude that u_0 is a non-negative bounded weak solution for problem (P) and by [26, Theorem 2.14, p. 7], we deduce that $u_0 \in C^1(\overline{\Omega})_+$. Using Proposition 3.1, we get that $u_0 \in \mathrm{int}(C^1(\overline{\Omega})_+)$. Similarly, using $\bar{J}_-$ and the truncation (3.64), we prove that problem (P) has a negative weak solution $v_0 \in -\mathrm{int}(C^1(\overline{\Omega})_+)$. □

4. Nodal solution

This section devoted for the existence of nodal solution. Our plan for the proof is, at first we prove that the set of all the positive solutions of problem (P) has a minimum u_* and we prove that the set of all negative solutions of problem (P) has a maximum v_* in the sense of Proposition 4.2. Next, we prove the existence of another solution y_0 between u_* and v_*. So, evidently, $y_0 = 0$ or y_0 is a nodal solution. Here, we compute the critical groups at the origin to prove that y_0 cannot be zero. In our proofs, we draw on arguments used in [16,21–23,35,36].

4.1. Some properties for the sets of fixed sign solutions

In what follow, under the assumptions of Theorem 1.4, we will show that problem (P) admits extremal constant sign solutions, that is, there exist a smallest positive solution $u_* \in \mathrm{int}(C^1(\overline{\Omega})_+)$ and a greatest negative solution $v_* \in -\mathrm{int}(C^1(\overline{\Omega})_+)$.

We introduce the following two sets

$$S_+ = \{u : u \text{ is a positive solution of problem (P)}\},$$

$$S_- = \{u : u \text{ is a negative solution of problem (P)}\}.$$

By Theorem 1.4, we have

$$\emptyset \neq S_+ \subseteq \mathrm{int}(C^1(\overline{\Omega})_+) \text{ and } \emptyset \neq S_- \subseteq -\mathrm{int}(C^1(\overline{\Omega})_+).$$

Hypotheses (f_1) and (f_3') imply, for a.a. $x \in \Omega$, that

$$c_2 q(s) - c_3 h(s) \leq f(x,s), \quad \text{for all } s \geq 0 \tag{4.72}$$

and

$$f(x,s) \leq c_2 q(s) - c_3 h(s), \quad \text{for all } s \leq 0, \tag{4.73}$$

for some $c_2, c_3 > 0$. Next, we consider the following auxiliary Robin problem

$$\begin{cases} -\mathrm{div}(a(|\nabla u(x)|)\nabla u(x)) + a(|u(x)|)u(x) = c_2\lambda q(u(x)) - c_3\lambda h(u(x)), & x \in \Omega \\ a(|\nabla u|)\frac{\partial u(x)}{dv} + b(x)|u(x)|^{p-2}u(x) = 0, & x \in \partial\Omega \end{cases} \tag{A}$$

The following result prove the existence and uniqueness of positive and negative weak solutions for the problem (A).

Proposition 4.1. *Under assumptions (f_1), (f_3') and $(g_1)-(g_4)$, problem (A) admits a unique positive weak solution $\tilde{u} \in int(C^1(\overline{\Omega})_+)$ and a unique negative weak solution $\tilde{v} = -\tilde{u} \in -int(C^1(\overline{\Omega})_+)$.*

Proof. First, we prove that problem (A) admits a non-negative smooth solution. For this, we introduce the C^1-functional $\vartheta_+ : W^{1,G}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\vartheta_+(u) = \mathcal{K}(u) + \frac{1}{p}\int_{\partial\Omega} b(x)|u|^p d\gamma - c_2\lambda\int_\Omega Q(u^+)dx + c_3\lambda\int_\Omega H(u^+)dx$$

for all $u \in W^{1,G}(\Omega)$.

Let $u \in W^{1,G}(\Omega)$ such that $\|u\| \geq 1$. Using Lemma 2.3, and the continuous embedding of $W^{1,G}(\Omega)$ in $L^Q(\Omega)$, we get

$$\begin{aligned}\vartheta_+(u) &\geq \mathcal{K}(u) - c_2\lambda \int_\Omega Q(|u^+|)dx \\ &\geq \|u\|^{g^-} - C\lambda\left(\|u\|^{q^+} + \|u\|^{q^-}\right)\end{aligned}$$

for some $C > 0$. Since $q^- \leq q^+ < g^-$, we conclude that ϑ_+ is coercive. Exploiting the compactness embedding and Fatou lemma, we prove that ϑ_+ is sequentially weakly lower semicontinuous. Then using the Weierstrass–Tonelli theorem (see [37]), we can find $\tilde{u} \in W^{1,G}(\Omega)$ such that

$$\vartheta_+(\tilde{u}) = \min\{\vartheta_+(u), u \in W^{1,G}(\Omega)\}. \tag{4.74}$$

Now, Let $u \in \operatorname{int}(C^1(\overline{\Omega})_+)$ and $t \in (0,1)$ small enough. By Lemma 2.3, we infer that

$$\begin{aligned}\vartheta_+(tu) &= \int_\Omega [G(|\nabla tu|) + G(|tu|)]\,dx + \frac{1}{p}\int_{\partial\Omega} b(x)|tu|^p d\gamma \\ &\quad - c_2\lambda \int_\Omega Q(|tu^+|)dx + c_3\lambda\int_\Omega H(|tu^+|)dx \\ &\leq |t|^{g^-}\mathcal{K}(u) + \frac{1}{p}|t|^p \int_{\partial\Omega} b(x)|u|^p d\gamma \\ &\quad - c_2\lambda|t|^{q^+}\int_\Omega Q(|u^+|)dx + c_3\lambda|t|^{h^-}\int_\Omega H(|u^+|)dx.\end{aligned}$$

Taking in mind that $q^- \leq q^+ < p < g^- \leq g^+ < h^- \leq h^+$, then we can choose $t \in (0,1)$ small enough such that $\vartheta_+(tu) < 0$. It follows, by (4.74), that

$$\vartheta_+(\tilde{u}) \leq \vartheta_+(tu) < 0 = \vartheta_+(0),$$

so, $\tilde{u} \neq 0$. Since $\tilde{u}$ is a global minimum of ϑ_+, we have

$$\langle \vartheta_+'(\tilde{u}), v\rangle = 0 \quad \text{for all } v \in W^{1,G}(\Omega)$$

which is equivalent to

$$\begin{aligned}\int_\Omega a(|\nabla\tilde{u}|)\nabla\tilde{u}.\nabla v\,dx + \int_\Omega a(|\tilde{u}|)\tilde{u}v\,dx + \int_{\partial\Omega} b(x)|\tilde{u}|^{p-2}\tilde{u}v\,d\gamma \\ - c_2\lambda\int_\Omega q(\tilde{u}^+)v\,dx + c_3\lambda\int_\Omega h(\tilde{u}^+)v\,dx = 0\end{aligned} \tag{4.75}$$

for all $v \in W^{1,G}(\Omega)$.

We act with $v = \tilde{u}^-$ in (4.75), we obtain

$$\int_\Omega a(|\nabla\tilde{u}^-|)\nabla\tilde{u}^-.\nabla\tilde{u}^-dx + \int_\Omega a(|\tilde{u}^-|)(\tilde{u}^-)^2dx \leq 0. \tag{4.76}$$

Using assumption (g_2) in (4.76), we see that

$$\mathcal{K}(\tilde{u}^-) \leq 0$$

which gives that $\tilde{u}^- = 0$ (by Lemma 2.3). We conclude that $\tilde{u} \geq 0$ and $\tilde{u} \neq 0$.

So $\tilde{u}$ is a non-negative weak solution for problem (A). From [26, Theorems 2.13 and 2.14, p. 7], we infer that $\tilde{u} \in L^\infty(\Omega)$ and $\tilde{u} \in C^1(\overline{\Omega})_+$. Hence, arguing as in Proposition 3.1, we find that $\tilde{u}$ is a positive solution for problem (A).

Next, we show the uniqueness of this positive solution. For this purpose, we consider the integral function $\sigma_+ : L^1(\Omega) \longrightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ defined by

$$\sigma_+(u) = \begin{cases} \displaystyle\int_\Omega G(|\nabla u^{\frac{1}{q^+}}|)dx + \frac{1}{p}\int_{\partial\Omega} b(x)|u^{\frac{1}{q^+}}|^p d\gamma & \text{if } u \geq 0,\ u^{\frac{1}{q^+}} \in W^{1,G}(\Omega), \\ +\infty & \text{otherwise.}\end{cases} \tag{4.77}$$

Let $w_1, w_2 \in \operatorname{dom}(\sigma_+) = \left\{u \geq 0,\ \sigma_+(u) < +\infty\right\}$ and let $w = (tw_1 + (1-t)w_2)^{\frac{1}{q^+}}$ with $t \in [0,1]$. From Diaz and Saa [38, Lemma 1, p. 522], we have that

$$|\nabla w| \leq \left(t|\nabla w_1^{\frac{1}{q^+}}|^{q^+} + (1-t)|\nabla w_2^{\frac{1}{q^+}}|^{q^+}\right)^{\frac{1}{q^+}}.$$

Then, by Lemma 2.17 and the fact that $t \longmapsto G(t)$ is increasing, we deduce

$$G(|\nabla w|) \leq G\left(\left(t|\nabla w_1^{\frac{1}{q^+}}|^{q^+} + (1-t)|\nabla w_2^{\frac{1}{q^+}}|^{q^+}\right)^{\frac{1}{q^+}}\right)$$

$$\leq tG\left(|\nabla w_1^{\frac{1}{q^+}}|\right)+(1-t)G\left(|\nabla w_2^{\frac{1}{q^+}}|\right).$$

Since $q^+ < p$, we have that the function $t \mapsto t^{\frac{p}{q^+}}$ is convex. Therefore, we deduce that σ_+ is convex. Moreover, via the Fatou lemma, we see that σ_+ is lower semi-continuous.

Suppose that $u_1, u_2 \in W^{1,G}(\Omega)$ are two nontrivial positive solutions of (A). From the first part of the proof, we have u_1 and $u_2 \in \text{int}(C^1(\overline{\Omega})_+)$. Therefore, $u_1^{q^+}$ and $u_2^{q^+} \in \text{dom}(\sigma_+)$. Using Proposition 4.1.22 of Papageorgiou–Rădulescu–Repŏvs [7, p. 274], we see that

$$\frac{u_1}{u_2} \in L^\infty(\Omega) \quad \text{and} \quad \frac{u_2}{u_1} \in L^\infty(\Omega). \tag{4.78}$$

Let $v = u_1^{q^+} - u_2^{q^+} \in C^1(\overline{\Omega})$. Then, from (4.78), for $t \in [-1,1]$ with $|t|$ small, we have $u_1^{q^+} + tv,\ u_2^{q^+} + tv \in \text{dom}(\sigma_+)$. Therefore, since σ_+ is convex, we have that the Gâteaux derivative of σ_+ at $u_1^{q^+}$ and at $u_2^{q^+}$ in the direction v exist. Moreover, via the chain rule and the nonlinear Green's identity, we have

$$\sigma'_+(u_1^{q^+})(v) = \frac{1}{q^+}\int_\Omega \frac{-\text{div}(a(|\nabla u_1|)\nabla u_1)}{u_1^{q^+-1}} v\,dx \quad \text{and} \quad \sigma'_+(u_2^{q^+})(v) = \frac{1}{q^+}\int_\Omega \frac{-\text{div}(a(|\nabla u_2|)\nabla u_2)}{u_2^{q^+-1}} v\,dx$$

The convexity of σ_+ implies that σ'_+ is increasing. Therefore, by Lemma 2.17, we have

$$\begin{aligned}
0 &\leq \int_\Omega \left(\frac{-\text{div}(a(|\nabla u_1|)\nabla u_1)}{u_1^{q^+-1}} + \frac{\text{div}(a(|\nabla u_2|)\nabla u_2)}{u_2^{q^+-1}}\right)(u_1^{q^+} - u_2^{q^+})dx \\
&= \int_\Omega \left(\frac{c_2\lambda q(u_1) - c_3\lambda h(u_1)}{u_1^{q^+-1}} - \frac{g(u_1)}{u_1^{q^+-1}} - \frac{c_2\lambda q(u_2) - c_3\lambda h(u_2)}{u_2^{q^+-1}} + \frac{g(u_2)}{u_2^{q^+-1}}\right)(u_1^{q^+} - u_2^{q^+})dx \\
&= c_2\lambda \int_\Omega \left(\frac{q(u_1)}{u_1^{q^+-1}} - \frac{q(u_2)}{u_2^{q^+-1}}\right)(u_1^{q^+} - u_2^{q^+})dx \\
&+ c_3\lambda \int_\Omega \left(\frac{h(u_2)}{u_2^{q^+-1}} - \frac{h(u_1)}{u_1^{q^+-1}}\right)(u_1^{q^+} - u_2^{q^+})dx \\
&+ \int_\Omega \left(\frac{g(u_2)}{u_2^{q^+-1}} - \frac{g(u_1)}{u_1^{q^+-1}}\right)(u_1^{q^+} - u_2^{q^+})dx \\
&< 0
\end{aligned}$$

which implies that $u_1 = u_2$ and this proves the uniqueness of the non-trivial positive weak solution $\tilde{u} \in \text{int}(C^1(\overline{\Omega})_+)$ for problem (A). Since the problem (A) is odd, then it has a unique non-trivial negative weak solution $\tilde{v} = \tilde{u} \in -\text{int}(C^1(\overline{\Omega})_+)$. This ends the proof. ☐

In the following result we prove that the weak solutions $\tilde{u}$ and $\tilde{v}$ of problem (A), provide bounds for the sets S_+ and S_-, where $\tilde{u}$ is a lower bound for S_+ and $\tilde{v}$ is an upper bound for S_-.

Proposition 4.2. *Under the assumptions (f_1), (f_3') and $(g_1)-(g_4)$, we have $\tilde{u} \leq u$ for all $u \in S_+$ and $v \leq \tilde{v}$ for all $v \in S_-$.*

Proof (*Proof*)**.** Let $u \in S_+$. We consider the Carathéodory function $k_+ : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$k_+(x,s) = \begin{cases} 0 & \text{if } s < 0, \\ c_2\lambda q(s) - c_3\lambda h(s) & \text{if } 0 \leq s \leq u(x), \\ c_2\lambda q(u(x)) - c_3\lambda h(u(x)) & \text{if } u(x) < s. \end{cases} \tag{4.79}$$

We set $K_+(x,s) = \int_0^s k_+(x,t)dt$ and consider the C^1-functional $\widehat{\vartheta}_+ : W^{1,G}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\widehat{\vartheta}_+(w) = \mathcal{K}(w) + \frac{1}{p}\int_{\partial\Omega} b(x)|w|^p d\gamma - \int_\Omega K_+(x,w)dx$$

for all $w \in W^{1,G}(\Omega)$.

From (4.79), we can see that $\widehat{\vartheta}_+$ is coercive. Exploiting Fatou lemma and the compactness embedding theorem, we prove that $\widehat{\vartheta}_+$ is sequentially weakly lower semicontinuous. Then, using the Weierstrass–Tonelli theorem, we can find $\tilde{u}_* \in W^{1,G}(\Omega)$ such that

$$\widehat{\vartheta}_+(\tilde{u}_*) = \min\{\widehat{\vartheta}_+(w) : w \in W^{1,G}(\Omega)\}. \tag{4.80}$$

As before, if $w \in \text{int}(C^1(\overline{\Omega})_+)$ and $t \in (0,1)$ small enough such that $tw \leq u$, we have $\widehat{\vartheta}_+(tw) < 0$. Due to (4.80), we have $\widehat{\vartheta}_+(\tilde{u}_*) < 0 = \widehat{\vartheta}_+(0)$. Hence, $\tilde{u}_* \neq 0$.

From (4.80), we have $(\widehat{\vartheta}_+)'(\tilde{u}_*) = 0$, that is,

$$\int_\Omega a(|\nabla \tilde{u}_*|)\nabla \tilde{u}_*.\nabla v\,dx + \int_\Omega a(|\tilde{u}_*|)\tilde{u}_*.v\,dx + \int_{\partial\Omega} b(x)|\tilde{u}_*|^{p-2}\tilde{u}_* v\,d\gamma - \int_\Omega k_+(x,\tilde{u}_*)v\,dx = 0 \tag{4.81}$$

for all $v \in W^{1,G}(\Omega)$.

We act with $v = \tilde{u}_*^- \in W^{1,G}(\Omega)$ in (4.81), we conclude that $\tilde{u}_* \geq 0$ and $\tilde{u}_* \neq 0$.

Next, we act with $v = (\tilde{u}_* - u)^+ \in W^{1,G}(\Omega)$ in (4.81), we get

$$\begin{aligned}\int_\Omega a(|\nabla\tilde{u}_*|)\nabla\tilde{u}_*.\nabla(\tilde{u}_* - u)^+dx + \int_\Omega a(|\tilde{u}_*|)\tilde{u}_*(\tilde{u}_* - u)^+dx + \int_{\partial\Omega} b(x)|\tilde{u}_*|^{p-2}\tilde{u}_*(\tilde{u}_* - u)^+d\gamma \\ = \int_\Omega k_+(x,\tilde{u}_*)(\tilde{u}_* - u)^+dx \\ = \int_\Omega \left[c_2\lambda q(u(x)) - c_3\lambda h(u(x))\right](\tilde{u}_* - u)^+dx \\ \leq \int_\Omega \lambda f(x,u)(\tilde{u}_* - u)^+dx \text{ (see (4.72))} \\ = \int_\Omega a(|\nabla u|)\nabla u.\nabla(\tilde{u}_* - u)^+dx + \int_\Omega a(|u|)u(\tilde{u}_* - u)^+dx \\ + \int_{\partial\Omega} b(x)|u|^{p-2}u(\tilde{u}_* - u)^+d\gamma, \quad (u \in S_+).\end{aligned} \tag{4.82}$$

It follows, by (4.82), that

$$\int_{\{\tilde{u}_*\geq u\}} \left[a(|\nabla\tilde{u}_*|)\nabla\tilde{u}_* - a(|\nabla u|)\nabla u\right].\nabla(\tilde{u}_* - u)dx + \int_{\{\tilde{u}_*\geq u\}} \left[a(|\tilde{u}_*|)\tilde{u}_* - a(|u|)u\right](\tilde{u}_* - u)dx \leq 0. \tag{4.83}$$

Using (4.83) and Lemmas 2.3, 2.11, we infer that $\tilde{u}_* \leq u$. So, we have proved that

$$\tilde{u}_* \in [0,u], \ \tilde{u}_* \neq 0. \tag{4.84}$$

From (4.79), (4.84) and Proposition 4.1, it follows that $\tilde{u}_* = \tilde{u}$. Thus, see (4.84), $\tilde{u} \leq u$ for all $u \in S_+$.

Similarly, by using (4.72) and the following Carathéodory function

$$k_-(x,s) = \begin{cases} c_2\lambda q(v(x)) - c_3\lambda h(v(x)) & \text{if } s < v(x), \\ c_2\lambda q(s) - c_3\lambda h(s) & \text{if } v(x) \leq s \leq 0, \\ 0 & \text{if } 0 < s, \end{cases} \tag{4.85}$$

where $v \in S_-$, with a C^1-functional $\widehat{\vartheta}_- : W^{1,G}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\widehat{\vartheta}_-(w) = \mathcal{K}(w) + \frac{1}{p}\int_{\partial\Omega} b(x)|w|^p d\gamma - \int_\Omega K_-(x,w)dx$$

for all $w \in W^{1,G}(\Omega)$, where $K_-(x,s) = \int_0^s k_-(x,t)dt$, we show that $v \leq \tilde{v}$ for all $v \in S_-$. □

Proposition 4.3. *Assume that assumptions of Theorem 1.4 hold. Then, if $u_1,\ u_2 \in W^{1,G}(\Omega)$ are two nontrivial upper solutions for problem (P), then $\hat{u} = \min\{u_1,u_2\} \in W^{1,G}(\Omega)$ is an upper solution for (P). And if $v_1,\ v_2 \in W^{1,G}(\Omega)$ are two nontrivial lower solutions for problem (P), then $\hat{v} = \max\{v_1,v_2\} \in W^{1,G}(\Omega)$ is a lower solution for (P).*

Proof. First we prove that if $u_1,\ u_2$ are two upper solutions for the problem (P), then $\hat{u} = \min\{u_1,u_2\} \in W^{1,G}(\Omega)$ is upper solution for (P).

Let $\varepsilon > 0$ and consider the truncation function $\xi_\varepsilon : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\xi_\varepsilon(s) = \begin{cases} -\varepsilon & \text{if } s < -\varepsilon \\ s & \text{if } -\varepsilon \leq s \leq \varepsilon \\ \varepsilon & \text{if } \varepsilon < s. \end{cases} \tag{4.86}$$

Clearly ξ_ε is Lipschitz continuous. Hence, we have

$$\xi_\varepsilon((u_1 - u_2)^-) \in W^{1,G}(\Omega) \tag{4.87}$$

and

$$D\xi_\varepsilon((u_1 - u_2)^-) = \xi_\varepsilon'((u_1 - u_2)^-)D(u_1 - u_2)^-. \tag{4.88}$$

Let $\psi \in C_c^1(\Omega)$ a test function such that $\psi \geq 0$. Then

$$\xi_\varepsilon((u_1 - u_2)^-)\psi \in W^{1,G}(\Omega) \cap L^\infty(\Omega) \tag{4.89}$$

and

$$\begin{aligned}D(\xi_\varepsilon((u_1 - u_2)^-)\psi) &= \psi D\xi_\varepsilon((u_1 - u_2)^-) + \xi_\varepsilon((u_1 - u_2)^-)D\psi \\ &= \xi_\varepsilon'((u_1 - u_2)^-)D(u_1 - u_2)^-\psi + \xi_\varepsilon((u_1 - u_2)^-)D\psi.\end{aligned} \tag{4.90}$$

Since by hypothesis $u_1,\ u_2 \in\ W^{1,G}(\Omega)$ are upper solution for problem (P), then from Definition 2.14, we have

$$\begin{aligned}\lambda\int_\Omega f(x,u_1)\xi_\varepsilon((u_1-u_2)^-)\psi dx \le& \int_\Omega a(|\nabla u_1|)\nabla u_1.\nabla(\xi_\varepsilon((u_1-u_2)^-)\psi)dx\\ &+\int_\Omega a(|u_1|)u_1\xi_\varepsilon((u_1-u_2)^-)\psi dx\\ &+\int_{\partial\Omega} b(x)|u_1|^{p-2}u_1\xi_\varepsilon((u_1-u_2)^-)\psi d\gamma \end{aligned}\tag{4.91}$$

and

$$\begin{aligned}\lambda\int_\Omega f(x,u_2)(\varepsilon-\xi_\varepsilon((u_1-u_2)^-))\psi dx \le& \int_\Omega a(|\nabla u_2|)\nabla u_2.\nabla((\varepsilon-\xi_\varepsilon((u_1-u_2)^-))\psi)dx\\ &+\int_\Omega a(|u_2|)u_2(\varepsilon-\xi_\varepsilon((u_1-u_2)^-))\psi dx\\ &+\int_{\partial\Omega} b(x)|u_2|^{p-2}u_2(\varepsilon-\xi_\varepsilon((u_1-u_2)^-))\psi d\gamma. \end{aligned}\tag{4.92}$$

Putting together (4.91) and (4.92), we get

$$\begin{aligned}\lambda\int_\Omega f(x,u_1)\xi_\varepsilon((u_1-u_2)^-)\psi dx + \lambda\int_\Omega f(x,u_2)(\varepsilon-\xi_\varepsilon((u_1-u_2)^-))\psi dx&\\ \le \int_\Omega a(|\nabla u_1|)\nabla u_1.\nabla(\xi_\varepsilon((u_1-u_2)^-)\psi)dx&\\ +\int_\Omega a(|u_1|)u_1\xi_\varepsilon((u_1-u_2)^-)\psi dx&\\ +\int_{\partial\Omega} b(x)|u_1|^{p-2}u_1\xi_\varepsilon((u_1-u_2)^-)\psi d\gamma&\\ +\int_\Omega a(|\nabla u_2|)\nabla u_2.\nabla(\varepsilon-\xi_\varepsilon((u_1-u_2)^-)\psi)dx&\\ +\int_\Omega a(|u_2|)u_2(\varepsilon-\xi_\varepsilon((u_1-u_2)^-))\psi dx&\\ +\int_{\partial\Omega} b(x)|u_2|^{p-2}u_2(\varepsilon-\xi_\varepsilon((u_1-u_2)^-))\psi d\gamma.& \end{aligned}\tag{4.93}$$

Note that

$$\begin{aligned}\int_\Omega a(|\nabla u_1|)\nabla u_1.\nabla(\xi_\varepsilon((u_1-u_2)^-)\psi)dx + \int_\Omega a(|u_1|)u_1\xi_\varepsilon((u_1-u_2)^-)\psi dx&\\ +\int_{\partial\Omega} b(x)|u_1|^{p-2}u_1\xi_\varepsilon((u_1-u_2)^-)\psi d\gamma&\\ =\int_\Omega a(|\nabla u_1|)\nabla u_1.\xi'_\varepsilon((u_1-u_2)^-)\nabla(u_1-u_2)^-\psi dx&\\ +\int_\Omega a(|\nabla u_1|)\nabla u_1.\xi_\varepsilon((u_1-u_2)^-)\nabla\psi dx&\\ +\int_\Omega a(|u_1|)u_1\xi_\varepsilon((u_1-u_2)^-)\psi dx&\\ +\int_{\partial\Omega} b(x)|u_1|^{p-2}u_1\xi_\varepsilon((u_1-u_2)^-)\psi d\gamma&\\ =-\int_{\{-\varepsilon\le u_1-u_2\le 0\}} a(|\nabla u_1|)\nabla u_1.\nabla(u_1-u_2)\psi dx&\\ +\int_\Omega a(|\nabla u_1|)\nabla u_1.\xi_\varepsilon((u_1-u_2)^-)\nabla\psi dx&\\ +\int_\Omega a(|u_1|)u_1\xi_\varepsilon((u_1-u_2)^-)\psi dx&\\ +\int_{\partial\Omega} b(x)|u_1|^{p-2}u_1\xi_\varepsilon((u_1-u_2)^-)\psi d\gamma& \end{aligned}\tag{4.94}$$

and

$$\begin{aligned}\int_\Omega a(|\nabla u_2|)\nabla u_2.\nabla((\varepsilon-\xi_\varepsilon((u_1-u_2)^-))\psi)dx + \int_\Omega a(|u_2|)u_2(\varepsilon-\xi_\varepsilon((u_1-u_2)^-))\psi dx&\\ +\int_{\partial\Omega} b(x)|u_2|^{p-2}u_2(\varepsilon-\xi_\varepsilon((u_1-u_2)^-))\psi d\gamma&\\ =-\int_\Omega a(|\nabla u_2|)\nabla u_2.\xi'_\varepsilon((u_1-u_2)^-)\nabla(u_1-u_2)^-\psi dx& \end{aligned}$$

$$
\begin{aligned}
&+\int_\Omega a(|\nabla u_2|)\nabla u_2.(\varepsilon-\xi_\varepsilon((u_1-u_2)^-))\nabla\psi\,dx\\
&+\int_\Omega a(|u_2|)u_2(\varepsilon-\xi_\varepsilon((u_1-u_2)^-))\psi\,dx\\
&+\int_{\partial\Omega} b(x)|u_2|^{p-2}u_2(\varepsilon-\xi_\varepsilon((u_1-u_2)^-))\psi\,d\gamma\\
&=\int_{\{-\varepsilon\le u_1-u_2\le 0\}} a(|\nabla u_2|)\nabla u_2.\nabla(u_1-u_2)\psi\,dx\\
&+\int_\Omega a(|\nabla u_2|)\nabla u_2.(\varepsilon-\xi_\varepsilon((u_1-u_2)^-))\nabla\psi\,dx\\
&+\int_\Omega a(|u_2|)u_2(\varepsilon-\xi_\varepsilon((u_1-u_2)^-))\psi\,dx\\
&+\int_{\partial\Omega} b(x)|u_2|^{p-2}u_2(\varepsilon-\xi_\varepsilon((u_1-u_2)^-))\psi\,d\gamma. \qquad (4.95)
\end{aligned}
$$

Adding (4.94) and (4.95) and using Lemma 2.9 and the fact that $\psi \geq 0$, we obtain

$$
\begin{aligned}
&\int_\Omega a(|\nabla u_1|)\nabla u_1.\nabla(\xi_\varepsilon((u_1-u_2)^-)\psi)dx+\int_\Omega a(|u_1|)u_1\xi_\varepsilon((u_1-u_2)^-)\psi\,dx\\
&\quad+\int_{\partial\Omega} b(x)|u_1|^{p-2}u_1\xi_\varepsilon((u_1-u_2)^-)\psi\,d\gamma\\
&\quad+\int_\Omega a(|\nabla u_2|)\nabla u_2.\nabla((\varepsilon-\xi_\varepsilon((u_1-u_2)^-))\psi)dx\\
&\quad+\int_\Omega a(|u_2|)u_2(\varepsilon-\xi_\varepsilon((u_1-u_2)^-))\psi\,dx\\
&\quad+\int_{\partial\Omega} b(x)|u_2|^{p-2}u_2(\varepsilon-\xi_\varepsilon((u_1-u_2)^-))\psi\,d\gamma\\
&\quad=-\int_{\{-\varepsilon\le u_1-u_2\le 0\}} a(|\nabla u_1|)\nabla u_1.\nabla(u_1-u_2)\psi\,dx\\
&\quad+\int_\Omega a(|\nabla u_1|)\nabla u_1.\xi_\varepsilon((u_1-u_2)^-)\nabla\psi\,dx\\
&\quad+\int_\Omega a(|u_1|)u_1\xi_\varepsilon((u_1-u_2)^-)\psi\,dx\\
&\quad+\int_{\partial\Omega} b(x)|u_1|^{p-2}u_1\xi_\varepsilon((u_1-u_2)^-)\psi\,d\gamma\\
&\quad+\int_{\{-\varepsilon\le u_1-u_2\le 0\}} a(|\nabla u_2|)\nabla u_2.\nabla(u_1-u_2)\psi\,dx\\
&\quad+\int_\Omega a(|\nabla u_2|)\nabla u_2.(\varepsilon-\xi_\varepsilon((u_1-u_2)^-))\nabla\psi\,dx\\
&\quad+\int_\Omega a(|u_2|)u_2(\varepsilon-\xi_\varepsilon((u_1-u_2)^-))\psi\,dx\\
&\quad+\int_{\partial\Omega} b(x)|u_2|^{p-2}u_2(\varepsilon-\xi_\varepsilon((u_1-u_2)^-))\psi\,d\gamma\\
&\quad\le\int_\Omega a(|\nabla u_1|)\nabla u_1.\xi_\varepsilon((u_1-u_2)^-)\nabla\psi\,dx\\
&\quad+\int_\Omega a(|u_1|)u_1\xi_\varepsilon((u_1-u_2)^-)\psi\,dx\\
&\quad+\int_{\partial\Omega} b(x)|u_1|^{p-2}u_1\xi_\varepsilon((u_1-u_2)^-)\psi\,d\gamma\\
&\quad+\int_\Omega a(|\nabla u_2|)\nabla u_2.(\varepsilon-\xi_\varepsilon((u_1-u_2)^-))\nabla\psi\,dx\\
&\quad+\int_\Omega a(|u_2|)u_2.(\varepsilon-\xi_\varepsilon((u_1-u_2)^-))\psi\,dx\\
&\quad+\int_{\partial\Omega} b(x)|u_2|^{p-2}u_2(\varepsilon-\xi_\varepsilon((u_1-u_2)^-))\psi\,d\gamma. \qquad (4.96)
\end{aligned}
$$

We return to (4.93), use (4.96) and then divide by $\varepsilon > 0$, we obtain

$$
\lambda\int_\Omega f(x,u_1)\frac{1}{\varepsilon}\xi_\varepsilon((u_1-u_2)^-)\psi\,dx+\lambda\int_\Omega f(x,u_2)(1-\frac{1}{\varepsilon}\xi_\varepsilon((u_1-u_2)^-))\psi\,dx
$$

$$\begin{aligned}
&\leq \int_{\Omega} a(|\nabla u_1|)\nabla u_1 . \frac{1}{\varepsilon}\xi_{\varepsilon}((u_1-u_2)^-)\nabla\psi dx \\
&+ \int_{\Omega} a(|u_1|)u_1 . \frac{1}{\varepsilon}\xi_{\varepsilon}((u_1-u_2)^-)\psi dx \\
&+ \int_{\partial\Omega} b(x)|u_1|^{p-2}u_1 \frac{1}{\varepsilon}\xi_{\varepsilon}((u_1-u_2)^-)\psi d\gamma \\
&+ \int_{\Omega} a(|\nabla u_2|)\nabla u_2 .(1-\frac{1}{\varepsilon}\xi_{\varepsilon}((u_1-u_2)^-))\nabla\psi dx \\
&+ \int_{\Omega} a(|u_2|)u_2 .(1-\frac{1}{\varepsilon}\xi_{\varepsilon}((u_1-u_2)^-))\psi dx \\
&+ \int_{\partial\Omega} b(x)|u_2|^{p-2}u_2(1-\frac{1}{\varepsilon}\xi_{\varepsilon}((u_1-u_2)^-))\psi d\gamma . \qquad (4.97)
\end{aligned}$$

Let us observe that

$$\frac{1}{\varepsilon}\xi_{\varepsilon}((u_1-u_2)^-(x)) \to \chi_{\{u_1<u_2\}}(x) \text{ a.e. on } \Omega \text{ as } \varepsilon \searrow 0$$

and

$$\chi_{\{u_1\geq u_2\}} = 1-\chi_{\{u_1<u_2\}}.$$

Therefore, if we pass to the limit as $\varepsilon \longrightarrow 0^+$ in (4.97), we get

$$\begin{aligned}
\lambda \int_{\{u_1<u_2\}} f(x,u_1)\psi dx + \lambda \int_{\{u_1\geq u_2\}} f(x,u_2)\psi dx \leq & \int_{\{u_1<u_2\}} a(|\nabla u_1|)\nabla u_1.\nabla\psi dx + \int_{\{u_1\geq u_2\}} a(|\nabla u_2|)\nabla u_2.\nabla\psi dx \\
&+ \int_{\{u_1<u_2\}} a(|u_1|)u_1\psi dx + \int_{\{u_1\geq u_2\}} a(|u_2|)u_2\psi dx \\
&+ \int_{\{x\in\partial\Omega,\ u_1<u_2\}} b(x)|u_1|^{p-2}u_1\psi d\gamma + \int_{\{x\in\partial\Omega,\ u_1\geq u_2\}} b(x)|u_2|^{p-2}u_2\psi d\gamma . \qquad (4.98)
\end{aligned}$$

Recall that $\hat{u} = \min\{u_1,u_2\} \in W^{1,G}(\Omega)$ and

$$D\hat{u} = \begin{cases} Du_1(x) & \text{a.e. on } \{u_1 < u_2\}, \\ Du_2(x) & \text{a.e. on } \{u_1 \geq u_2\}. \end{cases}$$

Using this in (4.98), we obtain

$$\lambda \int_{\Omega} f(x,\hat{u})\psi dx \leq \int_{\Omega} a(|\nabla\hat{u}|)\nabla\hat{u}.\nabla\psi dx + \int_{\Omega} a(|\hat{u}|)\hat{u}\psi dx + \int_{\partial\Omega} b(x)|\hat{u}|^{p-2}\hat{u}\psi d\gamma . \qquad (4.99)$$

Since $C^1_c(\Omega)_+$ is dense in $W^{1,G}(\Omega)_+$, from (4.99), we conclude that $\hat{u} = \min\{u_1,u_2\} \in W^{1,G}(\Omega)$ is an upper solution for problem (P).

Using a similar argument, we can show that if $v_1,\ v_2 \in W^{1,G}(\Omega)$ are two lower solutions of problem (P), then $\hat{v} = \max\{v_1,v_2\} \in W^{1,G}(\Omega)$ is a lower solution of problem (P). □

Proposition 4.4. *Under the assumptions of Theorem 1.4, the set S_+ is downward directed (see Definition 2.7) and the set S_- is upward directed (see Definition 2.7).*

Proof. For this purpose, let $u_1,\ u_2 \in S_+$. Both u_1 and u_2 are also upper solutions for problem (P). So by virtue of Proposition 4.3, $\hat{u} = \min\{u_1,u_2\} \in W^{1,G}(\Omega)$ is an upper solution for problem (P).

We consider the Carathéodory function $l_+ : \Omega\times\mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$l_+(x,s) = \begin{cases} 0 & \text{if } s<0 \\ \lambda f(x,s) & \text{if } 0\leq s\leq \hat{u}(x) \\ \lambda f(x,\hat{u}(x)) & \text{if } \hat{u}(x)\leq s. \end{cases} \qquad (4.100)$$

We set $L_+(x,s) = \int_0^s l_+(x,t)dt$ and the C^1-functional $\mathcal{L}_+ : W^{1,G}(\Omega) \longrightarrow \mathbb{R}$ defined by

$$\mathcal{L}_+(u) = \mathcal{K}(u) + \frac{1}{p}\int_{\partial\Omega} b(x)|u|^p d\gamma - \int_{\Omega} L_+(x,u)dx$$

for all $w \in W^{1,G}(\Omega)$. From (4.100), we can see that $\mathcal{L}_+$ is coercive. Exploiting Fatou lemma and the compactness embedding theorem, we prove that $\mathcal{L}_+$ is sequentially weakly lower semicontinuous. Then, using the Weierstrass–Tonelli theorem, we can find $y_* \in W^{1,G}(\Omega)$ such that

$$\mathcal{L}_+(y_*) = \min\{\mathcal{L}_+(u) :\ u\in W^{1,G}(\Omega)\}. \qquad (4.101)$$

As before, if $u \in \operatorname{int}(C^1(\overline{\Omega})_+)$ and $t \in (0,1)$ small enough such that $tu \leq \hat{u}$, we have $\mathcal{L}_+(tu) < 0$. Due to (4.101), we have $\mathcal{L}_+(y_*) < 0 = \mathcal{L}_+(0)$. Hence, $y_* \neq 0$.

From (4.101), we have $(\mathcal{L}_+)'(y_*) = 0$, that is,

$$\int_\Omega a(|\nabla y_*|)\nabla y_*.\nabla v dx + \int_\Omega a(|y_*|)y_* v dx + \int_{\partial\Omega} b(x)|y_*|^{p-2} y_* v d\gamma - \int_\Omega l_+(x, y_*) v dx = 0 \tag{4.102}$$

for all $v \in W^{1,G}(\Omega)$.

We act with $v = y_*^- \in W^{1,G}(\Omega)$ in (4.102), we conclude that $y_* \geq 0$ and $y_* \neq 0$. Next, we act with $v = (y_* - \hat{u})^+ \in W^{1,G}(\Omega)$ in (4.102), we get

$$\begin{aligned}\int_\Omega a(|\nabla y_*|)\nabla y_*.\nabla(y_* - \hat{u})^+ dx + &\int_\Omega a(|y_*|)y_*(y_* - \hat{u})^+ dx \\ &+ \int_{\partial\Omega} b(x)|y_*|^{p-2} y_*(y_* - \hat{u})^+ d\gamma \\ &= \int_\Omega l_+(x, y_*)(y_* - \hat{u})^+ dx \\ &= \lambda \int_\Omega f(x, \hat{u})(y_* - \hat{u})^+ dx \\ &\leq \int_\Omega a(|\nabla \hat{u}|)\nabla \hat{u}.\nabla(y_* - \hat{u})^+ dx \\ &+ \int_\Omega a(|\hat{u}|)\hat{u}(y_* - \hat{u})^+ dx \\ &+ \int_{\partial\Omega} b(x)|\hat{u}|^{p-2}\hat{u}(y_* - \hat{u})^+ d\gamma, \quad (\hat{u} \text{ is an "upper solution"}).\end{aligned} \tag{4.103}$$

It follows, by (4.103), that

$$\int_{\{y_* \geq \hat{u}\}} \left[a(|\nabla y_*|)\nabla y_* - a(|\nabla \hat{u}|)\nabla \hat{u}\right].\nabla(y_* - \hat{u}) dx + \int_{\{y_* \geq \hat{u}\}} \left[a(|y_*|)y_* - a(|\hat{u}|)\hat{u}\right](y_* - \hat{u}) dx \leq 0. \tag{4.104}$$

Using (4.104) and Lemmas 2.3, 2.11, we infer that $y_* \leq \hat{u} = \min\{u_1, u_2\}$. Hence, from (4.100), we have that $y_* \in W^{1,G}(\Omega)$ is a positive solution of problem (P). So, we conclude that S_+ is downward directed. Arguing similarly, we show that S_- is upward directed. □

Now, we are able to generate extremal constant sign solutions of (P).

Proposition 4.5. *Under the assumptions $(f_1)-(f_2)$, $(f_3')-(f_4')$ and $(g_1)-(g_4)$, the problem (P) has a smallest non-trivial positive solution $u_* \in int(C^1(\overline{\Omega})_+)$ and a biggest non-trivial negative solution $v_* \in -int(C^1(\overline{\Omega})_+)$.*

Proof. Recall that S_+ is the set of non-trivial positive solutions of problem (P). From Proposition 4.4, we know that the set S_+ is downward directed. Then Lemma 3.10 of Hu–Papageorgiou [39, p. 178] implies that there exists a decreasing sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq S_+$ such that

$$\inf_{n\in\mathbb{N}} u_n = \inf S_+.$$

From [26, Theorem 2.13, p. 7] and the fact that $\{u_n\}_{n\in\mathbb{N}}$ is a positive decreasing sequence, we can see that $\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,G}(\Omega)$ is bounded. Since $W^{1,G}(\Omega)$ is a reflexive space, we can find $u_* \in W^{1,G}(\Omega)$ such that

$$\begin{aligned}&u_n \rightharpoonup u_* \ \text{ in } W^{1,G}(\Omega),\\ &u_n \to u_* \ \text{ in } L^H(\Omega) \ \text{ and in } \ L^r(\partial\Omega), \text{ for } 1 \leq r < g_*^-\\ &\text{and } \ u_n(x) \to u_*(x) \ \text{ a.a. } x \in \Omega.\end{aligned} \tag{4.105}$$

Taking the fact that $u_n \in S_+$, we get

$$\int_\Omega a(|\nabla u_n|)\nabla u_n.\nabla v dx + \int_\Omega a(|u_n|)u_n v dx + \int_{\partial\Omega} b(x)|u_n|^{p-2} u_n v d\gamma = \lambda \int_\Omega f(x, u_n) v dx, \tag{4.106}$$

for all $v \in W^{1,G}(\Omega)$ and $n \in \mathbb{N}$.

In (4.106), we act with $v = u_n - u_*$, we obtain

$$\begin{aligned}\int_\Omega a(|\nabla u_n|)\nabla u_n.\nabla(u_n - u_*) dx + &\int_\Omega a(|u_n|)u_n(u_n - u_*) dx \\ &+ \int_{\partial\Omega} b(x)|u_n|^{p-2} u_n(u_n - u_*) d\gamma \\ &= \lambda \int_\Omega f(x, u_n)(u_n - u_*) dx.\end{aligned} \tag{4.107}$$

Note that, from (4.105), we have

$$\lim_{n\to+\infty} \int_\Omega f(x, u_n)(u_n - u_*) dx = 0, \tag{4.108}$$

and

$$\lim_{n\to+\infty}\int_{\partial\Omega} b(x)|u_n|^{p-2}u_n(u_n-u_*)d\gamma = 0. \tag{4.109}$$

Passing to the limit in (4.107) as $n\to+\infty$ and using (4.108) and (4.109), we get

$$\lim_{n\to+\infty}\int_{\Omega} a(|\nabla u_n|)\nabla u_n.\nabla(u_n-u_*) + a(|u_n|)u_n(u_n-u_*)dx = 0. \tag{4.110}$$

It follows, by Lemma 2.12, that

$$u_n \longrightarrow u_* \ \text{ in } \ W^{1,G}(\Omega). \tag{4.111}$$

Passing to the limit as $n\to+\infty$ in (4.106) and using (4.111), we infer that

$$\int_\Omega a(|\nabla u_*|)\nabla u_*.\nabla v dx + \int_\Omega a(|u_*|)u_* v dx + \int_{\partial\Omega} b(x)|u_*|^{p-2}u_* v d\gamma = \lambda\int_\Omega f(x,u_*)v dx, \tag{4.112}$$

for all $v\in W^{1,G}(\Omega)$.

Furthermore, since $u_n\in S_+$ for all $n\geq 0$, from Proposition 4.2, we have

$$0\leq \tilde{u}\leq u_n, \quad \text{for all } n\geq 0. \tag{4.113}$$

In light of (4.105) and (4.113), we see that $0\leq\tilde{u}\leq u_*$. Hence, from Proposition 3.1, we deduce that $u_*\in S_+$ and $u_* = \inf S_+$. Similarly, we prove that there exists $v_*\in S_-$ such that $v\leq v_*$ for all $v\in S_-$. □

4.2. Critical groups at the origin

As we mentioned in the beginning of the section, we need to compute the critical groups at zero. So, in the next result, we prove that $C_k(J,0)=0$ for all $k\in\mathbb{N}$.

Proposition 4.6. *Under the assumptions* (f_1), (f_3') *and* $(g_1)-(g_4)$, *we have* $C_k(J,0)=0$ *for all* $k\in\mathbb{N}$.

Proof. The critical groups at zero for J is defined by

$$C_k(J,0) = H_k(U\cap J^0, (U\cap J^0)\backslash\{0\})$$

where U is a neighborhood of zero. We can take $U=\bar{B}_\varrho = \{u\in W^{1,G}(\Omega): \ \|u\|\leq\varrho\}$, with $\varrho\in(0,1)$.

Let $\mathbb{X}$ be a Banach space and $Y_2\subseteq Y_1\subseteq\mathbb{X}$. From [40, Proposition 4.9 and 4.10, p. 389], if Y_1 and Y_2 are contractible, then the singular homology groups for the pair (Y_1,Y_2)

$$H_k(Y_1,Y_2)=0, \quad \text{for all } k\geq 0.$$

So, to prove our proposition, we just need to show that $\bar{B}_\varrho\cap J^0$ and $(\bar{B}_\varrho\cap J^0)\backslash\{0\}$ are contractible. We divide the proof into two steps.

•**Step 1:** We prove that $\bar{B}_\varrho\cap J^0$ is contractible.

Let $u\in W^{1,G}(\Omega)$ and $0<t\leq 1$, we have

$$J(tu) = \int_\Omega [G(|\nabla tu|)+G(|tu|)]\,dx + \frac{1}{p}\int_{\partial\Omega} b(x)|tu|^p d\gamma - \lambda\int_\Omega F(x,tu)dx. \tag{4.114}$$

From assumptions (f_1) and (f_3'), we get

$$F(x,s)\geq c_1 Q(|s|) - c_0 H(|s|) \ \text{ for all } s\in\mathbb{R} \tag{4.115}$$

and

$$q^+F(x,s) - f(x,s)s \geq -\tilde{c}_3 H(|s|) \text{ for almost all } x\in\Omega \text{ and } s\in\mathbb{R}. \tag{4.116}$$

Using (4.114), (4.115) and Lemma 2.3, we find that

$$\begin{aligned} J(tu) \leq\ & |t|^{g^-}\int_\Omega [G(|\nabla u|)+G(|u|)]\,dx + \frac{1}{p}|t|^p\int_{\partial\Omega} b(x)|u|^p d\gamma \\ & - c_1\lambda\int_\Omega Q(|tu|)dx + c_0\lambda\int_\Omega H(|tu|)dx \\ \leq\ & |t|^{g^-}\mathcal{K}(u) + \frac{1}{p}|t|^p\int_{\partial\Omega} b(x)|u|^p d\gamma \\ & - c_1\lambda|t|^{q^+}\int_\Omega Q(|u|)dx + c_0\lambda|t|^{h^-}\int_\Omega H(|u|)dx. \end{aligned}$$

Since $q^-\leq q^+ < p < g^-\leq g^+ < h^-\leq h^+$, it is clear that we can find $t^*\in(0,1)$ such that

$$J(tu)<0 \text{ for all } t\in(0,t^*). \tag{4.117}$$

Now, let $u \in W^{1,G}(\Omega)$ such that $0 < \|u\| \leq 1$ and $J(u) = 0$, we have

$$\begin{aligned}
\frac{d}{dt}J(tu)\Big|_{t=1} &= \langle J'(u), u\rangle \\
&= \int_\Omega a(|\nabla u|)\nabla u.\nabla u dx + \int_\Omega a(|u|)u^2 dx + \int_{\partial\Omega} b(x)|u|^p d\gamma - \lambda \int_\Omega f(x,u)u dx - q^+ J(u) \\
&= \int_\Omega a(|\nabla u|)\nabla u.\nabla u dx + \int_\Omega a(|u|)u^2 dx + \int_{\partial\Omega} b(x)|u|^p d\gamma - \lambda \int_\Omega f(x,u)u dx \\
&\quad - q^+ \int_\Omega [G(|\nabla u|) + G(|u|)]\, dx - \frac{q^+}{p}\int_{\partial\Omega} b(x)|u|^p d\gamma + q^+\lambda \int_\Omega F(x,u) dx \\
&\geq (g^- - q^+)\mathcal{K}(u) + (1 - \frac{q^+}{p})\int_{\partial\Omega} b(x)|u|^p d\gamma + \lambda \int_\Omega q^+ F(x,u) - f(x,u)u dx.
\end{aligned} \tag{4.118}$$

Exploiting (4.116), (4.118) and continuous embedding result, we get

$$\begin{aligned}
\frac{d}{dt}J(tu)\Big|_{t=1} &\geq (g^- - q^+)\mathcal{K}(u) - \tilde{c}_3\lambda \int_\Omega H(|u|)dx \\
&\geq (g^- - q^+)\|u\|^{g^+} - \tilde{c}_3\lambda \max\{\|u\|_{(H)}^{h^-}, \|u\|_{(H)}^{h^+}\} \\
&\geq (g^- - q^+)\|u\|^{g^+} - c_4\lambda \max\{\|u\|^{h^-}, \|u\|^{h^+}\} \text{ (since } W^{1,G}(\Omega) \hookrightarrow L^H(\Omega)).
\end{aligned}$$

Since $q^+ < p < g^- \leq g^+ < h^- \leq h^+$, there exists some $\varrho \in (0,1)$ small such that

$$\frac{d}{dt}J(tu)\Big|_{t=1} > 0, \tag{4.119}$$

for all $u \in W^{1,G}(\Omega)$ with $0 < \|u\| \leq \varrho$ and $J(u) = 0$.

Claim.

$$J(tu) \leq 0 \text{ for all } t \in [0,1], \tag{4.120}$$

for all $u \in W^{1,G}(\Omega)$, with $0 < \|u\| \leq \varrho$ and $J(u) \leq 0$.

Proof of Claim. Arguing by contradiction, we suppose that there is some $t_0 \in (0,1)$ such that $J(t_0 u) > 0$. Since J is continuous and $J(u) \leq 0$, by Bolzano's theorem, we can find $t_1 \in (t_0, 1]$ such that $J(t_1 u) = 0$.

Let $t_* = \min\{t \in [t_0,1] : J(tu) = 0\} > t_0 > 0$. Then

$$J(tu) > 0 \text{ for all } t \in [t_0, t_*). \tag{4.121}$$

Let $v = t_* u$. We have $0 < \|v\| \leq \|u\| \leq \varrho$ and $J(v) = 0$. Therefore, from (4.119) it follows that

$$\frac{d}{dt}J(tv)\Big|_{t=1} > 0. \tag{4.122}$$

From (4.121), we have

$$J(v) = J(t_* u) = 0 < J(tu) \text{ for all } t \in [t_0, t_*)$$

so,

$$\begin{aligned}
\frac{d}{dt}J(tv)\Big|_{t=1} &= \lim_{t\to 1}\frac{J(tv) - J(v)}{t-1} = \lim_{t\to 1}\frac{J(tv)}{t-1} = \lim_{t\to 1}\frac{J(tt_* u)}{t-1} \\
&= t_* \lim_{t\to 1}\frac{J(tt_* u)}{t_* t - t_*} = t_* \lim_{s\to t_*^-}\frac{J(su)}{s - t_*} \quad (s = t_* t) \\
&\leq 0.
\end{aligned} \tag{4.123}$$

Comparing (4.122) and (4.123), we get a contradiction. This proves (4.120).

Now, Taking $\varrho \in (0,1)$ even smaller if necessary, such that $K_J \cap \bar{B}_\varrho = \{0\}$.

Let $\varphi : [0,1] \times (\bar{B}_\varrho \cap J^0) \to \bar{B}_\varrho \cap J^0$ be a continuous function defined by

$$\varphi(t,u) = (1-t)u \text{ for all } (t,u) \in [0,1] \times (\bar{B}_\varrho \cap J^0).$$

From (4.120) we see that $\varphi(.,.)$ is well-defined. This deformation proves that $\bar{B}_\varrho \cap J^0$ is contractable in itself.

- **Step 2:** we show that $(\bar{B}_\varrho \cap J^0)\backslash\{0\}$ is contractible.

Fix $u \in \bar{B}_\varrho$ with $J(u) > 0$. We shall prove that there exists a unique $\tilde{t} \in (0,1)$ such that

$$J(\tilde{t}u) = 0. \tag{4.124}$$

From (4.117), we have

$$J(tu) < 0, \quad \text{for all } t \in (0, t^*)$$

Recall that $t \mapsto J(tu)$ is continuous and we have $J(u)J(tu) < 0$ for all $t \in (0, t^*)$. So, the existence of some $\tilde{t} \in (0,1)$ follows from Bolzano's theorem. Next we prove the uniqueness of $\tilde{t}$. Suppose that there are $0 < \tilde{t}_1 < \tilde{t}_2 < 1$ such that $J(\tilde{t}_1 u) = J(\tilde{t}_2 u) = 0$. From (4.120), we have

$$k(t) = J(t\tilde{t}_2 u) \le 0 \quad \text{for all } t \in [0,1].$$

Hence $\frac{\tilde{t}_1}{\tilde{t}_2} \in (0,1)$ is a maximizer of $k(.)$, so, using the same computation in (4.123), we get

$$\frac{d}{dt}k(t)\bigg|_{t=\frac{\tilde{t}_1}{\tilde{t}_2}} = 0 \Longrightarrow \frac{\tilde{t}_1}{\tilde{t}_2}\,\frac{d}{dt}J(t\tilde{t}_2 u)\bigg|_{t=\frac{\tilde{t}_1}{\tilde{t}_2}} = \frac{d}{dt}J(t\tilde{t}_1 u)\bigg|_{t=1} = 0,$$

which is contradict with (4.119). Then there exists a unique $\tilde{t} \in (0,1)$ such that $J(\tilde{t}u) = 0$.

From the uniqueness of $\tilde{t}$ and the fact that $J(u) > 0$, we get

$$J(tu) < 0 \quad \text{if } t \in (0,\tilde{t}) \quad \text{and} \quad J(tu) > 0 \quad \text{if } t \in (\tilde{t}, 1]. \tag{4.125}$$

Now, let $\varphi_1 : \bar{B}_\varrho \backslash \{0\} \longrightarrow (0,1]$ defined by

$$\varphi_1(u) = \begin{cases} 1 & \text{if } u \in \bar{B}_\varrho \backslash \{0\},\ J(u) \le 0, \\ \tilde{t} & \text{if } u \in \bar{B}_\varrho \backslash \{0\},\ J(u) > 0. \end{cases} \tag{4.126}$$

We shall prove that $\varphi_1(.)$ is continuous, so, we just need to see the continuity at $u \in \bar{B}_\varrho \backslash \{0\}$ with $J(u) = 0$. Let $u_n \longrightarrow u$ with $J(u_n) > 0$ for all $n \ge 1$. Arguing by contradiction, suppose that by passing to a subsequence if necessary, we have $\tilde{t}_n \le \tilde{t} < 1$ for all $n \ge 1$. From (4.125), we get

$$J(tu_n) > 0 \quad \text{for all } t \in (\tilde{t}, 1] \quad \text{and all } n \ge 1,$$

which gives us

$$J(tu) \ge 0 \quad \text{for all } t \in (\tilde{t}, 1].$$

Hence, from (4.120), we have

$$J(tu) = 0 \quad \text{for all } t \in (\tilde{t}, 1] \tag{4.127}$$

It follows, from (4.127), that

$$\frac{d}{dt}J(tu)\bigg|_{t=1} = 0,$$

which is contradiction with (4.119). This proves that $\varphi_1(.)$ is continuous.

Next, we consider the map $\varphi_2 : \bar{B}_\varrho \backslash \{0\} \longrightarrow (\bar{B}_\varrho \cap J^0)\backslash \{0\}$ defined by

$$\varphi_2(u) = \begin{cases} u & \text{if } u \in \bar{B}_\varrho \backslash \{0\},\ J(u) \le 0, \\ \varphi_1(u)u & \text{if } u \in \bar{B}_\varrho \backslash \{0\},\ J(u) > 0. \end{cases} \tag{4.128}$$

Evidently, $\varphi_2(.)$ is continuous and

$$\varphi_2|_{(\bar{B}_\varrho \cap J^0)\backslash\{0\}} = id|_{(\bar{B}_\varrho \cap J^0)\backslash\{0\}}.$$

Therefore, $\varphi_2(.)$ is a retraction of $\bar{B}_\varrho \backslash \{0\}$ into $(\bar{B}_\varrho \cap J^0)\backslash\{0\}$. But $\bar{B}_\varrho \backslash \{0\}$ is contractible in itself. Hence, $(\bar{B}_\varrho \cap J^0)\backslash\{0\}$ is contractible, too.

From Steps 1 and 2, we see that $(\bar{B}_\varrho \cap J^0)\backslash\{0\}$ and $\bar{B}_\varrho \cap J^0$ are contractible, then

$$C_k(J,0) = H_k(\bar{B}_\varrho \cap J^0, (\bar{B}_\varrho \cap J^0)\backslash\{0\}) = 0 \quad \text{for all } k \ge 0. \quad \square$$

4.3. Proof of Theorem 1.5

Using the extremal constant sign solutions of problem (P), we produce a nodal solution. Precisely, via a suitable truncation and variational methods coupled with Morse theory, we show that problem (P) admits a solution in $[v_*, u_*]$ distinct from $0, v_*, v_*$. Evidently, this is a nodal solution.

Let u_* and v_* be the two extremal constant sign solutions produced in Proposition 4.5. We introduce the following Carathéodory function $\hat{f} : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$\hat{f}(x,s) = \begin{cases} \lambda f(x, v_*(x)) & \text{if } s < v_*(x), \\ \lambda f(x,s) & \text{if } v_*(x) \le s \le u_*(x), \\ \lambda f(x, u_*(x)) & \text{if } u_*(x) < s. \end{cases} \tag{4.129}$$

We also consider the following Carathéodory functions $\hat{f}_\pm : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$\hat{f}_\pm(x,s) = \hat{f}(x, \pm s^\pm). \tag{4.130}$$

We set $\hat{F}(x,s)=\int_0^s \hat{f}(x,t)dt$ and $\hat{F}_\pm(x,s)=\int_0^s \hat{f}_\pm(x,t)dt$ and consider the C^1-functional $\mu,\mu_\pm : W^{1,G}(\Omega)\to\mathbb{R}$ defined by

$$\mu(u)=\mathcal{K}(u)+\frac{1}{p}\int_{\partial\Omega} b(x)|u|^p d\gamma-\int_\Omega \hat{F}(x,u)dx$$

and

$$\mu_\pm(u)=\mathcal{K}(u)+\frac{1}{p}\int_{\partial\Omega} b(x)|u|^p d\gamma-\int_\Omega \hat{F}_\pm(x,u)dx$$

for all $u\in W^{1,G}(\Omega)$.

From (4.129) and (4.130), we infer that

$$K_\mu\subseteq[v_*,u_*]\cap C^1(\overline{\Omega}),$$
$$K_{\mu_+}\subseteq[0,u_*]\cap C^1(\overline{\Omega})_+,$$
$$K_{\mu_-}\subseteq[v_*,0]\cap(-C^1(\overline{\Omega})_+).$$

The extremality of the solutions u_* and v_* implies that

$$K_\mu\subseteq[v_*,u_*]\cap C^1(\overline{\Omega}),\quad K_{\mu_+}=\{0,u_*\}\ \text{ and }\ K_{\mu_-}=\{v_*,0\}. \tag{4.131}$$

Due to (4.129) and (4.130), we can see that μ_+ is coercive and it is sequentially weakly lower semicontinuous. Hence, by the Weierstrass–Tonelli theorem, we can find $\hat{u}_*\in W^{1,G}(\Omega)$ such that

$$\mu_+(\hat{u}_*)=\min\{\mu_+(u):\ u\in W^{1,G}(\Omega)\}. \tag{4.132}$$

As before, we prove that $\mu_+(\hat{u}_*)<0=\mu_+(0)$, then $\hat{u}_*\neq 0$. so $\hat{u}_*=u_*$ (see (4.131)).

It is clear that

$$\mu\,|_{C^1(\overline{\Omega})_+}=\mu_+\,|_{C^1(\overline{\Omega})_+}$$

Since $u_*\in \operatorname{int}(C^1(\overline{\Omega})_+)$, it follows that u_* is a local $C^1(\overline{\Omega})$-minimizer of μ. Using [26, Theorem 2.14, p. 7], we get

$$u_*\ \text{ is a local }\ W^{1,G}(\Omega)-\text{minimizer of }\ \mu. \tag{4.133}$$

Similarly, working with the functional μ_-, we show that

$$v_*\ \text{ is a local }\ W^{1,G}(\Omega)-\text{minimizer of }\ \mu. \tag{4.134}$$

Without lose of generality, we assume that $\mu(v_*)\le\mu(u_*)$. From (4.131), if K_μ is not finite, then we have an infinity of smooth nodal solutions and so we are done. So, we may assume that K_μ is finite. Then (4.133) and Theorem 5.7.6 of Papageorgiou–Rădulescu–Repŏvs [7, p. 449], we can find $\rho\in(0,1)$ small enough such that

$$\mu(v_*)\le\mu(u_*)<\inf\{\mu(u):\ \|u-u_*\|=\rho\}=m. \tag{4.135}$$

By (4.129), we have that μ is coercive. These implies that μ satisfies the C-condition, see [7, Proposition 5.1.15, p. 369]. This fact coupled with (4.135) permit the use of the mountain pass theorem. So, there is $y_0\in W^{1,G}(\Omega)$ such that

$$y_0\in K_\mu\subseteq[v_*,u_*]\cap C^1(\overline{\Omega})\ \text{ and }\ m\le\mu(y_0). \tag{4.136}$$

From (4.135) and (4.136), we conclude that $y_0\notin\{v_*,u_*\}$.

Using Corollary 6.6.9 of Papageorgiou–Rădulescu–Repovs [7, p. 533], we get that

$$C_1(\mu,y_0)\neq 0. \tag{4.137}$$

From (4.129), it is clear that $\mu\,|_{[v_*,u_*]}=J\,|_{[v_*,u_*]}$, from the homotopy invariance of critical groups and Proposition 4.6, we infer that

$$C_k(\mu,0)=C_k(J,0)=0\ \text{ for all }\ k\ge 0. \tag{4.138}$$

Comparing (4.137) and (4.138), we conclude that $y_0\neq 0$. Hence, y_0 is a sign changing solution for the problem (P). Combining this with the result of Theorem 1.4, we get our result.

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