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Singular non-autonomous (p,q)-equations with competing nonlinearities

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ABSTRACT

We consider a parametric non-autonomous (p,q)-equation with a singular term and competing nonlinearities, a parametric concave term and a Carathéodory perturbation. We consider the cases where the perturbation is $(p-1)$-linear and where it is $(p-1)$-superlinear (but without the use of the Ambrosetti–Rabinowitz condition). We prove an existence and multiplicity result which is global in the parameter $\lambda > 0$ (a bifurcation type result). Also, we show the existence of a smallest positive solution and show that it is strictly increasing as a function of the parameter. Finally, we examine the set of positive solutions as a function of the parameter (solution multifunction). First, we show that the solution set is compact in $C_0^1(\bar{\Omega})$ and then we show that the solution multifunction is Vietoris continuous and also Hausdorff continuous as a multifunction of the parameter.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following parametric nonlinear Dirichlet problem

$$\left\{ \begin{array}{l} -\Delta_p^{\alpha_1} u(z) - \Delta_q^{\alpha_2} u(z) = \lambda[u(z)^{-\eta} + u(z)^{\tau-1}] + f(z,u(z)) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, 1 < \tau < q < p, 0 < \eta < 1, \lambda > 0, u > 0. \end{array} \right\} \tag{1}$$

For $\alpha \in C^{0,1}(\bar{\Omega})$ with $\alpha(z) \geq \hat{c} > 0$ for all $z \in \bar{\Omega}$ and for $s \in (1,\infty)$, we denote by Δ_s^α the non-autonomous s-Laplace differential operator defined by

$$\Delta_s^\alpha u = \operatorname{div}(\alpha(z)|Du|^{s-2}Du) \text{ for all } u \in W_0^{1,s}(\Omega).$$

The features of problem (1) are the following:

(i) The presence of two non-autonomous differential operators with different growth, which generates a double phase associated energy.

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(ii) The problem combines the effects generated by a smooth nonlinearity, a singular reaction, and an unbalanced operator.
(iii) The analysis is developed with respect to the values of the positive parameter associated with the power-type and singular nonlinear terms.

Since the content of the paper is closely concerned with unbalanced growth, we briefly introduce in what follows the related background and applications and we recall some pioneering contributions to these fields. Eq. (1) is driven by a differential operator with unbalanced growth due to the presence of the (p,q)-Laplace type operator. This kind of problem comes from a general reaction–diffusion system:

$$u_t = \operatorname{div}[A(\nabla u)\nabla u] + c(x,u), \quad \text{and} \quad A(\nabla u) = |\nabla u|^{p-2} + |\nabla u|^{q-2},$$

where the function u is a state variable and describes the density or concentration of multicomponent substances, $\operatorname{div}[A(\nabla u)\nabla u]$ corresponds to the diffusion with coefficient $A(\nabla u)$ and $c(x,u)$ is the reaction and relates to source and loss processes. Originally, the idea to treat such operators comes from Zhikov [1] who introduced such classes to provide models of strongly anisotropic materials, see also the monograph of Zhikov et al. [2]. We refer to the remarkable works initiated by Marcellini [3,4], where the author investigated the regularity and existence of solutions of elliptic equations with unbalanced growth conditions. The (p,q)-Laplacian Eq. (1) is also motivated by numerous models arising in mathematical physics. For instance, we can refer to the following Born–Infeld equation [5] that appears in electromagnetism, electrostatics and electrodynamics as a model based on a modification of Maxwell's Lagrangian density:

$$-\operatorname{div}\left(\frac{\nabla u}{(1-2|\nabla u|^2)^{\frac{1}{2}}}\right) = h(u) \quad \text{in } \Omega.$$

Indeed, by the Taylor formula, we have

$$(1-x)^{-\frac{1}{2}} = 1 + \frac{x}{2} + \frac{3}{2\cdot 2^2}x^2 + \frac{5!!}{3!\cdot 2^3}x^3 + \cdots + \frac{(2n-3)!!}{(n-1)!\cdot 2^{n-1}}x^{n-1} + \cdots \quad \text{for } |x|<1.$$

Taking $x = 2|\nabla u|^2$ and adopting the first order approximation, we obtain problem (1) for $p=2$ and $q=4$. Furthermore, the nth order approximation problem is driven by the multi-phase differential operator

$$-\Delta u - \Delta_4 u - \frac{3}{2}\Delta_6 u - \cdots - \frac{(2n-3)!!}{(n-1)!}\Delta_{2n} u.$$

We also refer to the following fourth-order relativistic operator

$$u \mapsto \operatorname{div}\left(\frac{|\nabla u|^2}{(1-|\nabla u|^4)^{\frac{3}{4}}}\nabla u\right),$$

which describes large classes of phenomena arising in relativistic quantum mechanics. Again, by Taylor's formula, we have

$$x^2(1-x^4)^{-\frac{3}{4}} = x^2 + \frac{3x^6}{4} + \frac{21x^{10}}{32} + \cdots.$$

This shows that the fourth-order relativistic operator can be approximated by the following operator

$$u \mapsto \Delta_4 u + \frac{3}{4}\Delta_8 u.$$

For more details on the physical backgrounds and other applications, we refer to Bahrouni et al. [6](for phenomena associated with transonic flows) and to Benci et al. [7](for models arising in quantum physics).

Problem (1) is driven by the sum of two such operators with different exponents and weight functions. So, the differential operator in (1) is not homogeneous. In the reaction (right hand side) of (1) we have the combined effects of a parametric singular term $u \to \lambda u^{-\eta}$, of a parametric concave term $u \to \lambda u^{\tau-1}$ (recall that $\tau < q < p$) and of a perturbation $f(x,z)$. This perturbation is a Carathéodory function (that is, $z \to f(z,x)$ is measurable and $x \to f(z,x)$ is continuous) which is either $(p-1)$-linear or $(p-1)$-superlinear as $x \to +\infty$ (the second case corresponds to concave-convex nonlinearities). Our aim is to prove the existence and multiplicity of positive solutions and we want the result to be global in the parameter $\lambda > 0$. Problems with competition phenomena, but without singular term, were first studied by Ambrosetti, Brezis, Cerami [8], for semilinear equations driven by the Laplacian. Their work was extended to p-Laplacian equations by Garcia Azorero, Peral Alonso, Manfredi [9] and Guo, Zhang [10]. Further generalizations can be found in the works of Leonardi, Papageorgiou [11], Liu, Papageorgiou [12], Marano, Marino, Papageorgiou [13], Papageorgiou, Rădulescu, Repovs [14] and the references therein. None of the aforementioned works involves a singular term. Problems with singular terms and concave-convex nonlinearities, were examined recently by Papageorgiou, Winkert [15] and Gasinski, Papageorgiou [16]. A common feature in these two works, is that the perturbation $f(z,x)$ is nonnegative. This makes the analysis of problems easier. Also, their hypotheses on $f(z,\cdot)$ near zero are more restrictive. Here, in contrast $f(z,x)$ can change sign. We consider both the cases of $(p-1)$-linear and $(p-1)$-superlinear perturbation and our hypotheses on $f(z,\cdot)$ are more general. We prove an existence and multiplicity result which is global in $\lambda > 0$ (a bifurcation-type theorem).

We also show the existence of a minimal positive solution and determine its monotonicity properties with respect to the parameter. Finally, we examine the dependence of solution set on the parameter $\lambda > 0$. We prove the continuity properties of this solution multifunction. Our result in this direction extends the recent works of Zeng, Gasinski, Nguyen, Bai [17] and Papageorgiou, Scapellato [18] (nonsingular equations) and by Bai, Motreanu, Zeng [19] (singular problems driven by the p-Laplacian). In all these works, the perturbation $f(z,x)$ is nonnegative and the overall hypotheses are more restrictive.

2. Mathematical background and hypothesis

The main spaces in the analysis of problem (1), are the Sobolev space $W_0^{1,p}(\Omega)$ and the Banach space $C_0^1(\bar{\Omega}) = \{u \in C^1(\bar{\Omega}) : u|_{\partial\Omega} = 0\}$. On account of the Poincaré inequality, on $W_0^{1,p}(\Omega)$ we consider the following equivalent norm

$$\|u\| = \|Du\|_p \text{ for all } u \in W_0^{1,p}(\Omega).$$

The space $C_0^1(\bar{\Omega})$ is an ordered Banach space with positive (order) cone $C_+ = \{u \in C_0^1(\bar{\Omega}) : u(z) \geq 0 \text{ for all } z \in \bar{\Omega}\}$. This cone has a nonempty interior given by

$$\text{int}\, C_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega, \frac{\partial u}{\partial n}|_{\partial\Omega} < 0\},$$

where $\frac{\partial u}{\partial n} = (Du, n)_{\mathbb{R}^N}$ with $n(\cdot)$ being the outward unit normal on $\partial\Omega$.

If $u : \Omega \to \mathbb{R}$ is a measurable function, then we define

$$u^+(z) = \max\{u(z), 0\}, u^-(z) = \max\{-u(z), 0\} \text{ for all } z \in \Omega.$$

Both are measurable functions and $u = u^+ - u^-$, $|u| = u^+ + u^-$. Moreover, if $u \in W_0^{1,p}(\Omega)$, then $u^\pm \in W_0^{1,p}(\Omega)$. If $u, v : \Omega \to \mathbb{R}$ are measurable functions such that $u(z) \leq v(z)$ for a.a. $z \in \Omega$, then we define

$$[u, v] = \{h \in W_0^{1,p}(\Omega) : u(z) \leq h(z) \leq v(z) \text{ for a.a. } z \in \Omega\},$$

$$[u) = \{h \in W_0^{1,p}(\Omega) : u(z) \leq h(z) \text{ for a.a. } z \in \Omega\},$$

$$\text{int}_{C_0^1(\bar{\Omega})}[u, v] = \{\text{the interior in } C_0^1(\bar{\Omega}) \text{ of } [u, v] \cap C_0^1(\bar{\Omega})\}.$$

If X is a Banach space and $\varphi \in C^1(X, \mathbb{R})$, then by K_φ we denote the critical set of $\varphi(\cdot)$, that is, $K_\varphi = \{u \in X : \varphi'(u) = 0\}$. Also, if $u : \Omega \to \mathbb{R}$ is measurable, we write "$0 \prec u$" if for all $K \subseteq \Omega$ compact, we have $0 < c_K \leq u(z)$ for a.a. $z \in K$.

A useful tool in the study of singular boundary value problems, is the so-called "Hardy's inequality" which we recall next (see, for example, Papageorgiou, Rădulescu, Repovs [20, p. 66]).

Proposition 1. *If $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with Lipschitz boundary and $p \in (1, \infty)$, then $\|\frac{u}{\hat{d}}\|_p \leq c\|Du\|_p$ for all $u \in W_0^{1,p}(\Omega)$, some $c > 0$ and with $\hat{d}(z) = d(z, \partial\Omega)$ for all $z \in \Omega$.*

Conversely, we have

$$\text{“}u \in W^{1,p}(\Omega) \text{ and } \frac{u}{\hat{d}} \in L^p(\Omega) \Rightarrow u \in W_0^{1,p}(\Omega)\text{”}.$$

Let $\alpha \in C^{0,1}(\bar{\Omega})$ with $\alpha(z) \geq \hat{c} > 0$ for all $z \in \bar{\Omega}$ and $s \in (1, \infty)$. We consider the following nonlinear eigenvalue problem

$$-\Delta_s^\alpha u(z) = \hat{\lambda}|u(z)|^{s-2}u(z) \text{ in } \Omega, u|_{\partial\Omega} = 0. \tag{2}$$

This problem was studied by Liu, Papageorgiou [21] (see the Appendix of [21]). They proved that (2) has a smallest eigenvalue $\hat{\lambda}_1^\alpha(s) > 0$ which has the following variational characterization.

$$\hat{\lambda}_1^\alpha(s) = \inf\left\{\frac{\rho_{\alpha,s}(Du)}{\|u\|_s^s} : u \in W_0^{1,s}(\Omega), u \neq 0\right\} \text{ with } \rho_{\alpha,s}(Du) = \int_\Omega \alpha(z)|Du|^s \mathrm{d}x. \tag{3}$$

This eigenvalue is isolated in the spectrum of (2) and simple (that is, if $\hat{u}, \hat{v}$ are two eigenfunctions corresponding to $\hat{\lambda}_1^\alpha(s) > 0$, then $\hat{u} = \theta\hat{v}$ with $\theta \in \mathbb{R}\backslash\{0\}$). The infimum in (3) is realized on the corresponding one dimensional eigenspace. It is clear from (3) that the elements of this eigenspace, have fixed sign. By $\hat{u}_1(s)$ we denote the positive, $L^s(\Omega)$ normalized (that is, $\|\hat{u}_1(s)\|_s = 1$) eigenfunction corresponding to $\hat{\lambda}_1^\alpha(s)$. The nonlinear regularity theory of Lieberman [22] and the nonlinear maximum principle of Pucci, Serrin [23] (pp. 111, 120), imply that $\hat{u}_1(\cdot) \in \text{int}\, C_+$. Using these properties we obtain the following useful inequality (see Liu, Papageorgiou [21], Proposition 4.2).

Proposition 2. *If $\theta \in L^\infty(\Omega)$, $\theta(z) \leq \hat{\lambda}_1^\alpha$ for a.a. $z \in \Omega$, $\theta \not\equiv \hat{\lambda}_1^\alpha(s)$, then there exists $c^* > 0$ such that*

$$c^*\|Du\|_s^s \leq \int_\Omega \alpha(z)|Du|^s \mathrm{d}z - \int_\Omega \theta(z)|u|^s \mathrm{d}z \text{ for all } u \in W_0^{1,s}(\Omega).$$

We mention that $\hat{\lambda}_1^\alpha(s) > 0$ is the only eigenvalue with eigenfunctions of constant sign. We will also encounter a weighted version of (2). So, let $\eta \in L^\infty(\Omega)\backslash\{0\}$, $\eta(z) \geq 0$ for a.a. $z \in \Omega$ and consider the following nonlinear eigenvalue problem

$$-\Delta_s^\alpha u(z) = \tilde{\lambda}\eta(z)|u(z)|^{s-2}u(z) \text{ in } \Omega, u|_{\partial\Omega} = 0.$$

For this problem, we have the same results as for (2). So, there is a smallest eigenvalue $\tilde{\lambda}_1^\alpha(s, \eta) > 0$ which has the same properties as $\hat{\lambda}_1^\alpha(s)$. Moreover, $\tilde{\lambda}_1^\alpha(s, \eta)$ has the following monotonicity property with respect to the weight η. If $\eta, \tilde{\eta} \in L^\infty(\Omega)\backslash\{0\}$, $0 \leq \eta(z) \leq \tilde{\eta}(z)$ for a.a. $z \in \Omega$, $\eta \neq \tilde{\eta}$, then

$$\tilde{\lambda}_1^\alpha(s, \tilde{\eta}) < \tilde{\lambda}_1^\alpha(s, \eta).$$

Let $A_s^{\alpha} : W_0^{1,s}(\Omega) \to W^{-1,s'}(\Omega) = W_0^{1,s}(\Omega)^*$ $(\frac{1}{s} + \frac{1}{s'} = 1)$ be the nonlinear operator defined by

$$\langle A_s^{\alpha}(u), h\rangle = \int_{\Omega} \alpha(z)|Du|^{s-2}(Du, Dh)_{\mathbb{R}^N}\,dz \text{ for all } u, h \in W_0^{1,s}(\Omega).$$

This operator has the following properties (see, for example, Gasinski, Papageorgiou [24], p. 279).

Proposition 3. *The operator $A_s^{\alpha} : W_0^{1,s}(\Omega) \to W^{-1,s'}(\Omega)$ is bounded (that is, maps bounded sets to bounded sets), continuous, strictly monotone (thus, maximal monotone too) and of type $(S)_+$, that is,*

"if $u_n \xrightarrow{w} u$ in $W_0^{1,s}(\Omega)$ and $\limsup_{n\to\infty}\langle A_s^{\alpha}(u_n), u_n - u\rangle \le 0$,

then $u_n \to u$ in $W_0^{1,s}(\Omega)$."

Our hypotheses on the weight functions and the exponents are the following:

H_0 : $\alpha_1, \alpha_2 \in C^{0,1}(\bar{\Omega}), 0 < \hat{c} \le \alpha_1(z), \alpha_2(z)$ for all $z \in \bar{\Omega}$ and $0 < \eta < 1 < \tau < q < p$.

We set $V(u) = A_p^{\alpha_1}(u) + A_p^{\alpha_2}(u)$ for all $u \in W_0^{1,p}(\Omega)$. Then on account of Proposition 3, $V : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^*$ $\left(\frac{1}{p} + \frac{1}{p'} = 1\right)$ is bounded, continuous, strictly monotone (thus maximal monotone too) and of type $(S)_+$.

In the last section, we will study the dependence of the solution set of (1) on the parameter $\lambda > 0$. For this purpose, we will need some continuity notions from multivalued analysis, which we recall below. For more details we refer to Hu, Papageorgiou [25].

So, let X, Y be Hausdorff topological spaces and $S : X \to 2^Y\setminus\{\emptyset\}$ a multifunction (set-valued function).

- $S(\cdot)$ is "lower semicontinuous" (lsc for short), if for every $U \subseteq Y$ open the set $S^-(U) = \{x \in X : S(x) \cap U \neq \emptyset\}$ is open.
- $S(\cdot)$ is "upper semicontinuous" (usc for short), if for all $U \subseteq Y$ open, the set $S^+(U) = \{x \in X : S(x) \subseteq U\}$ is open.

Suppose Y is a metric space and let $d(\cdot,\cdot)$ be its metric. For $A, C \subseteq Y$ nonempty sets, we define

$$\begin{aligned} h^*(A, C) &= \sup\{d(a, C) : a \in A\} \\ &= \inf\{\epsilon > 0 : A \subseteq C_\epsilon\}, \end{aligned}$$

where $C_\epsilon = \{x \in X : d(x, C) < \epsilon\}$ (the ϵ-enlargement of C) . It is easy to see that

$$h^*(A, C) = \sup\{d(x, C) \cdot d(x, A) : x \in X\}.$$

The "Hausdorff distance" between A and C is defined by

$$\begin{aligned} h(A, C) &= \max\{h^*(A, C), h^*(C, A)\} \\ &= \inf\{\epsilon > 0 : A \subseteq C_\epsilon \text{ and } C \subseteq A_\epsilon\}. \end{aligned}$$

If follows that

$$h(A, C) = \sup\{|d(x, C) - d(x, A)| : x \in X\}.$$

Let $P_f(Y)$ (resp. $P_k(Y)$) denote the family of nonempty, closed (resp. compact) subsets of Y. We know that $h(\cdot,\cdot)$ is a (generalized) metric on $P_f(Y)$ and if Y is complete, then so is $(P_f(Y), h)$. Let $s : X \to 2^Y\setminus\{\emptyset\}$ be a multifunction (Y a metric space)

- $S(\cdot)$ is "h-lower semicontinuous" (h-lsc for short), if for all $x \in X, u \to h^*(S(x), S(u))$ is continuous on X.
- $S(\cdot)$ is "h-upper semicontinuous" (h-usc for short), if for all $x \in X, u \to h^*(S(u), S(x))$ is continuous on X.

In general we have

"$h - lsc \Longrightarrow lsc$ and $usc \Longrightarrow h - usc$".

If $S(\cdot)$ is $P_k(Y)$-valued, then

"$h - lsc \Longleftrightarrow lsc$ and $h - usc \Longrightarrow usc$".

A multifunction $S(\cdot)$ which is both lsc and usc, is said to be continuous (or Vietoris continuous). A multifunction $S(\cdot)$ which is both h-lsc and h-usc, is said to be h-continuous (or Hausdorff continuous). From the previous remarks, we see that a $P_k(Y)$-valued multifunction. $S(\cdot)$ is continuous if and only if it is h-continuous.

If (Y, d) is a metric space and $\{C_n\}_{n\in\mathbb{N}} \subseteq 2^Y\setminus\{\emptyset\}$, then we define

$$\begin{aligned} \liminf_{n\to\infty} C_n &= \{y \in Y : y = \lim y_n, y_n \in C_n, n \in \mathbb{N}\} \\ &= \{y \in Y : \lim_{n\to\infty} d(y, C_n) = 0\}. \end{aligned}$$

We say that $S : X \to 2^Y\setminus\{\emptyset\}$ is locally compact, if for every $x \in X$, we can find U an open neighborhood of x such that $\overline{S(U)}$ is compact in Y.

A set $D \subseteq W_0^{1,p}(\Omega)$ is said to be downward directed, if given $u_1, u_2 \in D$ we can find $u \in D$ such that $u \leq u_1, u \leq u_2$.

Next we introduce our hypotheses on the perturbation $f(z,x)$. As we already mentioned in the introduction, we will present a unified treatment of both the $(p-1)$-linear and of the $(p-1)$-superlinear cases.

For the $(p-1)$-linear case the hypotheses on the perturbation $f(z,x)$ are the following:

$H_1 : f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

(i) for every $\rho > 0$, there exists $\hat{\alpha}_\rho \in L^\infty(\Omega)$ such that

$$|f(z,x)| \leq \hat{\alpha}_\rho(z) \text{ for a.a. } z \in \Omega, \text{all } 0 \leq x \leq \rho;$$

(ii) there exist functions $\eta, \hat{\eta} \in L^\infty \Omega$ such that

$$\hat{\lambda}_1^{\alpha_1}(p) \leq \eta(z) \text{ for a.a. } z \in \Omega, \eta \not\equiv \hat{\lambda}_1^{\alpha}(\rho),$$

$$\eta(z) \leq \liminf_{x\to+\infty} \frac{f(z,x)}{x^{p-1}} \leq \limsup_{x\to+\infty} \frac{f(z,x)}{x^{p-1}} \leq \hat{\eta}(z) \text{ uniformly for a.a. } z \in \Omega,$$

(iii) there exist a function $\theta \in L^\infty(\Omega)$ and $\delta > 0$ such that

$$\theta(z) \leq \hat{\lambda}_1^{\alpha_2}(q) \text{ for a.a. } z \in \Omega, \theta \not\equiv \hat{\lambda}_1^{\alpha_2}(q).$$

$$\limsup_{x\to 0^+} \frac{f(z,x)}{x^{q-1}} \leq \theta(z) \text{ uniformly for a.a. } z \in \Omega,$$

$$f(z,x) \geq 0 \text{ for a.a. } z \in \Omega, \text{ all } 0 \leq x \leq \delta;$$

(iv) for every $\rho > 0$, there exists $\hat{\xi}_\rho > 0$, such that for a.a. $z \in \Omega$, the function

$$x \to f(z,x) + \hat{\xi}_\rho x^{p-1}$$

is nondecreasing on $[0,\rho]$.

Remark 1. Note that hypothesis $H_1(iii)$ implies that $f(z,0) = 0$ for a.a. $z \in \Omega$. Also since we search for positive solutions and the above hypotheses concern the positive semiaxis, we may assume without any the loss of generality that $f(z,x) = 0$ for a.a. $z \in \Omega$, all $x \leq 0$.

For the $(p-1)$-superlinear case, the hypotheses on the perturbation $f(z,x)$ are the following:

$H_1' : f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

(i) $|f(z,x)| \leq \hat{\alpha}(z)(1 + x^{r-1})$ for a.a. $z \in \Omega$, all $x \geq 0$, with $\hat{\alpha} \in L^\infty(\Omega)$, $p < r < p^*$ (recall that $p^* = \frac{Np}{N-p}$ if $p < N$ and $p^* = +\infty$ if $N \leq p$);

(ii) If $F(z,x) = \int_0^x f(z,s)\mathrm{d}s$, then

$$\lim_{x\to+\infty} \frac{F(z,x)}{x^p} = +\infty \text{ uniformy for a.a. } z \in \Omega;$$

and there exists $s \in \left((r-p)\max\left\{\frac{N}{P}, 1\right\}, p^*\right)$ such that

$$0 < \hat{\beta} \leq \liminf_{x\to+\infty} \frac{f(z,x)x - pF(z,x)}{x^s} \text{ uniformly for a.a. } z \in \Omega;$$

(iii) same as hypothesis $H_1(iii)$;

(iv) same as hypothesis $H_1(iv)$.

Remark 2. Hypothesis $H_1'(ii)$ implies that

$$\lim_{x\to+\infty} \frac{f(z,x)}{x^{p-1}} = +\infty \text{ uniformly for a.a. } z \in \Omega.$$

So, the perturbation $f(z,\cdot)$ is $(p-1)$-superlinear. However, we do not employ the Ambrosetti–Rabinowitz (the AR-condition for short), which is common in the literature when dealing with superlinear problems (see Willem [26], p. 46). Hypothesis $H_1'(ii)$ is less restrictive and incorporates in our framework superlinear nonlinearities with "slower" growth near $+\infty$ which fail to satisfy the AR-condition (see the examples below).

Example 1. For the sake of simplicity, we drop the z-dependence. Consider the following function

$$f_1(x) = \begin{cases} \theta(x^+)^{q-1} - (x^+)^{r-1} & \text{if } x \leq 1, \\ \eta x^{p-1} + cx^{s-1} & \text{if } 1 < x, \end{cases}$$

with $\theta < \hat{\lambda}_1^{\alpha_2}(q)$, $\eta > \hat{\lambda}_1^{\alpha_1}(p)$, $c = \eta + 1 - \theta$ and $r > q$, $1 < s < p$. This function satisfies hypotheses H_1. Also consider the function

$$f_2(x) = \begin{cases} \theta(x^+)^{p-1} & \text{if } x \le 1, \\ x^{p-1}\ln x + \theta x^{s-1} & \text{if } 1 < x, \end{cases}$$

with $\theta < \hat{\lambda}_1^{\alpha_2}(q)$, and $1 < s \le p$. This function satisfies hypotheses H_1', but fails to satisfy the AR-condition.

3. An auxiliary problem

When dealing with singular equations, the problem that we face is that due to the presence of the singular term, the energy functional of the problem is not C^1 and so we cannot use the minimax results of the critical point theory. We have to find a way to bypass the singularity and deal with C^1-functionals. We will be able to do this using the solution of the following auxiliary Dirichlet problem:

$$\left\{ \begin{array}{l} -\Delta_p^{\alpha_1} u(z) - \Delta_q^{\alpha_2} u(z) = \lambda u(z)^{\tau-1} \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, 1 < \tau < q < p, \lambda > 0, u > 0. \end{array} \right\} \tag{4}$$

For this problem, we have the following result.

Proposition 4. *If hypotheses H_0 hold and $\lambda > 0$, then problem (4) has a unique positive solution $\bar{u}_\lambda \in \operatorname{int} C_+$, $\{\bar{u}_\lambda\}_{\lambda>0}$ is nondecreasing and $\bar{u}_\lambda \to 0$ in $C_0^1(\bar{\Omega})$ as $\lambda \to 0^+$.*

Proof. Consider the C^1-functional $\hat{\sigma}_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\sigma}_\lambda(u) = \frac{1}{p}\rho_{\alpha_1,p}(Du) + \frac{1}{q}\rho_{\alpha_2,q}(Du) - \frac{\lambda}{\tau}\|u^+\|_\tau^\tau \text{ for all } u \in W_0^{1,p}(\Omega).$$

Evidently, $\hat{\sigma}(\cdot)$ is coercive (recall that $\tau < q < p$). Also, using the Sobolev embedding theorem, we see that $\hat{\sigma}(\cdot)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass–Tonelli theorem, we can find $\bar{u}_\lambda \in W_0^{1,p}(\Omega)$ such that

$$\hat{\sigma}(\bar{u}_\lambda) = \inf\{\sigma_\lambda(u) : u \in W_0^{1,p}(\Omega)\}. \tag{5}$$

Let $u \in \operatorname{int} C_+$ and $t \in (0,1)$. Then

$$\begin{aligned} \hat{\sigma}_\lambda(tu) &= \frac{t^p}{\rho}p_{\alpha_1,p}(Du) + \frac{t^q}{\rho}p_{\alpha_2,q}(Du) - \frac{t^\tau}{\tau}\|u\|_\tau^\tau \\ &\le c_1 t^q - c_2 t^\tau \text{ for some } c_1, c_2 > 0 \text{ (recall that } 0 < t < 1, q < p). \end{aligned}$$

But $\tau < q$. So, choosing $t \in (0,1)$ even smaller if necessary, we have

$$\begin{aligned} &\hat{\sigma}_\lambda(tu) < 0, \\ &\Rightarrow \hat{\sigma}_\lambda(\bar{u}_\lambda) < 0 = \hat{\sigma}_\lambda(0) \text{ (see (5))}, \\ &\Rightarrow \bar{u}_\lambda \ne 0. \end{aligned}$$

From (5) we have

$$\begin{aligned} &\langle \hat{\sigma}_\lambda'(\bar{u}_\lambda), h\rangle = 0 \text{ for all } h \in W_0^{1,p}(\Omega), \\ &\Rightarrow \langle V(\bar{u}_\lambda), h\rangle = \int_\Omega \lambda(\bar{u}_\lambda')^{\tau-1} h dz \text{ for all } h \in W_0^{1,p}(\Omega). \end{aligned} \tag{6}$$

Choosing $h = -\bar{u}_\lambda^- \in W_0^{1,p}(\Omega)$ in (6) and using hypotheses H_0, we obtain

$$\begin{aligned} &\hat{c}\|D\bar{u}_\lambda^-\|_p^p \le 0, \\ &\Rightarrow \bar{u}_\lambda \ge 0, \bar{u}_\lambda \ne 0. \end{aligned}$$

Then from (6) we infer that $\bar{u}_\lambda$ is a positive solution of problem (4). By a standard Moser iteration process we show that $\bar{u}_\lambda \in L^\infty(\Omega)$ (see [27]). So, we can apply the nonlinear regularity theory of Lieberman [22] and have that $\bar{u}_\lambda \in C_+\backslash\{0\}$. We have (see Pucci, Serrin [23], pp. 111, 120)

$$\begin{aligned} &\Delta_p^{\alpha_1}\bar{u}_\lambda + \Delta_q^{\alpha_2}\bar{u}_\lambda \le 0 \text{ in } \Omega, \\ &\Rightarrow \bar{u}_\lambda \in \operatorname{int} C_+. \end{aligned}$$

We show that this positive solution of (4) is unique. Suppose $\bar{v}_\lambda \in W_0^{1,p}(\Omega)$ is another positive solution of (4). Again we have $\bar{v}_\lambda \in \operatorname{int} C_+$ and so using Proposition 4.1.22, p. 274, of Papageorgiou, Rădulescu, Repovs [20], we have

$$\frac{\bar{u}_\lambda}{\bar{v}_\lambda} \in L^\infty(\Omega) \text{ and } \frac{\bar{v}_\lambda}{\bar{u}_\lambda} \in L^\infty(\Omega).$$

Using the Diaz-Saa inequality (see Diaz, Saa [28] and also Papageorgiou, Rădulescu [29], proof of Proposition 3.5), we have

$$0 \le \int_\Omega \left[\frac{1}{\bar{u}_\lambda^{q-\tau}} - \frac{1}{\bar{v}_\lambda^{q-\tau}}\right](\bar{u}_\lambda^\tau - \bar{v}_\lambda^\tau)\mathrm{d}z \le 0,$$
$$\Rightarrow \bar{u}_\lambda = \bar{v}_\lambda.$$

This proves the uniqueness of the positive solution of (4).

Next, we show that the family $\{\bar{u}_\lambda\}_{\lambda>0}$ is nondecreasing. Suppose $0 < \psi < \lambda$. We introduce the Carathéodory function $\ell_p(z,x)$ defined by

$$\ell_\psi(z,x) = \begin{cases} \psi(x^+)^{\tau-1} & \text{if } x \le \bar{u}_\lambda(z), \\ \psi\bar{u}_\lambda(z)^{\tau-1} & \text{if } \bar{u}_\lambda(z) < x. \end{cases} \tag{7}$$

Let $L_\psi(z,x) = \int_0^x \ell_\psi(z,s)\mathrm{d}s$ and consider the C^1-functional $\sigma_\psi : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\sigma_\psi(u) = \frac{1}{p}\rho_{\alpha_1,p}(Du) + \frac{1}{q}\rho_{\alpha_2,q}(Du) - \int_\Omega L_\psi(z,u)\mathrm{d}z \text{ for all } u \in W_0^{1,p}(\Omega).$$

As before, using the Weierstrass–Tonelli theorem, we can find $\tilde{u}_\psi \in W_0^{1,p}(\Omega)$ such that

$$\sigma_\psi(\tilde{u}_\psi) = \inf\{\sigma_\psi(u) : u \in W_0^{1,p}(\Omega)\}. \tag{8}$$

Let $u \in \operatorname{int} C_+$ and choose $t \in (0,1)$ small so that $tu \le \bar{u}_\lambda$ (recall that $\bar{u}_\lambda \in \operatorname{int} C_+$ and use Proposition 4.1.22, p. 274, of [20] to see that such a $t \in (0,1)$ exists). Then as before since $\tau < q < p$, by taking $t \in (0,1)$ even small if necessary we can have

$$\sigma_\psi(tu) < 0,$$
$$\Rightarrow \sigma_\psi(\tilde{u}_\psi) < 0 = \sigma_\psi(0) \text{ (see (8))},$$
$$\Rightarrow \tilde{u}_\psi \ne 0.$$

From (8) we have

$$\begin{aligned} &\langle \sigma'_\psi(\tilde{u}_\psi), h\rangle = 0 \text{ for all } h \in W_0^{1,p}(\Omega), \\ &\Rightarrow \langle V(\tilde{u}_\psi), h\rangle = \int_\Omega \ell_\psi(z,\tilde{u}_\psi)h\mathrm{d}z \text{ for all } h \in W_0^{1,p}(\Omega). \end{aligned} \tag{9}$$

In (9) we choose $h = -\tilde{u}_\psi \in W_0^{1,p}(\Omega)$ and obtain

$$\hat{c}\|D\tilde{u}_\psi\|_p^p \le 0,$$
$$\Rightarrow \tilde{u}_\psi \ge 0, \tilde{u}_\psi \ne 0.$$

Also in (9) we use the test function $h = (\tilde{u}_\psi - u_\lambda)^+ \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned} \langle V(\tilde{u}_\psi), (\tilde{u}_\psi - \bar{u}_\lambda)^+\rangle &= \int_\Omega \psi\bar{u}_\lambda^{\tau-1}(\tilde{u}_\psi - \bar{u}_\lambda)^+\mathrm{d}z \text{ (see (7))} \\ &\le \int_\Omega \lambda\bar{u}_\lambda^{\tau-1}(\tilde{u}_\psi - \bar{u}_\lambda)^+\mathrm{d}z \text{ (since } \psi < \lambda) \\ &= \langle V(\bar{u}_\lambda), (\tilde{u}_\psi - \bar{u}_\lambda)^+\rangle, \\ &\Rightarrow \langle V(\tilde{u}_\psi) - V(\bar{u}_\lambda), (\tilde{u}_\psi - \bar{u}_\lambda)^+\rangle \le 0, \\ &\Rightarrow \tilde{u}_\psi \le \bar{u}_\lambda. \end{aligned}$$

So, we have proved that

$$\tilde{u}_\psi \in [0,\bar{u}_\lambda], \tilde{u}_\psi \ne 0. \tag{10}$$

Then from (10), (7) and (9), we see that $\tilde{u}_\psi$ is a positive solution of (4). Hence $\tilde{u}_\psi = \bar{u}_\psi$ and we infer that

$$\bar{u}_\psi \le \bar{u}_\lambda \text{ (see (8))},$$
$$\Rightarrow \{\bar{u}_\lambda\}_{\lambda>0} \text{ is nondecreasing.}$$

Finally, we show that $\bar{u}_\lambda \to 0$ in $C_0^1(\bar{\Omega})$ as $\lambda \to 0^+$. We have

$$0 \le \bar{u}_\lambda \le \bar{u}_1 \text{ for all } 0 < \lambda \le 1.$$

It follows that

$$\hat{c}\|D\bar{u}_\lambda\|_p^p \le \lambda c_3\|\bar{u}_\lambda\| \text{ for all } 0 < \lambda \le 1, \text{some } c_3 > 0 \text{ (see (7))},$$

$$\Rightarrow \bar{u}_\lambda \to 0 \text{ in } W_0^{1,p}(\Omega) \text{ as } \lambda \to 0^+. \tag{11}$$

Moreover, from the nonlinear regularity theory of Lieberman [22], we know that there exist $\alpha \in (0,1)$ and $c_4 > 0$ such that

$$\bar{u}_\lambda \in C_0^{1,\alpha}(\bar{\Omega}) = C^{1,\alpha}(\bar{\Omega}) \cap C_0^1(\bar{\Omega}), \|\bar{u}_\lambda\|_{C_0^{1,\alpha}(\bar{\Omega})} \le c_4 \text{ for all } 0 < \lambda \le 1. \tag{12}$$

But recall that $C_0^{1,\alpha}(\bar{\Omega}) \hookrightarrow C_0^1(\bar{\Omega})$ compactly (Arzela–Ascoli theorem). So, from (11) and (12), we conclude that

$$\bar{u}_\lambda \to 0 \text{ in } C_0^1(\bar{\Omega}) \text{ as } \lambda \to 0^+.$$

The proof is now complete. □

Remark 3. Although we will not need it in the sequel, we mention that we can improve the monotonicity of $\{\bar{u}_\lambda\}_{\lambda>0}$ and assert that $\{\bar{u}_\lambda\}_{\lambda>0}$ is strictly increasing, that is,

$$\text{“}0 < \psi < \lambda \Longrightarrow \bar{u}_\lambda - \bar{u}_\psi \in \operatorname{int} C_+\text{”}.$$

We already know that $\bar{u}_\psi \le \bar{u}_\lambda$. Then (see Gasinski, Papageorgiou [30], Proposition 3.4)

$$\begin{aligned} -\Delta_p^{\alpha_1}\bar{u}_\psi - \Delta_q^{\alpha_2}\bar{u}_\psi &= \psi\bar{u}_\psi^{\tau-1} \\ &\le \lambda\bar{u}_\lambda^{\tau-1} = -\Delta_p^{\alpha_1}\bar{u}_\lambda - \Delta_q^{\alpha_2}\bar{u}_\lambda \text{ in } \Omega \end{aligned}$$

$$\Rightarrow \bar{u}_\lambda - \bar{u}_\psi \in \operatorname{int} C_+ .$$

We will use this solution $\bar{u}_\lambda \in \operatorname{int} C_+$ to “neutralize” the singularity.

4. Positive solutions

We introduce the following two sets

$$\mathcal{L} = \{\lambda > 0 : \text{problem (1) has a positive solution}\},$$

$$S_\lambda = \text{set of positive solutions for problem (1).}$$

Using Proposition 4, we see that for $\psi \in (0,1)$ small, we have

$$0 \le \bar{u}_\psi(z) \le \delta \text{ for all } z \in \bar{\Omega} \tag{13}$$

(here $\delta > 0$ is as in hypothesis $H_1(iii) = H_1'(iii)$),

Proposition 5. *If hypotheses H_0, H_1 hold, then $\mathcal{L} \neq \emptyset$ and for every $\lambda \in \mathcal{L}$, $\emptyset \neq S_\lambda \subseteq \operatorname{int} C_+$.*

Proof. Using $\bar{u}_\psi \in \operatorname{int} C_+$ from (13), we introduce the Carathéodory function $k_\lambda(z,x)$ defined by

$$k_\lambda(z,x) = \begin{cases} \lambda(\bar{u}_\psi(z)^{-\eta} + \bar{u}_\psi(z)^{\tau-1}) + f(z,x^+) & \text{if } x < \bar{u}_\psi(z), \\ \lambda(x^{-\eta} + x^{\tau-1}) + f(z,x) & \text{if } \bar{u}_\psi(z) < x. \end{cases} \tag{14}$$

We set $K_\lambda(z,x) = \int_0^x k_\lambda(z,s)\mathrm{d}s$ and consider the functional $\psi_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_\lambda(u) = \frac{1}{p}\rho_{\alpha_1,p}(Du) + \frac{1}{q}\rho_{\alpha_2,q}(Du) - \int_\Omega K_\lambda(z,u)\mathrm{d}z \text{ for all } u \in W_0^{1,p}(\Omega).$$

We know that $\psi_\lambda \in C^1(W_0^{1,p}(\Omega))$ (see Papageorgiou, Smyrlis ([31], Proposition 3)).

Claim 1. *For every $\lambda > 0$, the functional $\psi_\lambda(\cdot)$ satisfies the C-condition.*

We consider a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ such that $\{\psi_\lambda(u_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded and $(1+\|u_n\|\psi_\lambda'(u_n) \to 0$ in $W^{-1,p'(\Omega)} = W_0^{1,p}(\Omega)^*$ as $n \to \infty$. We have

$$\langle \psi_\lambda'(u_n), h\rangle \le \frac{\epsilon_n\|h\|}{1+\|u_n\|} \text{ for all } h \in W_0^{1,p}(\Omega), \text{all } n \in \mathbb{N}, \text{with } \epsilon_n \to 0^+. \tag{15}$$

In (15) we use the test function $h = -u_n^- \in W_0^{1,p}(\Omega)$. From (14) and hypotheses H_0, we have

$$\hat{c}\|Du_n^-\|_p^p \le \epsilon_n \text{ for all } n \in \mathbb{N},$$

$$\Rightarrow u_n^- \to 0 \text{ in } W_0^{1,p}(\Omega) \text{ as } n \to \infty. \tag{16}$$

We will show that $\{u_n^+\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ is bounded. If this is not true, then we may assume that $\|u_n^+\| \to \infty$. Let $y_n = \frac{u_n^+}{\|u_n^+\|}$ $n \in \mathbb{N}$. Then $\|y_n\| = 1$, $y_n \geq 0$ for all $n \in \mathbb{N}$. So, we may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,p}(\Omega),\ y_n \to y \text{ in } L^p(\Omega). \tag{17}$$

From (15), we have

$$|\langle A_p^{\alpha_1}(u_n), h\rangle + \langle A_p^{\alpha_2}(u_n), h\rangle - \int_\Omega k_\lambda(z, u_n)h\mathrm{d}z| \leq \frac{\epsilon_n\|h\|}{1+\|u_n\|} \text{ for all } h \in W_0^{1,p}(\Omega), \text{all } n \in \mathbb{N},$$

$$\begin{aligned} \Rightarrow &|\langle A_p^{\alpha_1}(y_n), h\rangle + \frac{1}{\|u_n^+\|^{p-q}}\langle A_q^{\alpha_2}(y_n), h\rangle - \int_{\{u_n^+\leq \bar{u}_\psi\}} \frac{\lambda[\bar{u}_\psi^{-\eta} + \bar{u}_\psi^{\tau-1}]}{\|u_n^+\|^{p-1}} h\mathrm{d}z \\ &- \int_{\{\bar{u}_\psi < u_n^+\}} \frac{\lambda[(u_n^+)^{-\eta} + (u_n^+)^{\tau-1}]}{\|u_n^+\|^{p-1}} h\mathrm{d}z - \int_\Omega F(z, u_n^+)\mathrm{d}z| \leq \epsilon_n'\|h\| \end{aligned} \tag{18}$$

for all $h \in W_0^{1,p}(\Omega)$, all $n \in \mathbb{N}$, with $\epsilon_n' \to 0^+$ (see (14), (16)).

Since $\bar{u}_\psi \in \operatorname{int} C_+$, we can find $c_5 > 0$ such that $c_5\hat{d} \leq \bar{u}_\psi$ (see Guo, Webb [32]). We have

$$\begin{aligned} \int_{\{u_n^+\leq \bar{u}_\psi\}} \frac{\lambda|h|}{\bar{u}_\psi^\eta}\mathrm{d}z &\leq \int_\Omega \frac{\lambda|h|}{\bar{u}_\psi^\eta}\mathrm{d}z \\ &\leq \lambda \int_\Omega \bar{u}_\psi^{1-\eta}\frac{|h|}{\bar{u}_\psi}\mathrm{d}z \\ &\leq \lambda c_6 \int_\Omega \frac{|h|}{\bar{u}_\psi}\mathrm{d}z \text{ for some } c_6 > 0 \text{ (recall } \bar{u}_\psi \in \operatorname{int} C_+) \\ &\leq \lambda \frac{c_6}{c_5}\int_\Omega \frac{|h|}{\hat{d}}\mathrm{d}z \\ &\leq \lambda c_7\|Dh\|_p \text{ for some } c_7 > 0 \text{ (see Proposition 1, Hardy's inequality)} \\ \Rightarrow \frac{1}{\|u_n^+\|^{p-1}} &\int_{\{u_n^+\leq \bar{u}_\psi\}} \frac{\lambda|h|}{\bar{u}_\psi^\eta}\mathrm{d}z \to 0 \text{ as } n \to \infty. \end{aligned} \tag{19}$$

Also

$$\begin{aligned} \frac{1}{\|u_n^+\|^{p-1}} \int_{\{\bar{u}_\psi < u_n^+\}} \frac{\lambda|h|}{(u_n^+)^\eta}\mathrm{d}z &\leq \frac{1}{\|u_n^+\|^{p-1}} \int_{\{\bar{u}_\psi < u_n^+\}} \frac{\lambda|h|}{\bar{u}_\psi^\eta}\mathrm{d}z \\ &\leq \frac{1}{\|u_n^+\|^{p-1}} \int_\Omega \frac{\lambda|h|}{\bar{u}_\psi^\eta}\mathrm{d}z \to 0 \text{ as } n \to \infty. \end{aligned} \tag{20}$$

In addition, note that

$$\frac{1}{\|u_n^+\|^{p-1}} \int_{\{u_n^+\leq \bar{u}_\psi\}} \lambda\bar{u}_\psi^{\tau-1}h\mathrm{d}z \to 0 \text{ as } n \to \infty, \tag{21}$$

$$\frac{1}{\|u_n^+\|^{p-1}} \int_{\{\bar{u}_\psi < u_n^+\}} \lambda(u_n^+)_\psi^{\tau-1}h\mathrm{d}z \to 0 \text{ as } n \to \infty. \tag{22}$$

Hypotheses $H_1(i), (ii)$ imply that

$$|f(z,x)| \leq \hat{a}_0(z)(1 + x^{p-1})| \text{ for a.a. } z \in \Omega, \text{all } x \geq 0, \text{with } \hat{a}_0 \in L^\infty(\Omega).$$

If follows that

$$\left\{\frac{f(\cdot, u_n^+(\cdot))}{\|u_n^+\|^{p-1}}\right\}_{n\in\mathbb{N}} \subseteq L^{p'}(\Omega) \text{ is bounded.} \tag{23}$$

If in (18) we choose the test function $h = y_n - y \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (19) → (23), we obtain

$$\begin{aligned} &\lim_{n\to\infty} \langle A_p^{\alpha_1}(y_n), y_n - y\rangle = 0 \text{ (recall } q < p, \|u_n^+\| \to \infty) \\ &\Rightarrow y_n \to y \text{ in } W_0^{1,p}(\Omega), \text{hence } \|y\| = 1, y \geq 0 \text{ (see Proposition 3)}. \end{aligned} \tag{24}$$

From (23) and hypotheses $H_1(ii)$, we see that we may assume that (see Aizicovici, Papageorgiou, Staicu [33], proof of Proposition 16)

$$\frac{f(\cdot, u_n^+(\cdot))}{\|u_n^+\|^{p-1}} \xrightarrow{w} \eta_*(\cdot)y^{p-1} \text{ in } L^{p'}(\Omega) \text{ with } \eta(z) \leq \eta_*(z) \leq \hat{\eta}(z) \text{ for a.a. } z \in \Omega, \tag{25}$$

Note that on $\{y > 0\}$ we have $u_n^+(z) \to +\infty$ and $\Omega = \{y > 0\} \cup \{y = 0\}$ (see (24)). Therefore if in (18) we pass to the limit as $n \to \infty$ and use (19) → (22), (24) and (25), we obtain

$$\begin{aligned} &\langle A_p^{\alpha_1}(y), h\rangle = \int_\Omega \eta_*(z)y^{p-1}h\mathrm{d}z \text{ for all } W_0^{1,p}(\Omega), \\ &\Rightarrow -\Delta_p^\alpha y(z) = \eta_*(z)y(z)^{p-1} \text{ in } \Omega, y|_{\partial\Omega} = 0. \end{aligned} \tag{26}$$

We know that

$$\tilde{\lambda}_1^{\alpha}(p,\eta_*) < \tilde{\lambda}_1^{\alpha}(p,\hat{\lambda}_1^{\alpha_1}(p)) = 1.$$

So, from (26) it follows that $y = 0$ or y is nodal (sign changing).

Both possibilities contradict (24). So, we infer that

$$\{u_n^+\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega) \text{ is bounded,}$$
$$\Rightarrow \{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega) \text{ is bounded (see (16)).}$$

We may assume that

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,p},\ u_n \to u \text{ in } L^p(\Omega).$$

From (15) we have

$$|\langle V(u_n), u_n - u\rangle - \int_\Omega k_\lambda(z,u_n)(u_n-u)\mathrm{d}z| \le \epsilon_n'' \text{ for all } n\in\mathbb{N}, \text{with } \epsilon_n'' \to 0^+.$$

Note that $\int_\Omega k_\lambda(z,u_n)(u_n-u)\mathrm{d}z \to 0$ (see (14)). Hence

$$\lim_{n\to\infty}\langle V(u_n),(u_n-u)\rangle = 0,$$
$$\Rightarrow u_n \to u \text{ in } W_0^{1,p}(\Omega).$$

Therefore $\psi_\lambda(\cdot)$ satisfies the C-condition and this proves Claim 1.

Claim 2. *There exists $\tilde{\lambda} > 0$ such that for all $\lambda \in (0,\tilde{\lambda})$ we can find $\rho = \rho(\lambda) > 0$ so that*

$$\psi_\lambda(u) \ge m_\lambda > 0 \textit{ for all } \|u\| = \rho.$$

Given $\epsilon > 0$ and $r \in (p,p^*)$, on account of hypotheses $H_1(i),(ii),(iii)$ we can find $c_8 = c_8(\epsilon,r) > 0$ such that

$$F(z,x) \le \frac{1}{q}[\theta(z)+\epsilon]x^q + c_8x^r \text{ for a.a. } z\in\Omega, \text{all } x \ge 0. \tag{27}$$

Using (14), we have

$$\begin{aligned}\psi_\lambda(u) = {} & \frac{1}{p}\rho_{\alpha_1,p}(Du) + \frac{1}{q}\rho_{\alpha_2,q}(Du) - \int_{\{u\le\bar{u}_\psi\}}\lambda(\bar{u}_\psi^{-\eta} + \bar{u}_\psi^{\tau-1})u\mathrm{d}z \\ & - \frac{\lambda}{1-\eta}\int_{\{\bar{u}_\psi<u\}}(u^{1-\eta} - \bar{u}_\psi^{1-\eta})\mathrm{d}z - \lambda\int_{\{\bar{u}_\psi<u\}}\bar{u}_\psi^{1-\eta}\mathrm{d}z \\ & - \frac{\lambda}{\tau}\int_{\{\bar{u}_\psi<u\}}(u^\tau - \bar{u}_\psi^\tau)\mathrm{d}z - \lambda\int_{\{\bar{u}_\psi<u\}}\bar{u}_\psi^\tau\mathrm{d}z - \int_\Omega F(z,u^+)\mathrm{d}z.\end{aligned} \tag{28}$$

Using the fact that $\bar{u}_\psi \in \operatorname{int} C_+$ and Hardy's inequality (see Proposition 1), as before, we show that

$$-\lambda\int_{\{u\le\bar{u}_\psi\}}\bar{u}_\psi^{-\eta}u \ge -\lambda c_9\|u\| \text{ for some } c_9 > 0. \tag{29}$$

Also, we have

$$-\frac{\lambda}{1-\eta}\int_{\{\bar{u}_\psi<u\}}u^{1-\eta}\mathrm{d}z \ge -\lambda c_{10}\|u\|^{1-\eta} \text{ for some } c_{10} > 0. \tag{30}$$

(see Theorem 13.17 of Hewitt-Stromberg [34, p. 196]),

$$-\lambda\int_{\{\bar{u}_\psi<u\}}\bar{u}_\psi^{1-\eta}\mathrm{d}z \ge -\lambda\int_{\{\bar{u}_\psi<u\}}u^{1-\eta}\mathrm{d}z \ge -\lambda c_{11}\|u\|^{1-\eta} \text{ for some } c_{11} > 0 \text{ (as above).} \tag{31}$$

In addition we have

$$-\lambda\int_{\{u\le\bar{u}_\psi\}}\bar{u}_\psi^{\tau-1}u\mathrm{d}z \ge -\lambda c_{12}\|u\| \text{ for some } c_{12} > 0 \text{ (since } \bar{u}_\psi \in \operatorname{int} C_+), \tag{32}$$

$$-\frac{\lambda}{\tau}\int_{\{\bar{u}_\psi<u\}}u^\tau\mathrm{d}z \ge -\lambda c_{13}\|u\|^\tau \text{ for some } c_{13} > 0, \tag{33}$$

$$-\lambda\int_{\{\bar{u}_\psi^\tau<u\}}\bar{u}_\psi^\tau\mathrm{d}z \ge -\lambda\int_{\{\bar{u}_\psi^\tau<u\}}u_\psi^\tau\mathrm{d}z - \lambda c_{14}\|u\|^\tau \text{ for some } c_{14} > 0. \tag{34}$$

Finally using (27), we see that

$$-\int_\Omega F(z,u^+)\mathrm{d}z \ge -\frac{1}{q}\int_\Omega(\theta(z)+\epsilon)|u|^q\mathrm{d}z - c_{15}\|u\|^r \text{ for some } c_{15} > 0. \tag{35}$$

We return to (28) and use (29) → (35). Assuming that $\|u\| \le 1$, we have

$$\begin{aligned}\psi_\lambda(u) \geq & \hat{c}\|u\|^p + \frac{1}{q}\left[\rho_{\alpha_2,q}(Du) - \int_\Omega \theta(z)|u|^q \mathrm{d}z - \frac{\epsilon\|\alpha_2\|_\infty}{\hat{\lambda}_1^{\alpha_2}(q)}\|Du\|_q^q\right] \\ & - \lambda c_{16}\|u\|^{1-\eta} - c_{15}\|u\|^r \text{ for some } c_{16}>0 \\ & \quad \text{(see (3) and note that } \|u\| \le 1 \Rightarrow \|u\|^\tau \le \|u\| \le \|u\|^{1-\eta}) \\ \geq & \hat{c}\|u\|^p + \frac{1}{q}\left[c_{17} - \frac{\epsilon\|\alpha_2\|_\infty}{\hat{\lambda}_1^{\alpha_2}(q)}\right]\|Du\|_q^q - \lambda c_{16}\|u\|^{1-\eta} - c_{15}\|u\|^r \\ & \quad \text{for some } c_{17}>0 \text{ (see Proposition 2).}\end{aligned}$$

Choosing $\epsilon \in \left(0, \frac{\hat{\lambda}_1^{\alpha_2}(q)c_{17}}{\|\alpha_2\|_\infty}\right)$ we obtain

$$\begin{aligned}\psi_\lambda(u) \geq & \hat{c}\|u\|^p - c_{18}(\lambda\|u\|^{1-\eta} + \|u\|^r) \text{ with } c_{18} = \max\{c_{16}, c_{15}\} > 0 \\ = & [\hat{c} - c_{18}(\lambda\|u\|^{1-\eta-p} + \|u\|^{r-p})]\|u\|^p.\end{aligned} \tag{36}$$

Let $\beta_\lambda(t) = \lambda t^{1-\eta-p} + t^{r-p}$ for all $t>0$. Since $0<\eta<1<p<r$, we see that

$$\beta_\lambda(t) \to +\infty \text{ as } t \to 0^+ \text{ and as } t \to +\infty$$

Note that $\beta_\lambda \in C^1(\mathbb{R}_+)$ with $\mathbb{R}_+ = (0,\infty)$. So, we can find $t_0>0$ such that

$$\begin{aligned}& \beta_\lambda(t_0) = \inf_{t>0} \beta_\lambda(t), \\ \Rightarrow & \beta_\lambda'(t_0) = 0, \\ \Rightarrow & (1-\eta-p)\lambda t_0^{-(p+\eta)} + (r-p)t_0^{r-p-1} = 0, \\ \Rightarrow & t_0 = t_0(\lambda) = \left(\frac{\lambda(p+\eta-1)}{r-p}\right)^{\frac{1}{r+\eta-1}}.\end{aligned} \tag{37}$$

Then we have

$$\beta_\lambda(t_0) = \lambda\left(\frac{r-p}{\lambda(p+\eta-1)}\right)^{\frac{p+\eta-1}{r+\eta-1}} + \left(\frac{\lambda(p+\eta-1)}{r-p}\right)^{\frac{r-p}{r+\eta-1}}$$

Since $\frac{p+\eta-1}{r+\eta-1} < 1$, we see that

$$\beta_\lambda(t_0) \to 0^+ \text{ as } \lambda \to 0^+. \tag{38}$$

Let $\lambda_0 > 0$ such that

$$t_0 = t_0(\lambda) \le 1 \text{ for all } 0<\lambda\le\lambda_0 \text{ (see (37))}.$$

Then on account of (38), we see that we can find $\tilde{\lambda} \in (0,\lambda_0]$ such that

$$0 < \beta_\lambda(t_0) < \frac{\hat{c}}{c_{18}} \text{ for all } 0<\lambda<\tilde{\lambda}. \tag{39}$$

From (36) and (39) it follows that

$$\psi_\lambda(u) \ge m_\lambda > 0 \text{ for all } \|u\| = \rho = t_0(\lambda) \le 1.$$

This proves Claim 2.

Hypotheses $H_1(i),(ii)$ imply that given $\epsilon>0$, we can find $c_{19} = c_{19}(\epsilon) > 0$ such that

$$F(z,x) \ge \frac{1}{p}(\eta(z)-\epsilon)x^p - c_{19} \text{ for a.a. } z\in\Omega, \text{all } x \ge 0. \tag{40}$$

Then for $t>1$, we have

$$\begin{aligned}\psi_\lambda(t\hat{u}_1(p)) \leq & \frac{t^q}{q}\left(\int_\Omega [\hat{\lambda}_1^{\alpha_1}(p) - \eta(z)]\hat{u}_1(p)^p \mathrm{d}z - \epsilon\right) + \frac{t^q}{q}\rho_{\alpha_2,q}(D\hat{u}_1(p)) \\ & - \lambda t \int_{\{\hat{u}_1(p)\le\bar{u}_\psi\}} (\bar{u}_\psi^{-\eta} + \bar{u}_\psi^{\tau-1})\hat{u}_1(p)\mathrm{d}z \\ & - \lambda t^{1-\eta}\int_{\{\bar{u}_\psi<\hat{u}_1(p)\}} (\hat{u_1}(p)^{1-\eta} + \hat{u}_1(p)^\tau \mathrm{d}z) + c_{20} \\ & \quad \text{for some } c_{20} = c_{20}(\lambda) > 0 \text{ (recall } \|\hat{u}_1(p)\|_p = 1).\end{aligned} \tag{41}$$

Recalling that $\hat{u}_1(p) \in \operatorname{int} C_+$, we see that

$$\xi = \int_\Omega [\eta(z) - \hat{\lambda}_1^{\alpha_1(p)}]\hat{u}_1(p)^p \mathrm{d}z > 0.$$

So choosing $\epsilon \in (0, \xi)$, from (41) we see that

$$\begin{aligned} &\psi_\lambda(t\hat{u}_1(p)) \le -c_{21}t^p + c_{22}t^q - \lambda c_{23}t^{1-\eta} \\ &\qquad \text{for some } c_{21}, c_{22}, c_{23} > 0 \text{ (recall } t > 1, \eta < 1), \\ &\Rightarrow \psi_\lambda(t\hat{u}_1(p)) \to -\infty \text{ (since } 1-\eta < 1 < q < p). \end{aligned} \tag{42}$$

Then (42) together with Claims 1 and 2, permit the use of the Mountain Pass Theorem. So, we can find $u_\lambda \in W_0^{1,p}(\Omega)$ $(0 < \lambda < \tilde{\lambda})$ such that

$$u_\lambda \in K_{\psi_\lambda} \text{ and } \psi_\lambda(0) = 0 < m_\lambda \le \psi_\lambda(u_\lambda).$$

Therefore $u_\lambda \neq 0$ and we have

$$\langle V(u_\lambda), h\rangle = \int_\Omega k_\lambda(z, u_\lambda) h \mathrm{d}z \text{ for all } h \in W_0^{1,p}(\Omega). \tag{43}$$

In (43) above we use the test function $h = (\bar{u}_\psi - u_\lambda)^+ \in W_0^{1,p}(\Omega)$. Note that on $\{u_\lambda \le \bar{u}_\psi\}$ we have $f(z, u_\lambda(\tau)) \ge 0$ for a.a. $z \in \Omega$ (see (13) and hypothesis $H_1(iii)$). Therefore using (14), we have

$$\begin{aligned} &\langle V(u_\lambda), (\bar{u}_\psi - u_\lambda)^+\rangle \\ &= \int_\Omega (\lambda(\bar{u}_\psi^{-\eta} + \bar{u}_\psi^{\tau-1}) + f(z, u_\lambda))(\bar{u}_\psi - u_\lambda)^+ \mathrm{d}z \\ &\ge \int_\Omega \lambda \bar{u}_\psi^{-\eta}(\bar{u}_\psi - u_\lambda)^+ \mathrm{d}z \\ &= \langle V(\bar{u}_\psi), (\bar{u}_\psi - u_\lambda)^+\rangle \text{ (see Proposition 4)}, \\ &\Rightarrow \langle V(u_\lambda) - V(\bar{u}_\psi), (\bar{u}_\psi - u_\lambda)^+\rangle \ge 0, \\ &\Rightarrow \bar{u}_\psi \le u_\lambda. \end{aligned} \tag{44}$$

From (44), (14) and (43), we infer that

$$\begin{aligned} &u_\lambda \in S_\lambda \text{ for all } 0 < \lambda < \tilde{\lambda}, \\ &\Rightarrow (0, \tilde{\lambda}) \subseteq \mathcal{L} \neq \emptyset. \end{aligned}$$

Next we show that for $\lambda \in \mathcal{L}$, we have $\emptyset \neq S_\lambda \subseteq \operatorname{int} C_+$. To this end, we consider the following parametric purely singular Dirichlet problem

$$-\Delta_p^{\alpha_1} u(z) - \Delta_q^{\alpha_2} u(z) = \theta u(z)^{-\eta} \text{ in } \Omega, u|_{\partial\Omega} = 0, u > 0, \theta > 0. \tag{45}$$

From Papageorgiou, Zhang [35] (see the proof of Proposition 3.5), we know that (45) has a unique solution $\tilde{u}_\theta \in \operatorname{int} C_+$, $\{\tilde{u}_\theta\}_{\theta>0}$ is nondecreasing and $\tilde{u}_\theta \to 0$ in $C_0^1(\bar{\Omega})$ as $\theta \to 0^+$. We choose $\theta \in (0,1)$ small such that

$$0 \le \tilde{u}_\theta(z) \le \delta \text{ for all } z \in \bar{\Omega} \tag{46}$$

($\delta > 0$ as in hypothesis $H_1(iii)$). Suppose that $u \in S_\lambda$. We have

$$\begin{aligned} &\langle V(u), (\tilde{u}_\theta - u)^+\rangle \\ &= \int_\Omega [\lambda(u^{-\eta} + u^{\tau-1}) + f(z, u)](\tilde{u}_\theta - u)^+ \mathrm{d}z \\ &\ge \int_\Omega \lambda u^{-\eta}(\tilde{u}_\theta - u)^+ \mathrm{d}z \text{ (since } f(z,u) \ge 0 \text{ on } \{u \le \tilde{u}_\theta\}) \text{ see (46) and } H_1(iii) \\ &\ge \int_\Omega \lambda \tilde{u}_\theta^{-\eta}(\tilde{u}_\theta - u)^+ \mathrm{d}z \\ &= \langle V(\tilde{u}_\theta), (\tilde{u}_\theta - u)^+\rangle, \\ &\Rightarrow \langle V(\tilde{u}_\theta) - V(u), (\tilde{u}_\theta - u)^+\rangle \le 0, \\ &\Rightarrow \tilde{u}_\theta \le u. \end{aligned} \tag{47}$$

From Marino, Winkert [27], we know that $u \in L^\infty(\Omega)$. Therefore we have

$$\begin{aligned} |\lambda(u^{-\eta} + u^{\tau-1}) + f(z,u)| &\le \lambda u^{-\eta} + (\lambda+1)c_{24} \text{ for some } c_{24} > 0 \text{ (see hypothesis } H_1(ii)) \\ &\le \lambda \tilde{u}_\theta^{-\eta} + (\lambda+1)c_{24} \text{ (see (47))} \\ &\le \lambda c_{25}\hat{d}^{-\eta} + (\lambda+1)c_{24} \text{ for some } c_{25} \text{ (since } \tilde{u}_\theta \in \operatorname{int} C_+) \\ &\le \lambda c_{26}\hat{d}^{-\eta} \text{ for some } c_{26} > 0. \end{aligned}$$

So, we can apply Theorem 1.7 of Giacomoni, Kumar, Sreenadh [36] and infer that $u \in C_+ \backslash \{0\}$. Let $\rho = \|u\|_\infty$ and let $\hat{\xi}_\rho > 0$ be as postulated by hypothesis $H_1(iv)$. Then

$$\begin{aligned} &-\Delta_p^{\alpha_1} u(z) - \Delta_q^{\alpha_2} u(z) + \hat{\xi}_\rho u(z)^{p-1} - \lambda u(z)^{-\eta} \\ &= \lambda u(z)^{\tau-1} + f(z, u(z)) + \hat{\xi} u(z)^{p-1} \ge 0 \text{ in } \Omega. \end{aligned}$$

Then Proposition A2 of Papageorgiou, Rădulescu, Zhang [37] implies that $u \in \operatorname{int} C_+$. Therefore finally we conclude that for all $\lambda \in \mathcal{L}, \emptyset \neq S_\lambda \subseteq \operatorname{int} C_+$. □

Remark 4. In the above proof, we used twice that if $u \in \operatorname{int} C_+$, then for some $c > 0, c\hat{d} \leq u$ and we referred to Guo, Webb [32]. Here we provide a different proof of this fact. By Lemma 14.16 of Gilbarg-Trudinger [38, p. 355], we have that there exists $\beta \in (0,1)$ such that $\hat{d} \in C^2(\bar{\Omega}_\beta)$ with $\Omega_\beta = \{z \in \Omega : \hat{d}(z) < \beta\}$. Since $u \in \operatorname{int} C_+$, using Proposition 4.1.22, p. 274, of [20], we can find $\hat{c}_1 > 0$ such that $\hat{c}_1\hat{d} \leq u$ on Ω_β. On $\hat{\Omega}_\beta = \Omega \backslash \Omega_\beta$, we have $\hat{d}, u \in \operatorname{int} L^\infty(\Omega)_+$ ($L^\infty(\Omega)_+$ being the order cone of $L^\infty(\Omega)$), so we can find $\hat{c}_2 > 0$ such that $\hat{c}_2\hat{d} \leq u$ in $\hat{\Omega}_\beta$. Therefore if $\hat{c}_* = \min\{\hat{c}_1, \hat{c}_2\}$, then $\hat{c}_*\hat{d} \leq u$ on $\bar{\Omega}$. In fact since $\hat{d}$ belongs in the interior of the order cone of $\hat{C}^1(\bar{\Omega}_\beta) = \{u \in C^1(\bar{\Omega}_\beta) : u|_{\partial\Omega = 0}\}$, then we can find $\hat{c}^* > 0$ such that $u \leq \hat{c}^*\hat{d}$ on $\bar{\Omega}$, this is, $\hat{c}_*\hat{d} \leq u \leq \hat{c}^*\hat{d}$ on $\bar{\Omega}$.

Proposition 5 remains true if hypotheses H_1 are replaced by H_1' (superlinear case).

Proposition 6. *If hypotheses H_0, H_1' hold, then $\mathcal{L} \neq \phi$ and for all $\lambda \in \mathcal{L}, \emptyset \neq S_\lambda \subseteq \operatorname{int} C_+$.*

Proof. The proof remains the same as that of Proposition 5. The only part that changes is Claim 1 in that proof, where we show that $\psi_\lambda(\cdot)$ satisfies the C-condition. In this case the proof goes as follows:

Claim. *$\psi_\lambda(\cdot)$ satisfies the C-condition.*

We consider a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ such that

$$\left|\psi_\lambda\left(u_n\right)\right| \leqslant M \text{ for some } M > 0, \text{ all } n \in \mathbb{N}, \tag{48}$$

$$(1 + \|u_n\|)\psi_\lambda'\left(u_n\right) \to 0 \text{ in } W^{-1,p'}(\Omega) = W_0^{1,p}(n)^* \text{ as } n \to \infty. \tag{49}$$

From (49) we have

$$\left|\left\langle V\left(u_n\right), h\right\rangle - \int_\Omega k_\lambda\left(z, u_n\right) h \mathrm{d}z\right| \leqslant \frac{\varepsilon_n\|h\|}{1 + \|u_n\|} \text{ for all } h \in W_0^{1,p}(\Omega), \text{ with } \varepsilon_n \to 0^+. \tag{50}$$

In (50) we use the test function $h = u_n^- \in W_0^{1,p}(\Omega)$ and obtain

$$u_n^- \to 0 \text{ in } W_0^{1,p}(\Omega) \text{ as } n \to \infty. \tag{51}$$

From (48) and (51), we infer that

$$\rho_{\alpha_1,p}\left(Du_n^+\right) + \frac{p}{q}\rho_{\alpha_2,q}\left(Du_n^+\right) - \int_\Omega pK_\lambda\left(z, u_n^+\right)u_n^+ \mathrm{d}z \leqslant \hat{M} \text{ for some } \hat{M} > 0, \text{ all } n \in \mathbb{N}. \tag{52}$$

Also, if in (50) we choose the test function $h = u_n^+ \in W_0^{1,p}(\Omega)$, then

$$-\rho_{\alpha_1,p}\left(Du_n^+\right) - \rho_{\alpha_2,q}\left(Du_n^+\right) + \int_\Omega k_\lambda\left(z, u_n^+\right)u_n^+ \mathrm{d}z \leqslant \epsilon_n \text{ for some } M_0 > 0, \text{ all } n \in \mathbb{N}. \tag{53}$$

We add (52), (53) and obtain

$$\begin{aligned} &\int_\Omega \left[k_\lambda\left(z, u_n^+\right)u_n^+ - pK_\lambda\left(z, u_n^+\right)\right] \mathrm{d}z \leqslant M_0 \text{ for some } M_0 > 0, \text{ all } n \in \mathbb{N}(\text{ recall } q < p), \\ \Rightarrow &\int_\Omega \left[f\left(z, u_n^+\right)u_n^+ - pF\left(z, u_n^r\right) \mathrm{d}z\right] \leqslant M_\lambda[\|u_n^+\|_\tau^\tau + 1] \text{ for some } M_\lambda > 0, \text{ all } n \in \mathbb{N}. \end{aligned} \tag{54}$$

On account of hypotheses $H_1'(i)$, (ii), we can find $\hat{\beta}_0 \in (0, \hat{\beta})$ and $c_{27} > 0$ such that

$$\hat{\beta}_0 x^s - c_{27} \leqslant f(z,x)x - pF(z,x) \text{ for a.a. } z \in \Omega, \text{ all } x \geqslant 0. \tag{55}$$

Note that we can always assume that $r \in (p, p^*)$ in hypothesis $H_1'(i)$ is close to p^*. So, from hypothesis $H_1'(ii)$, we see that $\tau < s$. Then using (55) in (54), we obtain that

$$\{u_n^+\}_{n\in\mathbb{N}} \leq L^s(\Omega) \subseteq \text{ is bounded.} \tag{56}$$

From hypothesis $H_1'(ii)$, it is clear that we can always assume that $s < r < p^*$. First suppose $p \neq N$ and recall that if $N < p$, then $p^* = \infty$. We can find $t \in (0,1)$, such that

$$\frac{1}{r} = \frac{1-t}{s} + \frac{1}{p^*}. \tag{57}$$

Using the interpolation inequality (see Hu, Papageorgiou [10], p. 82), we have

$$\|u_n^+\|_r \leqslant \|u_n^+\|_s^{1-\frac{1}{t}} \|u_n^+\|_{p^*}^{\frac{1}{t}},$$

$$\Rightarrow \|u_n^+\|_r^r \leqslant c_{28} \|u_n^+\|^{tr} \text{ for some } c_{28} > 0, \text{ all } n \in \mathbb{N}. \tag{58}$$

Here we have used (56) and the fact that $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ continuously (Sobolev embedding theorem).

In (50) we use the test function $h = u_n^+ \in W_0^{1,p}(\Omega)$ and obtain

$$\begin{aligned}
&\rho_{a_1,p}\left(Du_n^+\right) + \rho_{a_2,q}\left(Du_n^+\right) - \int_\Omega k_\lambda\left(z, u_n^+\right) u_n^+ \mathrm{d}z \leqslant \epsilon_n \text{ for all } n \in \mathbb{N},\\
\Rightarrow\ &\rho_{a_{1,p}}\left(Du_n^+\right) + \rho_{a_{2,q}}\left(Du_n^+\right) - \int_{\{u_n^+ \leqslant \bar{u}_\psi\}} \lambda\left[\bar{u}_\psi^{-\eta} + \bar{u}_\psi^{\tau-1}\right] u_n^+ \mathrm{d}z\\
&\quad - \int_{\{\bar{u}_\psi < u_n^+\}} \left(\frac{\lambda}{1-\eta}\left((u_n^+)^{1-\eta} - \bar{u}_\psi^{1-\eta}\right) + \frac{\lambda}{\tau}\left((u_n^+)^\tau - \bar{u}_\psi^\tau\right)\right) \mathrm{d}z\\
&\quad - \int_{\{\bar{u}_\psi < u_n^+\}} \lambda\left(\bar{u}_\psi^{-\eta} + \bar{u}_\psi^{\tau-1}\right) \bar{u}_\psi \mathrm{d}z\\
&\quad - \int_\Omega f\left(z, u_n^+\right) u_n^+ \mathrm{d}z \le \epsilon_n \text{ for all } n \in \mathbb{N} \text{ (see (14))},\\
\Rightarrow\ &\hat{c}\,\|u_n^+\|^p \leqslant c_{29}\left[\|u_n^+\| + \|u_n\|^{1-\eta} + \|u_n^+\|^\tau + \|u_n^+\|^{tr} + 1\right]\\
&\quad \text{for some } c_{29} > 0, \text{ all } n \in \mathbb{N}\left(\text{ see (58) and } H_1'(i)\right).
\end{aligned} \tag{59}$$

By hypotheses H_0, we have $0 < \eta < 1 < \tau < q < p$ and from (57) and hypothesis $H_1'(ii)$, we see that $tr < p$. So, from (59) we see that

$$\{u_n^+\}_{n\in\mathbb{N}} \leqslant W_0^{1,p}(\Omega) \text{ is bounded}. \tag{60}$$

Combining (60) and (51), we conclude that

$$\{u_n\}_{n\in\mathbb{N}} \le W_0^{1,p}(\Omega) \text{ is bounded}. \tag{61}$$

Next, suppose that $p = N$. Then according to the Sobolev embedding theorem $W_0^{1,p}(\Omega) \hookrightarrow L^\gamma(\Omega)$ continuously for all $\gamma \in [1, \infty)$. Then let $s < r < \gamma$ and choose $t \in (0, 1)$ such that

$$\begin{aligned}
&\frac{1}{r} = \frac{1-t}{s} + \frac{t}{\gamma},\\
\Rightarrow\ &tr = \frac{\gamma(r-s)}{\gamma - s}.
\end{aligned}$$

We see that

$$\frac{\gamma(r-s)}{\gamma-s} \to r - s, \text{ as } \gamma \to +\infty.$$

Hypothesis $H_1'(ii)$ implies that $r - s < p$ (recall $p = N$). So, the previous argument remains valid if we choose $\gamma > r$ large so that $tr = \frac{\gamma(r-s)}{\gamma-s} < p$. Then we conclude that (61) holds.

On account of (61), we may assume that

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,p}(\Omega), u_n \to u \text{ in } L^r(\Omega) \text{ as } n \to \infty. \tag{62}$$

In (50) we use the test function $h = u_n - u \in W_0^{1,p}(\Omega)$ and pass to the limit as $n \to \infty$. Using (62), we see that

$$\begin{aligned}
&\int_\Omega k_\lambda(z, u_n)(u_n - u)\mathrm{d}z \to 0 \text{ as } n \to \infty,\\
\Rightarrow\ &\lim_{n\to\infty} \left\langle V\left(u_n\right), u_n - u\right\rangle = 0,\\
\Rightarrow\ &u_n \to u \text{ in } W_0^{1,p}(\Omega).
\end{aligned}$$

This proves the claim.

The rest of the proof of Proposition 5 can be used unchanged. Only note that in this case on account of the superlinearity of $F(z, \cdot)$, for any $u \in \operatorname{int} C_+$, we have

$$\psi_\lambda(tu) \to -\infty \text{ as } t \to \infty$$

and so we can apply the Mountain Pass Theorem. □

Next we show that $\mathcal{L}$ is connected (an interval).

Proposition 7. *If hypotheses H_0 and H_1 or H_1' hold, $\lambda \in \mathcal{L}$ and $0 < \gamma < \lambda$, then $\gamma \in \mathcal{L}$.*

Proof. since $\lambda \in \mathcal{L}$, we can have $u_\lambda \in S_\lambda \subseteq \operatorname{int} C_+$ (see Propositions 5 and 6). As before, we choose $\psi \in (0,1)$ small such that

$$0 \leq \bar{u}_\psi(z) \leq \min\left\{\delta, u_\lambda(z)\right\} \text{ for all } z \in \bar{\Omega},$$

with $\delta > 0$ as in hypothesis $H_1(iii) = H_1'(iii)$ and recall that $u_\lambda \in \operatorname{int} C_+$. We introduce the Carathéodory function $\hat{k}_\gamma(z,x)$ defined by

$$\hat{k}_\gamma(z,x) = \begin{cases} \gamma\left(\bar{u}_\psi^{-\eta} + \bar{u}_\psi^{\tau-1}\right) + f\left(z, \bar{u}_\psi\right) & \text{if } x < \bar{u}_\gamma(z). \\ \gamma\left(x^{-\eta} + x^{\tau-1}\right) + f(z,x) & \text{if } \bar{u}_\psi(z) \leq x \leq u_\lambda(z) \\ \gamma\left(u_\lambda^{-\eta} + u_\lambda^{\tau-1}\right) + f\left(z, u_\lambda\right) & \text{if } u_\lambda(z) < x. \end{cases} \tag{63}$$

We set $\hat{K}_\gamma(t,x) = \int_0^x k_\gamma(z,s)\mathrm{d}s$ and consider the C^1 functional $\hat{\psi}_\gamma : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by (see [31])

$$\hat{\psi}_\gamma(u) = \frac{1}{p}\rho_{a_1,p}(Du) + \frac{1}{q}\rho_{a_2,q}(Du) - \int_\Omega \hat{K}_\gamma(z,u)\mathrm{d}z \text{ for all } u \in W_0^{1,p}(\Omega).$$

From (63) and hypotheses H_0, we see $\hat{\psi}_\gamma(\cdot)$ is coercive. Also using the Sobolev embedding theorem, we see that $\hat{\psi}_\gamma(\cdot)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass–Tonelli theorem, we can find $u_\gamma \in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned} &\hat{\psi}_\gamma\left(v_\gamma\right) = \inf\left\{\hat{\psi}_\gamma(u) : u \in W_0^{1,p}(\Omega)\right\}, \\ &\Rightarrow \left\langle \hat{\psi}_\gamma'\left(u_\gamma\right), h\right\rangle = 0 \text{ for all } h \in W_0^{1,p}(\Omega), \\ &\Rightarrow \left\langle V\left(u_\gamma\right), h\right\rangle = \int_\Omega \hat{k}_\gamma\left(z, u_\gamma\right) h\mathrm{d}z \text{ for all } u \in W_0^{1,p}(\Omega). \end{aligned} \tag{64}$$

In (64) first we use the test function $h = (u_\gamma - u_\lambda)^+ \in W_0^{1,p}(\Omega)$. Then we have

$$\begin{aligned} &\left\langle V\left(u_\gamma\right), \left(u_\gamma - u_\lambda\right)^+\right\rangle \\ =&\int_\Omega \left[\gamma\left(u_\lambda^{-\eta} + u_\lambda^{\tau-1}\right) + f\left(z, u_\lambda\right)\right]\left(u_\gamma - u_\lambda\right)^+ \mathrm{d}z \text{ (see (63))} \\ \leq&\int_\Omega \left[\lambda\left(u_\lambda^{-\eta} + u_\lambda^{\tau-1}\right) + f\left(z, u_\lambda\right)\right]\left(u_\gamma - u_\lambda\right)^+ \mathrm{d}z \text{ (since } \gamma < \lambda) \\ =&\langle V\left(u_\lambda\right), \left(u_\gamma - u_\lambda\right)^+\rangle \text{ (since } u_\lambda \in S_\lambda), \\ \Rightarrow& u_\gamma \leqslant u_\lambda. \end{aligned}$$

Next in (64) we choose the test function $h = (\bar{u}_\psi - u_\gamma)^+ \in W_0^{1,p}(\Omega)$. We have

$$\begin{aligned} &\left\langle V\left(u_\gamma\right), \left(\bar{u}_\psi - u_\gamma\right)^+\right\rangle \\ =&\int_\Omega \left[\gamma\left(\bar{u}_\psi^{-\eta} + \bar{u}_\psi^{\tau-1}\right) + f\left(z, \bar{u}_\psi\right)\right]\left(\bar{u}_\psi - u_\gamma\right)^+ \mathrm{d}z \\ \geq&\int_\Omega \gamma\bar{u}_\gamma^{\tau-1}\left(\bar{u}_\psi - u_\gamma\right)^+ \mathrm{d}z \left(\text{ since } f\left(z, \bar{u}_\psi\right) \geqslant 0, \text{ see } H_1(\text{ iii }) = H_1'(iii)\right) \\ =&\left\langle V\left(\bar{u}_\psi\right), \left(\bar{u}_\psi - u_\gamma\right)^+\right\rangle \text{ (see Proposition 4)}, \\ \Rightarrow& \bar{u}_\gamma \leq u_\gamma. \end{aligned}$$

So, we have proved that

$$u_\gamma \in [\bar{u}_\gamma, u_\lambda]. \tag{65}$$

From (65), (63) and (64), we infer that $u_\gamma \in S_\gamma \leq \operatorname{int} C_+$ and so $\gamma \in \mathcal{L}$. □

An interesting byproduct of the above proof is the following corollary.

Corollary 1. *If hypotheses H_0 and H_1 or H_1' hold, $0 < \gamma < \lambda \in \mathcal{L}$ and $u_\lambda \in S_\lambda \subseteq \operatorname{int} C_+$, then $\gamma \in \mathcal{L}$ and we can find $u_\gamma \in S_\gamma \subseteq \operatorname{int} C_+$, such that $u_\gamma \leq u_\lambda$.*

This corollary says that the solution multifunction $\mathcal{L} \ni \lambda \to S_\lambda$ exhibits a kind of a weak monotonicity property. We can improve this monotonicity.

Proposition 8. *If hypotheses H_0 and H_1 or H_1^1 hold, $0 < \gamma < \lambda \in \mathcal{L}$ and $u_\lambda \in S_\lambda \subseteq \operatorname{int} S_+$, then $\gamma \in \mathcal{L}$ and we can find $u_\gamma \in S_\gamma \leq \operatorname{int} C_+$ such that*

$$u_\lambda - u_\gamma \in \operatorname{int} C_+.$$

Proof. From Corollary 1 we already have that $\gamma \in \mathcal{L}$ and there exists $u_\gamma \in S_\gamma \subseteq \operatorname{int} C_+$ such that $u_\gamma \leq u_\lambda$. Let $\rho = \|u_\lambda\|_\infty$ and let be $\hat{\xi}_\rho > 0$ as postulated by hypothesis $H_1(iv) = H_1'(iv)$. We have

$$\begin{aligned}
& -\Delta_p^{a_1} u_\gamma - \Delta_q^{a_2} u_\gamma + \hat{\xi}_\rho u_\gamma^{p-1} - \lambda u_\gamma^{-\eta} \\
& \leq \gamma u_\gamma^{\tau-1} + f\left(z, u_\gamma\right) + \hat{\xi}_\rho u_\gamma^{p-1} (\text{ since } \gamma < \lambda \text{ and } u_\gamma \in S_\gamma) \\
& \leq \lambda u_\lambda^{\tau-1} + f\left(z, u_\lambda\right) + \hat{\xi}_\rho u_\lambda^{p-1} \\
& = -\Delta_p^{a_1} u_\lambda - \Delta_q^{a_2} u_\lambda + \hat{\xi}_\rho u_\lambda^{p-1} - \lambda u_\lambda^{\eta} \text{ in } \Omega \quad (\text{ since } u_\lambda \in S_\lambda).
\end{aligned} \tag{66}$$

Since $u_\gamma \in \operatorname{int} C_+$, we see that $0 \prec (\lambda - \gamma) u_\gamma^{\tau-1}$. So, from (66) and Proposition 7 of Papageorgiou, Rădulescu, Repovs [39], we conclude that

$$u_\lambda - u_\gamma \in \operatorname{int} C_+.$$

The proof is now complete. □

Let $\lambda^* = \sup \mathcal{L}$.

Proposition 9. *If hypotheses H_0 and H_1 or H_1' hold, then $\lambda^* < \infty$.*

Proof. Hypotheses H_1 or H_1' imply that there exist $\hat{\lambda} > 0$ and $c \in (0,1)$ such that

$$\hat{\lambda} x^{\tau-1} + f(z,x) \geq c x^{p-1} \text{ for a.a. } z \in \Omega, \text{ all } x \geq 0. \tag{67}$$

Let $\lambda > \hat{\lambda}$ and suppose that $\lambda \in \mathcal{L}$. Then we can find $u_\lambda \in S_\lambda \subseteq \operatorname{int} C_+$. We consider $\hat{\Omega} \subseteq \Omega$ an open subset with C^2-boundary such that $\bar{\hat{\Omega}} \subseteq \Omega$. Since $u_\lambda \subseteq \operatorname{int} C_+$, we have $\hat{m}_\lambda = \min_{\bar{\hat{\Omega}}} u_\lambda > 0$. For $\epsilon > 0$ we set $\hat{m}_\lambda^\epsilon = \hat{m}_\lambda + \epsilon$. Let $\rho = \|\hat{u}_\lambda\|_\infty$ and let $\hat{\xi}_\rho > 0$ be as in hypothesis $H_1(iv) = H_1'(iv)$. Then in $\hat{\Omega}$, we have

$$\begin{aligned}
& -\Delta_p^{a_1}\left(\hat{m}_\lambda^\varepsilon\right) - \Delta_q^{a_2}\left(\hat{m}_\lambda^\varepsilon\right) + \hat{\xi}_p\left(\hat{m}_\lambda^\varepsilon\right)^{p-1} - \lambda\left(\hat{m}_\lambda^\varepsilon\right)^{-\eta} \\
\leq & \hat{\xi}_\rho \hat{m}_\lambda^{p-1} + \chi(\epsilon) \text{ with } \chi(\epsilon) \to 0^+ \text{ as } \epsilon \to 0^+ \\
\leq & \left[\hat{\xi}_\rho + c\right] \hat{m}_\lambda^{p-1} + \chi(\epsilon) \\
\leq & \hat{\lambda} \hat{m}_\lambda^{\tau-1} + f\left(z, \hat{m}_\lambda\right) + \hat{\xi}_\rho \hat{m}_\lambda^{p-1} + \chi(\epsilon) \quad (\text{ see } (67)) \\
\leq & \lambda u_\lambda^{\tau-1} + f(z,u_\lambda) + \hat{\xi}_\rho u_\lambda^{p-1} - (\lambda - \hat{\lambda}) \hat{m}^{\tau-1} + \chi(\epsilon) \quad (\text{ see hypothesis } H_1(iv) - H_1'(iv)) \\
= & -\Delta_p^{a_1} u_\lambda - \Delta_q^{a_2} u_\lambda + \hat{\xi}_\rho u_\lambda^{p-1} - \lambda v_\lambda^{-\eta} \quad \text{ in } \hat{\Omega} \text{ for } \epsilon \in (0,1) \text{ small }.
\end{aligned} \tag{68}$$

For $\epsilon \in (0,1)$ small, we have

$$0 < c_{30} \leqslant (\lambda - \hat{\lambda}) \hat{m}_\lambda^{\tau-1} - \chi(\epsilon).$$

Then from (68) and using Proposition 6 of Papageorgiou, Rădulescu, Repovs [39], we infer that for $\epsilon \in (0,1)$ small

$$\hat{m}_\lambda^\epsilon < u_\lambda(z) \text{ for all } z \in \hat{\Omega},$$

contradicting the definition of $\hat{m}_\lambda$. Therefore $\lambda^* \leq \hat{\lambda} < \infty$. □

Hence $\mathcal{L}$ is a bounded interval and we have

$$(0, \lambda^*) \subseteq \mathcal{L} \subseteq (0, \lambda^*].$$

We show that for $\lambda \in (0, \lambda^*)$ we have multiplicity of positive solutions.

Proposition 10. *If hypotheses H_0 and H_1 or H_1' hold and $\lambda \in (0, \lambda^*)$, then problem (1) has at least two positive solutions*

$$u_0, \hat{u} \in \operatorname{int} C_+.$$

Proof. Let $0 < \gamma < \lambda < \theta < \lambda^*$. On account of Proposition 8, we can find $u_\theta \in S_\theta \subseteq \operatorname{int} C_+$, $u_0 \in S_\theta \subseteq \operatorname{int} C_+$ and $u_\gamma \in S_\gamma \subseteq \operatorname{int} C_+$, such that

$$u_0 \in \operatorname{int}_{C_0^1(\bar{\Omega})} [v_\gamma, u_\theta]. \tag{69}$$

We may assume that

$$S_\lambda \cap \left[u_\gamma, u_\theta\right] = \left\{u_0\right\}. \tag{70}$$

Otherwise, we already have a second positive smooth solution and so we are done. We introduce the Carathéodory function $d_\lambda(z,x)$ defined by

$$d_\lambda(z,x) = \begin{cases} \lambda\left(u_\gamma^{-\eta} + u_\gamma^{\tau-1}\right) + f\left(z, u_\gamma\right) & \text{if } x < u_\gamma(z), \\ \lambda\left(x^{-\eta} + x^{\tau-1}\right) + f(z,x) & \text{if } u_\gamma(z) \leqslant x \leqslant u_\lambda(z), \\ \lambda\left(u_\theta^{-\eta} + u_\theta^{\tau-1}\right) + f\left(z, u_\theta\right) & \text{if } u_\lambda(z) < x. \end{cases} \tag{71}$$

We set $D_\lambda(z,x) = \int_0^x d_\lambda(z,s)\mathrm{d}s$ and consider the C^1-functional $\sigma_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\sigma_\lambda(u) = \frac{1}{p}\rho_{\alpha_{1,p}}(Du) + \frac{1}{q}\rho_{\alpha_2,q}(Du) - \int_\Omega D_\lambda(z,v)\mathrm{d}z \text{ for all } u \in W_0^{1,p}(\Omega).$$

Evidently $\sigma_\lambda(\cdot)$ is coercive (see (71)) and sequentially weakly lower semicontinuous. So, we can find $\tilde{u}_0 \in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned}&\sigma_\lambda\left(\tilde{u}_0\right) = \inf\left\{\sigma_\lambda(u) : u \in W_0^{1,p}(\Omega)\right\},\\ &\Rightarrow \tilde{u}_0 \in K_{\sigma_\lambda} \subseteq [u_\gamma, u_\theta] \cap \operatorname{int} C_+(\text{ as in the proof of Proposition 7}).\end{aligned} \tag{72}$$

Then from (71) and (72) if follows that

$$\begin{aligned}&\tilde{u}_0 = S_\lambda \cap \left[u_\gamma, u_\theta\right],\\ &\Rightarrow \tilde{u}_0 = u_0 \quad (\text{ see (70)}).\end{aligned} \tag{73}$$

Let $\hat{d}_\lambda(z,x)$ be the Carathéodory function defined by

$$\hat{d}_\lambda(z,x) = \begin{cases}\lambda\left(u_\gamma^{-\eta} + u_\gamma^{\tau-1}\right) + f\left(z,u_\gamma\right) & \text{if } x \leqslant u_\gamma(z)\\ \lambda\left(x^{-\eta} + x^{\tau-1}\right) + f(z,x) & \text{if } u_\gamma(z) < x.\end{cases} \tag{74}$$

Let $\hat{D}_\lambda(z,x) = \int_0^x \hat{d}_\lambda(z,s)ds$ and consider the C^1-functional $\hat{\sigma}_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\sigma}_\lambda(u) = \frac{1}{p}\rho_{\alpha_{1,p}}(Du) + \frac{1}{q}\rho_{\alpha_{2,q}}(Du) - \int_\Omega \hat{D}_\lambda(z,u)\mathrm{d}z \text{ for all } u \in W_0^{1,p}(\Omega).$$

From (71) and (74), we see that (see Papageorgiou, Rădulescu, Zhang [37], Proposition A3)

$$\begin{aligned}&\sigma_\lambda\big|_{[u_\gamma,u_\theta]} = \hat{\sigma}_\lambda\big|_{[u_\gamma,u_\theta]},\\ &\Rightarrow u_0 \text{ is a local } C_0^1(\bar{\Omega}) - \text{ minimizer of } \hat{\sigma}_\lambda(\cdot)(\text{ see (69) and (73)}),\\ &\Rightarrow u_0 \text{ is a local } W_0^{1,p}(\Omega) - \text{minimizer of } \hat{\sigma}_\lambda(\cdot).\end{aligned} \tag{75}$$

Using (71) we can easily check as before that

$$K_{\hat{\delta}_\lambda} \subseteq \left[u_\gamma\right) \cap \operatorname{int} C_+.$$

So, we may assume that $K_{\hat{\delta}_\lambda}$ is finite or otherwise we already have an infinite set of positive smooth solutions of (1) (see((74))). Then (75) and Theorem 5.7.6, p. 449, of Papageorgiou, Rădulescu, Repovs [20], we can find $\rho \in (0,1)$ small such that

$$\hat{\sigma}_\lambda(u_0) < \inf\left\{\hat{\sigma}_\lambda(u) : \left\|u - u_0\right\| = \rho\right\} = m_\lambda^*. \tag{76}$$

We know that:

- If hypotheses H_1 hold, then

$$\hat{\sigma}_\lambda(t\hat{u},(\rho)) \to -\infty \text{ as } t \to +\infty(\text{ see the proof of Proposition 5}). \tag{77}$$

- If hypotheses H_1' hold, then for every $u \in \operatorname{int} C_+$ we have

$$\sigma_\lambda(tu) \to -\infty \text{ as } t \to +\infty(\text{ see hypothesis } H_1'(ii) \text{ and the proof of Proposition 6}). \tag{78}$$

Moreover, reasoning as in the proof of Proposition 6 (if hypotheses H_1' hold) or as in the proof of Proposition 6 (if hypotheses H_1' hold), we show that

$$\hat{\sigma}_\lambda(\cdot) \text{ satisfies the } C-\text{condition}. \tag{79}$$

Then (76), (77) (if H_1 hold) or (78) (if H_1' hold) and (79), permit the use of the Mountain Pass Theorem. So, we can find $\hat{u} \in W_0^{1,p}(\Omega)$ such that

$$\hat{u} \in K_{\hat{\sigma}_\lambda} \subseteq \left[u_\gamma\right] \cap \operatorname{int} C_+, \hat{\sigma}_\lambda\left(u_0\right) < m_\lambda^* \leq \hat{\sigma}_\lambda(\hat{u}).$$

Therefore $\hat{u} \in \operatorname{int} C_+$ is a positive solution (1) $\lambda \in (0,\lambda)$ and see (74)), which is distinct from u_0. ☐

To complete our analysis of problem, we need to check the admissibility of the critical parameter λ^*. To this end, first we show that S_λ has a smallest element u_λ^* and prove some useful properties of the minimal solution map $\lambda \to u_\lambda^*$.

Proposition 11. *If hypotheses H_0 and H_1 or H_1' hold and $\lambda \in \mathcal{L}$, then*

(a) problem (1) has a smallest element $u_\lambda^* \in S_\lambda \subseteq \operatorname{int} C_+$, that is $u_\lambda^* \leq u$ for all $u \in S_\lambda$;
(b) $\mathcal{L} \ni \lambda \to u_\lambda^* \in C_0^1(\bar{\Omega})$ is strictly increasing, that is,

$$0 < \lambda_1 < \lambda_2 \Rightarrow u_{\lambda_2}^* - u_{\lambda_1}^* \in \operatorname{int} C_+.$$

Proof. (a) We know that S_λ is downward directed (see Filippakis, Papageorgiou [40] and Bai, Gasinski, Papageorgiou [41]). So, invoking Theorem 5.109, p.305, of Hu, Papageorgiou [25], we can find a decreasing sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq S_\lambda$ such that

$$\inf S_\lambda = \inf_{n\in\mathbb{N}} u_n.$$

We have

$$\left\langle V\left(u_n\right), h\right\rangle = \int_\Omega \left[\lambda\left(u_n^{-\eta} + u_n^{\tau-1}\right) + f\left(z, u_n\right)\right] h\mathrm{d}z \text{ for all } n \in \mathbb{N}, \tag{80}$$

$$0 \leq u_n \leq u_1 \text{ for all } n \in \mathbb{N}. \tag{81}$$

In (80) we use the test function $h = u_n \in W_0^{1,p}(\Omega)$. Using (81) and hypothesis $H_1(i)$ or $H_1'(i)$, we obtain

$$\begin{aligned}
&\rho_{a_1,p}\left(Du_n\right) + \rho_{a_2,q}\left(Du_n\right) \leq \int_\Omega \left(\lambda u_n^{1-\eta} + (\lambda+1)c_{31}\right)\mathrm{d}z \text{ for all } n \in \mathbb{N}, \text{ some } c_{31} > 0,\\
\Rightarrow\, &\hat{c}\left\|u_n\right\|^p \leq c_{32}\left(\lambda\left\|u_n\right\| + \lambda + 1\right) \text{ for all } n \in \mathbb{N}, \text{ some } c_{32} > 0,\\
\Rightarrow\, &\left\{u_n\right\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega) \text{ is bounded}.
\end{aligned}$$

We may assume that

$$u_n \xrightarrow{w} u_\lambda^* \text{ in } W_0^{1,p}(\Omega), u_n \to u_\lambda^* \in L^p(\Omega) \text{ as } n \to \infty. \tag{82}$$

Suppose $u_\lambda^* = 0$. Then from (80) with $h = u_n \in W_0^{1,p}(\Omega)$, we have

$$\begin{aligned}
\hat{c}\left\|u_n\right\|^p &\leq \int_\Omega \lambda u_n^{1-\eta}\mathrm{d}z + \lambda\left\|u_n\right\|_\tau^\tau + \int_\Omega f\left(z, u_n\right)u_n\mathrm{d}z\\
&\leq \lambda c_{33}\left\|u_n\right\|_p^{1-\eta} + \lambda\left\|u_n\right\|_\tau^\tau + \int_\Omega f\left(z, u_n\right)u_n\mathrm{d}z \text{ for some } c_{33} > 0, \text{ all } n \in \mathbb{N}.
\end{aligned}$$

Since $u_\lambda^* = 0$, from (81), (82) and since $\tau < p$, we see that the right hand side of the above inequality converges to zero as $n \to \infty$, hence (see [42])

$$\begin{aligned}
&u_n \to 0 \text{ in } W_0^{1,p}(\Omega),\\
\Rightarrow\, &u_n \to 0 \text{ in } L^\infty(\Omega).
\end{aligned}$$

So, we can find $n_0 \in \mathbb{N}$ such that

$$0 \leq u_n(z) \leq \delta \text{ for a.a. } z \in \Omega, \text{ all } n \geq n_0. \tag{83}$$

Fix $n \geq n_0$ otherwise arbitrary and consider the function

$$g_\lambda(z,x) = \begin{cases} \lambda(x)^{\tau-1} & \text{if } x \leqslant u_n(z)\\ \lambda u_n(z)^{\tau-1} & \text{if } u_n(z) < x. \end{cases} \tag{84}$$

We set $G_\lambda(z,x) = \int_0^x g_\lambda(z,s)\mathrm{d}s$ and consider the C^1-functional $w_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$w_\lambda(u) = \frac{1}{p}\rho_{a_{1,p}}(Du) + \frac{1}{q}\rho_{\alpha_2 q}(Du) - \int_\Omega G_\lambda(z,u)\mathrm{d}z \text{ for all } u \in W_0^{1,p}(\Omega).$$

Evidently, $w_\lambda(\cdot)$ is coercive (see (84))) and sequentially weakly lower semicontinuous. So, we can find $\tilde{u}_\lambda \in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned}
&w_\lambda\left\{\tilde{u}_\lambda\right\} = \inf\left\{w_\lambda(u) : u \in W_0^{1,p}(\Omega)\right\},\\
\Rightarrow\, &\left\langle w_\lambda'(\tilde{u}_\lambda), h\right\rangle = 0 \text{ for all } h \in W_0^{1,p}(\Omega),\\
\Rightarrow\, &\left\langle v\left(\tilde{u}_\lambda\right), h\right\rangle = \int_\Omega g_\lambda\left(z, \tilde{u_\lambda}\right)h\mathrm{d}z \text{ for all } h \in W_0^{1,p}(\Omega).
\end{aligned} \tag{85}$$

In (85) we choose $h = -\tilde{u}_\lambda^- \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned}
&\hat{c}\left\|\tilde{u}_\lambda^-\right\|^p \leqslant 0 \quad (\text{ see } (84)),\\
\Rightarrow\, &\tilde{u}_\lambda \geq 0 \text{ and since } \tau < q < p, \tilde{u}_\lambda \neq 0.
\end{aligned}$$

Also in (85) we use $h=(\tilde{u}_\lambda-u_n)^+\in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned}&\left\langle V\left(\tilde{u}_\lambda\right),\left(\tilde{u}_\lambda-u_n\right)'\right\rangle\\&=\int_\Omega \lambda u_n^{\tau-1}\left(\tilde{u}_\lambda-u_n\right)^+\mathrm{d}z\quad(\text{ see (84)})\\&\le\int_\Omega\left(\lambda\left(u_n^{-\eta}+u_n^{\tau-1}\right)+f\left(z,u_n\right)\right)\left(\tilde{u}_\lambda-u_n\right)^+\mathrm{d}z\\&\quad(\text{ on account of (83)}, f(z,u_n)\ge 0 \text{ for a.e. } z\in\Omega)\\&=\left\langle V\left(u_n\right),\left(\tilde{u}_\lambda-u_n\right)^+\right\rangle\quad(\text{ since } u_n\in S_\lambda),\\&\Rightarrow\tilde{u}_\lambda\le u_n,\\&\Rightarrow\tilde{u}_\lambda=\bar{u}_\lambda\quad(\text{see (84), (85) and Proposition 4}),\\&\Rightarrow\bar{u}_\lambda\le u_n\quad\text{ for all } n\ge n_0,\end{aligned}$$

which contradicts our hypothesis that $u_\lambda^*=0$ (see (82)). So, $u_\lambda^*\neq 0$.

From (80) with $h=u_n-u_\lambda^*\in W_0^{1,p}(\Omega)$, we have

$$\left\langle V\left(u_n\right),u_n-u_\lambda^*\right\rangle=\int_\Omega\left(\lambda\left(\frac{u_n-u_\lambda^*}{u_n^\eta}+u_n^{\tau-1}\left(u_n-u_\lambda^*\right)\right)+f(z,u_n)\left(u_n-u_\lambda^*\right)\right)\mathrm{d}z \text{ for all } n\in\mathbb{N}. \tag{86}$$

From (82) we see that

$$\int_\Omega u_n^{\tau-1}\left(u_n-u_\lambda^*\right)\mathrm{d}z\to 0,\ \int_\Omega f(z,u_n)\left(u_n-u_\lambda^*\right)\mathrm{d}z\to 0 \text{ as } n\to\infty. \tag{87}$$

Also we have

$$\begin{aligned}0\leqslant\left|\frac{u_n-u_\lambda^*}{u_n^\eta}\right|&=\frac{u_n-u_\lambda^*}{u_n^\eta}\quad(\text{recall that } \left\{u_n\right\}_{n\in\mathbb{N}} \text{ is decreasing})\\&\le u_n^{1-\eta}\le g\quad\text{ for a.a. } z\in\Omega,\ \text{with } g\in L^p(\Omega) \text{ (see (82))}.\end{aligned}$$

In addition we know that

$$\frac{u_n-u_\lambda^*}{u_\lambda^\eta}\to 0 \text{ for a.a. } z\in\Omega,\ \text{ as } n\to\infty.$$

Hence the Lebesgue dominated convergence theorem, implies that

$$\int_\Omega\frac{u_n-u_\lambda^*}{u_n^\eta}\mathrm{d}z\to 0 \text{ as } n\to\infty. \tag{88}$$

If in (86) we pass to the limit as $n\to\infty$ and use (87), (88), we obtain

$$\begin{aligned}&\lim_{n\to\infty}\left\langle V\left(u_n\right),u_n-u_\lambda^*\right\rangle=0,\\&\Rightarrow u_n\to u_\lambda^* \text{ in } W_0^{1,p}(\Omega) \text{ as } n\to\infty.\end{aligned} \tag{89}$$

In (80) we pass to the limit as $n\to\infty$ and using the monotone convergence theorem and (89), we obtain

$$\begin{aligned}&\left\langle V\left(u_\lambda^*\right),h\right\rangle=\int_\Omega\left[\lambda\left(\left(u_\lambda^*\right)^{-\eta}+\left(u_\lambda^*\right)^{\tau-1}\right)+f\left(z,u_\lambda^*\right)\right]h\mathrm{d}z \text{ for all } h\in W_0^{1,p}(\Omega),\\&\Rightarrow u_\lambda^*\in S_\lambda\subseteq \operatorname{int} C_+,\quad u_\lambda^*=\operatorname{int} S_\lambda.\end{aligned}$$

(b) Suppose $\lambda_1,\lambda_2\in\mathcal{L}$ with $0<\lambda_1<\lambda_2$. Let $u_{\lambda_2}^*\in S_{\lambda_2}\subseteq\operatorname{int} C_+$ be the minimal solution of problem ρ_{λ_2} produced in part (a). From Proposition 8 we know that there exists $u_{\lambda_1}\in S_{\lambda_1}\subseteq\operatorname{int} C_1$ such that

$$u_{\lambda_2}^*-u_{\lambda_1}\in\operatorname{int} C_+.$$

Let $u_{\lambda_1}^*\in S_{\lambda_1}\subseteq\operatorname{int} C_+$ be the minimal solution of ρ_{λ_1}. We have

$$\begin{aligned}&u_{\lambda_2}^*-u_{\lambda_1}\le u_{\lambda_2}^*-u_{\lambda_1}^*,\\&\Rightarrow u_{\lambda_2}^*-u_{\lambda_1}\in\operatorname{int} C_+.\end{aligned}\quad\square$$

Using the extremal positive solution, we can prove the admissibility of the critical parameter λ^*.

Proposition 12. *If hypotheses H_0 and H_1 or H_1' hold, then $\lambda^*\in\mathcal{L}$.*

Proof. We consider a sequence $\left\{\lambda_n\right\}_{n\in\mathbb{N}}\subseteq(0,\lambda^*)$ such that $\lambda_n\to(\lambda^*)^-$. From Proposition 11, we know that $u_{\lambda_1}^*\le u_{\lambda_n}^*$ for all $n\in\mathbb{N}_0$. Using Proposition 4 and since $u_{\lambda_1}^*\in\operatorname{int} C_+$, we can find $\psi\in(0,\lambda_1)$ small such that $\bar{u}_\psi(z)\le\min\left\{\delta,u_{\lambda_1}^*(z)\right\}$ for all $z\in\bar{\Omega}$ ($\delta>0$ as in hypothesis $H_1(iii) = H_1'(iii)$). We introduce the Carathéodory functions $e_{\lambda_1}(z,x)$ and $\hat{e}_{\lambda_1}(z,x)$ defined by

$$e_{\lambda_1}(z,x) = \begin{cases} \lambda_1\left(\bar{u}_\psi^{-\eta} + \bar{u}_\psi^{\tau-1}\right) + f(z,\bar{u}_\psi) & \text{if } x \le \bar{u}_\psi(z), \\ \lambda_1\left(x^{-\eta} + x^{\tau-1}\right) + f(z,x) & \text{if } \bar{u}_\psi(z) < x, \end{cases} \tag{90}$$

$$\hat{e}_{\lambda_1}(z,x) = \begin{cases} e_{\lambda_1}(z,x) & \text{if } x \leqslant u_{\lambda_1}^*(z), \\ e_{\lambda_2}\left(z,u_\lambda^*\right) & \text{if } u_{\lambda_2}^*(z) < x. \end{cases} \tag{91}$$

We set $E_{\lambda_1}(z,x) = \int_0^x e_{\lambda_1}(z,s)\mathrm{d}s$ and $\hat{E}_{\lambda_1}(z,x) = \int_0^x \hat{e}_{\lambda_1}(z,x)\mathrm{d}s$ and consider the C^1 -functionals $\varphi_{\lambda_1}, \hat{\varphi}_{\lambda_1} : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_{\lambda_1}(u) = \frac{1}{p}\rho_{a_1,p}(Du) + \frac{1}{q}\rho_{a_2,q}(Du) - \int_\Omega E_{\lambda_1}(z,u)\,\mathrm{d}z,$$

$$\hat{\varphi}_{\lambda_1}(u) = \frac{1}{p}\rho_{a_1,p}(Du) + \frac{1}{q}\rho_{a_2,q}(Du) - \int_\Omega \hat{E}_{\lambda_1}(z,u)\mathrm{d}z \text{ for all } u \in W_0^{1,p}(\Omega).$$

From (90) and (91), we see that

$$\varphi_{\lambda_1}\Big|_{\left[\bar{u}_\psi, u_{\lambda_2}^*\right]} = \hat{\varphi}_{\lambda_1}\Big|_{\left[\bar{u}_\psi, u_{\lambda_2}^*\right]}. \tag{92}$$

From (91) we see that $\hat{\varphi}_{\lambda_1}$ is coercive. Also it is sequentially weakly lower semicontinuous. So, we can find $u_{\lambda_1} \in W_0^{1,p}(\Omega)$ such that

$$\hat{\varphi}_{\lambda_1}\left(u_{\lambda_1}\right) = \inf\left\{\hat{\varphi}_{\lambda_1}(u) : u \in W_0^{1,p}(\Omega)\right\}, \tag{93}$$

$$\begin{aligned} &\Rightarrow \langle \varphi_{\lambda_1}'(u_{\lambda_1}), h\rangle = 0 \text{ for all } h \in W_0^{1,p}(\Omega), \\ &\Rightarrow \langle V(u_{\lambda_1}), h\rangle = \int_\Omega \hat{e}_{\lambda_1}(z,u_{\lambda_1})h\mathrm{d}z \text{ for all } h \in W_0^{1,p}(\Omega). \end{aligned} \tag{94}$$

In (94) use the test function $h = (u_{\lambda_1} - u_{\lambda_2}^*)^+ \in W_0^{1,p}$. Then

$$\begin{aligned} &\langle V(u_{\lambda_1}), (u_{\lambda_1} - u_{\lambda_2}^*)^+\rangle \\ &= \int_\Omega (\lambda_1((u_{\lambda_2}^*)^{-\eta}) + (u_{\lambda_2}^*)^{\tau-1}) + f(z,u_{\lambda_2}^*)(u_{\lambda_1} - u_{\lambda_2}^*)^+\mathrm{d}z \text{ (see (90), (91))} \\ &\le \int_\Omega (\lambda_2((u_{\lambda_2}^*)^{-\eta}) + (u_{\lambda_2}^*)^{\tau-1}) + f(z,u_{\lambda_2}^*)(u_{\lambda_1} - u_{\lambda_2}^*)^+\mathrm{d}z \text{ (since } \lambda_1 < \lambda_2) \\ &= \langle V(u_{\lambda_2}^*), (u_{\lambda_1} - u_{\lambda_2}^*)^+\rangle, \\ &\Rightarrow u_{\lambda_1} \le u_{\lambda_2}^*. \end{aligned}$$

Next in (94) we choose $h = (\bar{u}_\psi - u_{\lambda_1})^+ \in W_0^{1,p}(\Omega)$. We have

$$\begin{aligned} &\langle V(u_{\lambda_1}), (\bar{u}_\psi - u_{\lambda_1})^+\rangle \\ &= \int_\Omega \lambda_1(\bar{u}_\psi^{-\eta} + \bar{u}_\psi^{\tau-1}) + f(z,\bar{u}_\psi)(\bar{u}_\psi - u_{\lambda_1})^+\mathrm{d}z \text{ (see (90), (91))} \\ &\ge \int_\Omega (\lambda_1(\bar{u}_\psi^{-\eta}(\bar{u}_\psi - u_{\lambda_1})^+\mathrm{d}z \text{ (since } f(z,\bar{u}_\psi) \ge 0))) \\ &\ge \int_\Omega \psi(\bar{u}_\psi^{-\eta}(\bar{u}_\psi - u_{\lambda_1})^+\mathrm{d}z \text{ (since } \psi < \lambda_1)) \\ &= \langle V(\bar{u}_\psi), (\bar{u}_\psi - u_{\lambda_1})^+\rangle \text{ (see Proposition 4)}, \\ &\Rightarrow \bar{u}_\psi \le u_{\lambda_1}. \end{aligned}$$

So, we have proved that

$$u_{\lambda_1} \in [\bar{u}_\psi, u_{\lambda_2}^*]. \tag{95}$$

From (93), we have

$$\begin{aligned} &\hat{\varphi}_{\lambda_1}(u_{\lambda_1}) \le \hat{\varphi}_{\lambda_1}(\hat{u}_\psi), \\ &\Rightarrow \varphi_{\lambda_1}(u_{\lambda_1}) \le \varphi_{\lambda_1}(\hat{u}_\psi) \text{ see (91), (94)} \\ &\qquad = \frac{1}{p}\rho_{\alpha_{1,p}}(D\bar{u}_\psi) + \frac{1}{q}\rho_{\alpha_2,q}(D\bar{u}_\psi) - \int_\Omega E_{\lambda_1}(z,\bar{u}_\psi)\mathrm{d}z \\ &\qquad \le \rho_{\alpha_{1,p}}(D\bar{u}) + \rho_{\alpha_2,q}(D\bar{u}_\psi) - \int_\Omega \lambda_1\bar{u}_\psi^{1-\eta}\mathrm{d}z \\ &\qquad = \langle V(\bar{u}_\psi, \bar{u}_\psi)\rangle - \int_\Omega \lambda_1(\bar{u}_\psi^{-\eta})\bar{u}_\psi\mathrm{d}z \\ &\qquad = 0 \text{ (see Proposition 4)}. \end{aligned}$$

We can say that

$$u_{\lambda_1} \in S_{\lambda_1} \subseteq \operatorname{int} C_+ \text{ see (95)}, \ \varphi_{\lambda_1}(u_{\lambda_1}) \leq 0.$$

We repeat the same argument with λ_1 replaced by λ_2 and $u^*_{\lambda_2}$ replaced by $u^*_{\lambda_3}$. We produce $u_{\lambda_2} \in W^{1,p}_0(\Omega)$ such that

$$u_{\lambda_2} \in S_{\lambda_2} \subseteq \operatorname{int} C_+, \varphi_{\lambda_2}(u_{\lambda_2}) \leq 0.$$

We continue this way and generate a sequence $\{u_n = u_{\lambda_n}\}_{n\in\mathbb{N}}$ such that

$$u_n = u_{\lambda_n} \in S_{\lambda_n} \subseteq \operatorname{int} C_+, \bar{u}_\psi \leq u_n, \varphi_{\lambda_n}(u_n) \leq 0 \text{ for all } n \in \mathbb{N}. \tag{96}$$

Using (96) and reasoning as in Claim 1 in the proof of Proposition 5 (if hypotheses H_1 hold) and as in the Claim in the proof of Proposition 6 (f hypotheses H_1' hold), we show that

$$\{u_n\}_{n\in\mathbb{N}} \subseteq W^{1,p}_0(\Omega) \text{ is bounded.}$$

So, we may assume that

$$u_n \xrightarrow{w} u_* \text{ in } W^{1,p}_0(\Omega), u_n \to u_* \text{ in } L^r(\Omega)\ (r > p). \tag{97}$$

From (96), we have

$$\langle V(u_n), h\rangle = \int_\Omega [\lambda_n(u_n^{-\eta} + u_n^{\tau-1}) + f(z, u_n)]h\mathrm{d}z \text{ for all } h \in W^{1,p}_0(\Omega), \text{ all } h \in \mathbb{N}. \tag{98}$$

In (98), we use $h = u_n - u_* \in W^{1,p}_0(\Omega)$. Then an account of (97) we see that

$$\int_\Omega (\lambda_n u_n^{\tau-1} + f(z, u_n))(u_n - u_*)\mathrm{d}z \to 0 \text{ as } n \to \infty. \tag{99}$$

Also note that

$$\begin{aligned} &\left|\int_\Omega \lambda_n u_n^{\eta}(u_n - u_*)\mathrm{d}z\right| \\ \leq&\lambda^* \int_\Omega \bar{u}_\psi^{-\eta}|u_n - u_*|\mathrm{d}z \text{ (see (96))} \\ \leq&\lambda^* c_{34} \int_\Omega \frac{|u_n - u_*|}{\hat{d}}\mathrm{d}z \text{ for some } c_{34} > 0 \text{ (since } \bar{u}_\psi \in \operatorname{int} C_+). \end{aligned}$$

Using Proposition 1 (Hardy's inequality), we infer that

$$\begin{aligned} &\left\{\frac{u_n - u_*}{\hat{d}}\right\}_{n\in\mathbb{N}} \subseteq L^p(\Omega) \text{ is bounded. (see (96))}, \\ \Rightarrow &\left\{\frac{u_n - u_*}{\hat{d}}\right\}_{n\in\mathbb{N}} \text{ is uniformly integrable.} \end{aligned}$$

Also, we have

$$\frac{(u_n - u_*)(z)}{\hat{d}(z)} \to 0 \text{ for a.a. } z \in \Omega \text{ as } n \in \infty \text{ (see (97))}.$$

Hence by Vitali's Theorem (see [25], Theorem 2.147, p.91), we have

$$\begin{aligned} &\int_\Omega \frac{|u_n - u_*|}{\hat{d}}\mathrm{d}z \to 0, \\ \Rightarrow &\lambda_n \int_\Omega u_n^{-\eta}(u_n - u_*)\mathrm{d}z \to 0. \end{aligned} \tag{100}$$

Therefore from (98) with $h = u_n - u_* \in W^{1,p}_0(\Omega)$, using (99), (100), we obtain

$$\begin{aligned} &\lim_{n\to\infty} \langle V(u_n), u_n - u_*\rangle = 0, \\ \Rightarrow &u_n \to u_* \text{ in } W^{1,p}_0(\Omega) \text{ as } n \to \infty. \end{aligned} \tag{101}$$

So, if in (98) we pass to the limit as $n \to \infty$, arguing as above and using (101), we obtain

$$\langle V(u_*), h\rangle = \int_\Omega [\lambda^*(u_*^{-\eta} + u_*^{\tau-1}) + f(z, u_*)]h\mathrm{d}z \text{ for all } h \in W^{1,p}_0(\Omega), \bar{u}_\psi \leq u_* \text{ (see (96))}.$$

We conclude that $u_* \in S_{\lambda^*} \subseteq \operatorname{int} C_+$ and $\lambda^* \in \mathcal{L}$. □

Summarizing, we can state the following result concerning the positive solutions of problem (1). The result is global in the parameter $\lambda > 0$ (bifurcation type theorem).

Theorem 1. *If hypotheses H_0 and H_1 or H_1' hold, then there exists $\lambda^* > 0$ such that* *(a) for all $\lambda \in (0, \lambda^*)$ problem (1) has at least two positive solutions*

$$u_0, \hat{u} \in \operatorname{int} C_+;$$

(b) for $\lambda = \lambda^*$ problem (1) has at least one positive solution

$$u_* \in \operatorname{int} C_+;$$

(c) for all $\lambda > \lambda^*$ problem (1) has no positive solutions. Moreover, for every $\lambda \in (0, \lambda^*]$, problem (1) has a smallest positive solution $u^*_\lambda \in \operatorname{int} C_+$ and

$$\text{“}0 < \lambda_1 < \lambda_2 \le \lambda^* \Rightarrow u^*_{\lambda_2} - u^*_{\lambda_1} \in \operatorname{int} C_+\text{”}.$$

5. Solution multifunction

In this section, we examine the solution multifunction $\lambda \to S_\lambda$ and determine its continuity properties. Our result extends the works of [17,18] (for nonsingular equations) and of [19] (singular problems driven by the p-Laplacian).

We start with a basic topological property of the solution set S_λ.

Proposition 13. *If hypotheses H_0 and H_1 or H_1' hold, then for every $\lambda \in \mathcal{L}, S_\lambda \subseteq C^1_0(\bar{\Omega})$ is compact.*

Proof. Let $\lambda \in \mathcal{L}$. From Proposition 11 we know that S_λ has a smallest element $u^*_\lambda \in \operatorname{int} C_+$. Using Proposition 4, we can find $\psi \in (0, \lambda)$ small such that

$$\bar{u}_\psi \le u^*_\lambda \le u \text{ for all } u \in S_\lambda. \tag{102}$$

We consider the Carathéodory function $k_\lambda(z, x)$ from the proof of Proposition 5 (see (14)), and the corresponding C^1-functional $\psi_\lambda : W^{1,p}_0(\Omega) \to \mathbb{R}$ (see the proof of Proposition 5). If hypotheses H_1 hold, then from Claim 1 in the proof of Proposition 5, we have that $S_\lambda \subseteq W^{1,p}_0(\Omega)$ is bounded. Similarly, if hypotheses H_1' hold using this time the Claim in the proof of Proposition 6. The boundedness in $W^{1,p}_0(\Omega)$, implies that $S_\lambda \subseteq L^\infty(\Omega)$ and it is bounded (see [27,42]). Using (102) and recalling that $\bar{u}_\psi \in \operatorname{int} C_+$, we have

$$\begin{aligned} &|\lambda(u^{-\eta} + u^{\tau-1}) + f(z, u)| \\ \le& c_{35}(\lambda \hat{d}^\eta + 1) \text{ for some } c_{35} > 0 \\ \le& \lambda c_{36} \hat{d}^{-\eta} \text{ for some } c_{36} > 0, \text{ all } u \in S_\lambda. \end{aligned}$$

Thus we can use Theorem 1.7 of Giacomoni, Kumar, Sreenadh [36] and get $\alpha \in (0, 1)$ and $c_{37} > 0$ such that

$$u \in C^{1,\alpha}_0(\bar{\Omega}), \|u\|_{C^{1,\alpha}_0(\bar{\Omega})} \le c_{37} \text{ for all } u \in S_\lambda.$$

We know that $C^{1,\alpha}_0(\bar{\Omega}) \hookrightarrow C^1_0(\bar{\Omega})$ compactly (Arzela–Ascoli theorem). Therefore we infer that $S_\lambda \subseteq C^1_0(\bar{\Omega})$ is relatively compact. We can easily see that $S_\lambda \subseteq C^1_0(\bar{\Omega})$ is closed. Therefore $S_\lambda \subseteq C^1_0(\bar{\Omega})$ is compact. □

Remark 5. A careful reading of the above proof reveals that for every closed interval $[\tilde{\lambda}_0, \tilde{\lambda}_1] \subseteq \mathcal{L}$, we have that $\bigcup\{S_\lambda : \lambda \in [\tilde{\lambda}_0, \tilde{\lambda}_1]\}$ is relatively compact in $C^1_0(\bar{\Omega})$. So the solution multifunction $\lambda \to S_\lambda$ is locally compact (see [25], p. 275).

In what follows, $P_k(C^1_0(\bar{\Omega}))$ denotes the family of nonempty and compact subsets of $C^1_0(\bar{\Omega})$.

Proposition 14. *If hypotheses H_0 and H_1 or H_1' hold, then the multifunction $\mathcal{L} \ni \lambda \to S_\lambda \in P_k(C^1_0(\bar{\Omega}))$ is lsc and h-lsc.*

Proof. According to Proposition 5.6, p.274, of Hu, Papageorgiou [25], in order to obtain the lower semicontinuity of the solution multifunction, it suffices to show that if $\lambda_n \to \lambda \in \mathcal{L}$ in $\mathcal{L} \in (0, \lambda^*]$, then

$$S_\lambda \subseteq \liminf_{n\to\infty} S_{\lambda_n}. \tag{103}$$

Let $u \in S_\lambda \subseteq \operatorname{int} C_+$ and consider the following Dirichlet problem

$$-\Delta^{\alpha_1}_p v - \Delta^{\alpha_2}_q v = \lambda_n(u^\eta + u^{\tau-1}) + f(z, u) \text{ in } \Omega, u|_{\partial\Omega} = 0, n \in \mathbb{N}. \tag{104}$$

Let $\lambda_0 = \inf_{n\in\mathbb{N}} \lambda_n > 0$ and using Proposition 4, choose $\psi \in (0, \lambda_0)$ small so that

$$\begin{aligned} &\bar{u}_\psi \le u^*_{\lambda_0}, \\ \Rightarrow& \bar{u}_\psi \text{ for all } u \in S_{\lambda_n}, \text{all } n \in \mathbb{N}. \end{aligned} \tag{105}$$

Note that, if $\hat{\lambda}_0 = \sup_{n\in\mathbb{N}} \lambda_n \leq \lambda^*$, then for every $h \in W_0^{1,p}(\Omega)$ we have

$$\begin{aligned}
\left|\lambda_n \int_\Omega \frac{h}{u^\eta} dz\right| &\leq \hat{\lambda}_0 \int_\Omega \frac{|h|}{\bar{u}_\psi^\eta} dz \text{ (see (105))} \\
&\leq \hat{\lambda}_0 c_{38} \int_\Omega \frac{|h|}{\hat{d}} dz \text{ for some} c_{38} > 0 \text{ (since } \bar{u}_\psi \in \operatorname{int} C_+) \\
&\leq \hat{\lambda}_0 c_{39} \|h\| \text{ for some } c_{39} > 0, \\
\Rightarrow & \lambda_n u^{-\eta} \in W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^*.
\end{aligned}$$

We know that $V(\cdot)$ is maximal monotone and coercive. Hence, by Corollary 2.8.7, p. 135, of Papageorgiou, Rădulescu, Repovs [20], we have that $V(\cdot)$ is surjective. So, from (104) and since $\lambda_n u^{-\eta} + \lambda_n u^{\tau-1} + f(z,u) \in W_0^{-1,p'}(\Omega)$, we see that there exist $v_n \in W_0^{1,p}(\Omega)$ which solves (104). In fact the strict monotone city of $V(\cdot)$ implies that the solution v_n of (104) is unique. On (104) we act with $v_n \in W_0^{1,p}(\Omega)$ and using hypotheses H_0 and the local compactness of the solution multifunction, we obtain

$$\begin{aligned}
&\hat{c}\,\|v_n\|^p \leq c_{40}\left(1 + \|v_n\|\right) \text{ for some } c_{40} > 0, \text{ all } n \in \mathbb{N}, \\
&\Rightarrow \{v_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.}
\end{aligned}$$

From this, as before, via the nonlinear regularity theory (see [36]), we produce $\alpha_1 \in (0,1)$ and $c_{41} > 0$ such that

$$v_n \in C_0^{1,\alpha}(\bar{\Omega}), \|v_n\|_{C_0^{1,\alpha}(\bar{\Omega})} \leq c_{41}, \text{ for all } n \in \mathbb{N}.$$

Exploiting the compact embedding of $C_0^{1,\alpha}(\bar{\Omega})$ into $C_0^1(\bar{\Omega})$, we may assume that

$$\begin{aligned}
&v_n \to v \in C_0^1(\bar{\Omega}) \text{ as } n \to \infty, \\
&\Rightarrow -\Delta_p^{a_1} v - \Delta_q^{\alpha_2} v = \lambda\left(u^{-\eta} + u^{\tau-1}\right) + f(z,u) \text{ in } \Omega, v|_{\partial\Omega} = 0.
\end{aligned} \tag{106}$$

The solution of (106) is unique and clearly u solves (106). Therefore $v = u$ and we have

$$v_n \to u \text{ in } C_0^1(\bar{\Omega}) \text{ as } n \to \infty.$$

Let $v_n^0 = v_n \in \operatorname{int} C_+$ and consider the following Dirichlet problem

$$-\Delta_p^{a_1} v - \Delta_q^{a_2} v = \lambda_n\left(\left(v_n^0\right)^{-\eta} + \left(v_n^0\right)^{\tau-1}\right) + f\left(z, v_n^0\right) \text{ in } \Omega,\ v_n\big|_{\partial\Omega} = 0, n \in \mathbb{N}.$$

Reasoning as above, for every $n \in \mathbb{N}$ this problem has a unique solution $v_n^1 \in \operatorname{int} C_+$. Moreover, as in the proof of Proposition 12, using Vitali's theorem, we have

$$\int_\Omega \lambda_n \frac{h}{\left(v_n^0\right)^\eta} dz \to \int_\Omega \lambda \frac{h}{u^\eta} dz \text{ for all } h \in W_0^{1,p}(\Omega).$$

Therefore, we can say that

$$v_n^1 \to u \in C_0^1(\bar{\Omega}) \text{ as } n \to \infty.$$

Continuing this way, we generate a sequence $\{v_n^k\}_{k\in\mathbb{N}_0} \subseteq \operatorname{int} C_+, n \in \mathbb{N}$ such that

$$\left\{\begin{array}{c}
-\Delta_p^{a_1} v_n^k - \Delta_q^{a_2} v_n^k = \lambda_n\left[\left(v_n^{k-1}\right)^{-\eta} + \left(v_n^{k-1}\right)^{\tau-1}\right] + f\left(z, v_n^{k-1}\right) \text{ in } \Omega, \\
v_n^k\big|_{\partial\Omega} = 0 \text{ for all } k, n \in \mathbb{N}, \\
v_n^k \to u \text{ in } C_0^1(\bar{\Omega}) \text{ as } n \to \infty \text{ for all } k \in \mathbb{N}_0.
\end{array}\right\} \tag{107}$$

Claim. *For every $n \in \mathbb{N}$, the sequence $\{v_n^k\}_{k\in\mathbb{N}_0} \subseteq W_0^{1,p}(\Omega)$ is bounded.*

We argue indirectly. So suppose that the assertion of the Claim is not true. We may assume that

$$\left\|v_n^k\right\| \to \infty \text{ as } k \to \infty. \tag{108}$$

We set $y_k = \frac{v_n^k}{\|v_n^k\|}$ $k \in \mathbb{N}_0$. Then $\|y_k = 1\|, y_k = 0$ for all $k \in \mathbb{N}_0$. So, we may assume that

$$y_k \xrightarrow{w} y \text{ in } W_0^{1,p}(\Omega), y_k \to y \text{ in } L^r(\Omega). \tag{109}$$

From (107), we have

$$\begin{aligned}
&\langle A_p^{\alpha_1}(y_k), h\rangle + \frac{1}{\|v_n^k\|^{p-q}} \langle A_q^{\alpha_2}(y_n^k), h\rangle \\
&= \int_\Omega \left(\lambda_n\left(\frac{1}{\left(v_n^{k-1}\right)^\eta \|v_n^k\|^{p-1}} + \frac{1}{\|v_n^k\|^{p-\tau}} y_k^{\tau-1}\right) + \frac{f(z, v_n^{k-1})}{\|v_n^k\|^{p-1}}\right) h dz \\
&\qquad \text{for all } h \in W_0^{1,p}(\Omega), \text{ all } k \in \mathbb{N}_0.
\end{aligned} \tag{110}$$

First assume that hypotheses H_1 hold. If in (110) we use $h = y_k - y \in W_0^{1,p}(\Omega)$, then

$$\lim_{k\to\infty} \left\langle A_p^{a_1}(y_k), y_k - y\right\rangle = 0 \quad (\text{ see (108),(109)}), \\ \Rightarrow y_k \to y \text{ in } W_0^{1,p}(\Omega) \quad (\text{ see Proposition 3, so } \|y\| = 1, y \geq 0). \tag{111}$$

If in (110) we pass to the limit as $k \to \infty$ and use hypothesis $H_1(ii)$ and (111), we obtain

$$\langle A_p^{a_1}(y), h\rangle = \int_\Omega \eta_z(z) y^{p-1} h dz \quad \text{for all } h \in W_0^{1,p}(\Omega),$$

with $\eta(z) \leq \eta_k(z) \leq \hat{\eta}(z)$ for a.a. $z \in \Omega$ (see hypothesis $H_1(ii)$). Hence

$$-\Delta_p^{\alpha_1} y(z) = \eta_k(z) y(z)^{p-1} \text{ in } \Omega,\ y|_{\partial\Omega} = 0,$$
$$\Rightarrow y \text{ is nodal or } y = 0.$$

Both possibilities contradict (111).

Now assume that hypotheses H_1' hold. Then in (110) the left hand side is bounded. On the other hand, looking at the right hand side, on account of hypothesis $H_1'(ii)$ and Fatou's lemma, we see that we must have $y = 0$, again a contradiction to (111).

We conclude that for every $n \in \mathbb{N}$ $\left\{v_n^k\right\}_{k\in\mathbb{N}_0} \subseteq W_0^{1,p}(\Omega)$ is bounded. Then as before the nonlinear regularity theory implies that we may assume that for all $n \in \mathbb{N}$, we have

$$v_n^k \to v \text{ in } C_0^1(\Omega) \text{ as } k \to \infty. \tag{112}$$

Passing to the limit as $k \to \infty$ in (107) and using (112), we obtain

$$\left\{ \begin{array}{l} -\Delta_p^{\alpha_1} v_n - \Delta_q^{\alpha_2} v_n = \lambda_n \left(v_n^{-\eta} + v_n^{\tau-1}\right) + f\left(z, v_n\right) \text{ in } \Omega, \\ v_n|_{\partial\Omega} = 0. \end{array} \right\} \tag{113}$$

From (107), (110) and the double limit lemma (see Hu-Papageorgiou [25], p. 43), we can find a sequence $\{k(n)\}_{n\in\mathbb{N}}$ such that

$$u_n = v_n^{k(n)} \to u \text{ in } C_0^1(\bar{\Omega}). \tag{114}$$

But (113) and (114) imply $u \in \liminf_{n\to\infty} S_{\lambda_n}$. Hence

$$S_\lambda \subseteq \lim_{n\to\infty} \text{int}\, S_{\lambda_n},$$
$$\Rightarrow \lambda \to S_\lambda \text{ is lsc.}$$

Because its values are compact $\lambda \to S_\lambda$ is also h-lsc.

Proposition 15. *If hypotheses H_0 and H_1 or H_1' hold, Then $\mathcal{L} \ni \lambda \to S_\lambda \in P_k\left(C_0^1(\bar{\Omega})\right)$ is usc and h-usc.*

Proof. We know that the solution multifunction is locally compact. So, according to Proposition 5.13, p.273, of Hu, Papageorgiou [25], to show the upper semicontinuity of the solution multifunction, it suffices to show that it has a closed graph. So, let $\left\{\lambda_n\right\}_{n\in\mathbb{N}} \in \mathcal{L}$ be such that $\lambda_n \to \lambda_0$ and $u_n \in S_{\lambda_n} \subseteq \text{int}\, C_+$, such that $u_n \to u$ in $C_0^1(\Omega)$. As before, we set $\lambda_0 = \inf_{n\in\mathbb{N}} \lambda_n > 0$ and choose $\psi \in (0, \lambda_0)$ small so that

$$\bar{u}_\psi \leq u_{\lambda_0}^* \leq u_{\lambda_n}^* \leq u \text{ for all } n \in \mathbb{N}, \text{ all } u \in s_{\lambda_n}.$$

We have

$$\left\langle V\left(u_n\right), h\right\rangle = \int_\Omega k_{\lambda_n}\left(z, u_n\right) h dz \text{ for all } h \in W_0^{1,p}(\Omega), \text{ all } n \in \mathbb{N},$$
$$\Rightarrow \langle V(u), h\rangle = \int_\Omega k_\lambda(z, u) h dz \text{ for all } h \in W_0^{1,p}(\Omega).$$

As before, we check that $u \in [\bar{u}_\psi) \cap \text{int}\, C_+$ (see (12)) and so $u \in S_\lambda$. Therefore $\lambda \to S_\lambda$ has closed graph and so it is both usc and h-usc. □

Combining Propositions 14 and 15, we obtain the following theorem for the solution multifunction.

Theorem 2. *If hypotheses H_0 and H_1 or H_1' hold, then for all $\lambda \in \mathcal{L}$ $S_\lambda \in P_k(C_0^1(\bar{\Omega}))$ and the multifunction $\mathcal{L} \ni \lambda \to S_\lambda$ is continuous and h-continuous.*

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data sharing not applicable to this article as no data sets were generated or analyzed during the current study.

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