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Groundstates with prescribed mass for unbalanced problems with Trudinger–Moser reaction and lack of symmetry

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E-mail: radulescu@agh.edu.pl and xiangmingqi_hit@163.com**Keywords:** mixed local and nonlocal Laplacian, Trudinger–Moser nonlinearity, normalized solution, ground state**2020 Mathematics subject classification:** 35A15, 35R11, 35B38, 35Q92

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Abstract

In this paper, we investigate the existence of normalized ground state solutions for the following mixed local and nonlocal Laplacian with Trudinger–Moser nonlinearity and singular weight

$$\begin{cases} (-\Delta)_{n/\alpha}^\alpha v - \mu \Delta_p v = \lambda |v|^{\frac{n}{\alpha}-2} v + \frac{f(v)}{|x|^\beta}, & x \in \mathbb{R}^n, \\ \int_{\mathbb{R}^n} |v|^{\frac{n}{\alpha}} dx = m, \end{cases}$$

where $n \geq 2$, Δ_p denotes the p -Laplacian with $p \geq n$, $(-\Delta)_{n/\alpha}^\alpha$ represents the fractional n/α -Laplacian with $\alpha \in (0, 1)$, $\mu \geq 0$, $0 < \beta < n$, $m > 0$, $\lambda \in \mathbb{R}$ acts as a Lagrange multiplier, $f \in C(\mathbb{R}, \mathbb{R})$ satisfies the exponential critical growth. We first establish a Trudinger–Moser inequality with weights in fractional Sobolev spaces. Then under some general assumptions and without symmetry constraints, the existence of mountain pass type solutions is obtained. Finally, we show that the mountain pass type solution is a ground state. Key innovations include: 1. Our approach completely avoids radial symmetry assumptions. 2. We address challenges from both the weighted singularity and critical Trudinger–Moser growth. 3. New techniques are developed to circumvent the absence of Pohozaev-type identities. These results are novel even for the classical case $n = 2$, significantly extending existing studies on normalized solutions for mixed local and non-local operators.

1. Introduction and main results

In this paper, we consider the existence of normalized ground state solutions for the following mixed local and nonlocal problem with Trudinger–Moser nonlinearity

$$\begin{cases} (-\Delta)_{n/\alpha}^\alpha v - \mu \Delta_p v = \lambda |v|^{\frac{n}{\alpha}-2} v + \frac{f(v)}{|x|^\beta}, & x \in \mathbb{R}^n, \\ \int_{\mathbb{R}^n} |v|^{\frac{n}{\alpha}} dx = m, \end{cases} \quad (1.1)$$

where $n \geq 2$, Δ_p is the p -Laplacian with $p \geq n$, $\mu \geq 0$, $0 < \beta < n$, $m > 0$, $\lambda \in \mathbb{R}$ is a Lagrange multiplier, $f \in C(\mathbb{R}, \mathbb{R})$ satisfies the exponential critical growth. Here $(-\Delta)_{n/\alpha}^\alpha$ is the fractional n/α -Laplacian with $\alpha \in (0, 1)$ defined as: for each $x \in \mathbb{R}^n$,

$$(-\Delta)_{n/\alpha}^\alpha \varphi(x) = 2 \lim_{\delta \rightarrow 0^+} \int_{\mathbb{R}^n \setminus B_\delta(x)} \frac{|\varphi(x) - \varphi(y)|^{\frac{n}{\alpha}-2} (\varphi(x) - \varphi(y))}{|x - y|^{2n}} dy,$$

along any $\varphi \in C_0^\infty(\mathbb{R}^n)$, where $B_\delta(x) = \{y \in \mathbb{R}^n : |y - x| < \delta\}$. For more details involving the fractional Laplacian, we refer the reader to [33].

In physics, the prescribed norm (in a suitable Lebesgue space) of the solution is often a quantity conserved during the system's time evolution. Therefore, finding solutions with a fixed norm ensures that the solutions can be observed or realized physically with those exact properties. At the same time, a normalized solution usually appears as a critical point of an energy functional. Analysing the nature of this critical point (for instance, whether it is a local minimum) allows physicists to determine if the solution is orbitally stable; a stable solution is one that will persist against small perturbations, making it physically observable. Problems like (1.1) naturally emerge from the superposition of Brownian and Lévy processes. For the background of mixed local and nonlocal operators, we refer the interested readers to the monographs [33, 37]. The fractional Laplacian and generalized nonlocal operators have been received much more attention over the past decade, which arise from finance, quantum mechanics, anomalous diffusion, and so on, see [4, 11, 12, 26] and the references therein. The literature on studying fractional Laplacian via variational methods is rich and huge. In recent years, the mixed local and nonlocal problems have been received more and more attention. For example, Gou *et al* [21] considered the solitary waves of anisotropic Schrödinger involving mixed local and nonlocal operators in $\mathbb{R}^2$. Arora and Rădulescu in [5] discussed the combined effects in mixed local and nonlocal Laplace problems. Biagi and Vecchi [9] studied the multiplicity of positive solutions for mixed local and nonlocal singular critical problems. Molica Bisci *et al* [34] obtained the existence of local minimizers for a critical problem involving a superposition operator of mixed fractional order. For the regularity of solutions involving local and nonlocal operators, we refer to [19].

In recent years, research on normalized solutions to nonlinear elliptic problems has attracted increasing attention. From a physical perspective, normalized solution can satisfy the law of conservation of mass and physical constraints related to some average quantities. In certain theoretical models, such as those governing nonlinear optical phenomena and Bose–Einstein condensates (see [1, 6]), the physical dynamics are described by the following time-dependent nonlinear Schrödinger equation

$$i\psi_t - \Delta\psi = g(|\psi|^2)\psi, \quad x \in \mathbb{R}^N, \quad t > 0.$$

A main feature of the Schrödinger equation is that the mass $\int_{\mathbb{R}^N} |\psi|^2 dx$ maintains invariance under time evolution. In particular, the prescribed mass can be interpreted as power supply in nonlinear optics or as the total number of the particles in the Bose–Einstein condensates.

Recent studies have established significant progress in the existence of normalized solutions for the Schrödinger equation, particularly addressing polynomial-type and exponential critical nonlinearities. For instance, Jeanjean [23] considered the nonlinear Schrödinger equation

$$\begin{cases} -\Delta u = f(u) + \lambda u \text{ in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} u^2 dx = m, \quad u \in H^1(\mathbb{R}^N), \end{cases} \quad (1.2)$$

where $m > 0$, $\lambda \in \mathbb{R}$ arises as a Lagrange multiplier, the nonlinear term f is continuous and satisfies mass supercritical conditions. Jeanjean pioneered a kind of mountain-pass argument to obtain the existence of normalized solution. Bartsch and Soave in [8] proposed a natural constrained method to study the existence of normalized solutions via restricting the discussion on Nehari–Pohozaev manifold. Subsequent developments by Soave in [39, 40] extended these methodologies to the Schrödinger equation with the combined nonlinearities, achieving breakthroughs in characterizing the existence and asymptotic properties of normalized ground states under Sobolev-critical growth conditions. Alves, Ji and Miyagaki in [2] investigated the existence of normalized solutions to (1.2) with an exponential critical growth nonlinearity when $N = 2$. Without assuming the Ambrosetti–Rabinowitz condition, Chang, Liu and Yan [14] considered the existence of normalized ground state solution of (1.2) with critical exponential growth conditions. The following assumption in [2, 14] enable the authors to control upper bound estimate of ground state energy,

(H₁) there exist $p > 4$ and $\theta > 0$ such that

$$\operatorname{sgn}(t)f(t) \geq \theta|t|^{p-1}, \quad \forall t \in \mathbb{R} \setminus \{0\},$$

where $\operatorname{sgn}: \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ is defined by

$$\operatorname{sgn}(t) = \begin{cases} 1 & \text{if } t > 0, \\ -1 & \text{if } t < 0. \end{cases}$$

In particular, the radial symmetry assumption is crucial for verifying the compactness of energy functional.

Recently, under the following assumption,

(H₂) there exist $C_0, C_1 > 0$ and $T_0 > 0$ such that

$$C_0 t^{-2} e^{\alpha_0 t^2} \leq F(t) \leq C_1 |f(t)|, \quad \forall |t| > T_0$$

where $F(t) = \int_0^t f(\tau) \, d\tau$ and $\alpha_0 > 0$, Mo and Wang in [32] overcame the lack of compactness resulting from the exponential critical growth by using some new estimates and approaches, see also [18]. In particular, Lions' compactness Lemma in [42] was used in the argument, which relies on translational invariance of the energy functional. However, for singular problems, the relevant arguments will no longer be applicable. For more papers with respect to the topics of normalized solutions of Schrödinger equations, we refer to [10, 22, 24, 41].

It is noteworthy that problem (1.1) is driven by a mixed operator with unbalanced growth due to the presence of the n -Laplacian and the nonlocal n/α -Laplacian, which can be used to model strongly anisotropic materials, see [49]. Recently, substantial research efforts have focused on establishing the existence of normalized solutions for elliptic equations characterized by this type of unbalanced growth structure. Baldelli and Yang in [7] considered the following $(2, q)$ -Laplacian equation,

$$\begin{cases} -\Delta u - \Delta_q u = \lambda u + g(u), & x \in \mathbb{R}^N, \\ \int_{\mathbb{R}^N} u^2 \, dx = c^2. \end{cases} \quad (1.3)$$

In the L^2 -mass subcritical and supercritical regimes, the authors established the existence of ground state solutions while deriving non-existence results for both the L^2 -mass critical and L^q -mass critical cases. For nonlinear terms g exhibiting mass-subcritical growth near the origin and mass-supercritical growth at infinity, Ding *et al* [15] proved the existence of a locally least energy solution and constructed a second solution with higher energy level. In [43], Wei and Song investigated the existence and multiple of normalized solutions of the $(2, q)$ -Laplacian equation (1.3) with critical nonlinearity. More recently, Cai and Rădulescu [13] studied the following (p, q) -Laplacian equation with L^p -mass constraint,

$$\begin{cases} -\Delta_p u - \Delta_q u + \lambda |u|^{p-2} u = f(u), & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^p \, dx = c^p, \end{cases}$$

where $1 < p < q < N$, $f \in C(\mathbb{R}, \mathbb{R})$ and satisfies weak mass supercritical conditions, they established the existence of ground states and revealed the basic behaviour of the ground state energy E_c as $c > 0$ varies. For more papers about (p, q) -Laplace operator, we refer to [3, 16, 30, 36, 48].

Inspired by the above foundational works, we further consider the normalized solution of mixed local and nonlocal Laplacian problem (1.1) with exponential critical nonlinearity and singular weight, aiming to bridge gaps in the analysis of constrained variational problems with mixed local and non-local operators and Trudinger–Moser nonlinearity. We point out that the importance of PDEs with Trudinger–Moser type nonlinearity in physics stems from their ability to model systems where interactions are maximally strong without causing a mathematical blow-up, a situation that often corresponds to critical physical phenomena like phase transitions. Related patterns involving Trudinger–Moser reaction can be found in statistical hydrodynamics and 2D turbulence (describing the statistical mechanics of point vortices in a 2D fluid, following Onsager's theory), Chern–Simons gauge theory (the nonlinearity emerges from the interaction of matter fields with gauge fields; the related Toda systems, which rely on Moser–Trudinger inequalities, are mathematically challenging and physically motivated), or nonlinear Schrödinger equations (the standard model for describing Bose–Einstein condensates and nonlinear optical media).

Before giving our main results, we introduce our work space.

Definition 1.1. Let $\alpha \in (0, 1)$ and $1 < r < \infty$. Define the fractional Sobolev space

$$W^{\alpha,r}(\mathbb{R}^n) = \{u \in L^r(\mathbb{R}^n) : [u]_{\alpha,r} < \infty\},$$

where

$$[u]_{\alpha,r} = \left(\iint_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)|^r}{|x - y|^{n+r\alpha}} dx dy \right)^{1/r}.$$

Then it is a Banach space when equipped with the norm

$$\|u\|_{\alpha,r} = \|u\|_{L^r(\mathbb{R}^n)} + [u]_{\alpha,r}.$$

Definition 1.2. Let $\alpha \in (0, 1)$, $n \geq 2$ and $p \geq n$. Define

$$\mathcal{W}(\mathbb{R}^n) = \left\{ u \in W^{\alpha,n/\alpha}(\mathbb{R}^n) : \|\nabla u\|_p^p < \infty \right\}$$

and endow it with the norm

$$\|u\| = \|\nabla u\|_{L^p(\mathbb{R}^n)} + \|u\|_{\alpha,n/\alpha} = \|\nabla u\|_{L^p(\mathbb{R}^n)} + \left([u]_{\alpha,n/\alpha}^{n/\alpha} + \|u\|_{L^{n/\alpha}(\mathbb{R}^n)}^{n/\alpha} \right)^{\frac{\alpha}{n}}.$$

To get our main results, we make the following assumptions on the nonlinear term f . Set

$$\mathcal{K}_0 = \max \left\{ \frac{p\alpha + p - n}{\alpha}, \frac{n}{\alpha} \right\}, \quad \mathcal{K}_1 = \mathcal{K}_0 + \frac{n - \beta}{\alpha}.$$

(f_1) $f \in C(\mathbb{R})$ and there exists a constant $\vartheta_0 > 0$ such that

$$\lim_{t \rightarrow \infty} \frac{f(t)}{\exp(\vartheta |t|^{n/(n-\alpha)})} = \begin{cases} 0, & \forall \vartheta > \vartheta_0, \\ \infty, & \forall \vartheta < \vartheta_0. \end{cases}$$

(f_2)

$$\lim_{t \rightarrow 0} \frac{f(t)}{|t|^{\frac{2n}{\alpha}-1}} = 0.$$

(f_3) there exists $\mathcal{K} > \mathcal{K}_1$ such that

$$f(t)t \geq \mathcal{K}F(t) \geq 0$$

for all $t \in \mathbb{R}$, where $F(t) = \int_0^t f(\tau) d\tau$.

(f_4) there exist $M_0 > 0$ and $t_0 > 0$ such that

$$F(t) \leq M_0 |f(t)|, \quad \forall |t| > t_0.$$

(f_5) $\frac{\mathcal{H}(t)}{|t|^{\mathcal{K}_1-1}t}$ is nondecreasing on $(-\infty, 0) \cup (0, \infty)$, where $\mathcal{H}(t) := f(t)t - \frac{n-\beta}{\alpha}F(t)$.

(f_6) there exists $q > \mathcal{K}_0 + \frac{n}{\alpha}$ such that

$$F(t) \geq \mathcal{C}_q |t|^q, \quad \forall t \in \mathbb{R},$$

where

$$\mathcal{C}_q > \max \left\{ \left(\frac{2n\mu}{p\alpha} \left(\frac{m}{C_0} \right)^{\frac{\alpha p}{n}} \right)^{\frac{(q-p)\alpha - p + \beta}{p\alpha + p - n}} \frac{2\mu(p\alpha + p - n) \left(\frac{m}{C_0} \right)^{\frac{(p-q)\alpha}{n}}}{p(q\alpha - n + \beta)} \right. \\ \times \left(\frac{n\vartheta_0}{(n-\beta)\vartheta_{\alpha,n}^*} \right)^{\frac{(n-\alpha)((q-p)\alpha - p + \beta)}{(p\alpha + p - n)\alpha}} S_*^{\frac{p(q\alpha + n - \beta)}{p\alpha + p - n}}, \\ \left. \left(\frac{2m}{C_0} \right)^{\frac{q\alpha - n + \beta}{n}} \left(\frac{n\vartheta_0}{(n-\beta)\vartheta_{\alpha,n}^*} \right)^{\frac{(n-\alpha)(q\alpha - 2n + \beta)}{n\alpha}} \frac{2\alpha \left(\frac{m}{C_0} \right)^{\frac{n-q\alpha}{n}}}{q\alpha - n + \beta} S_*^{\frac{q\alpha - n + \beta}{\alpha}} \right\}$$

and

$$S_* = \inf_{u \in \mathcal{W}(\mathbb{R}^n)} \frac{\|\nabla u\|_p + \|u\|_{\alpha, n/\alpha}}{\left(\int_{\mathbb{R}^n} \frac{|u|^q}{|x|^\beta} dx \right)^{\frac{1}{q}}}.$$

Note that a typical example of f satisfying the assumptions (f_1) – (f_6) is given by

$$f(t) = C_q |t|^{q-2} t \left(e^{\theta_0 |t|^{\frac{n}{n-\alpha}}} - \sum_{k=0}^{j_{n,\alpha}-2} \frac{(\theta_0 |t|^{n/(n-\alpha)})^k}{k!} \right)$$

for all $t \in \mathbb{R}$ and $j_{n,\alpha} = \min\{j \in \mathbb{N}^+ : j \geq n/\alpha\}$.

As usual, a pair of $(u, \lambda) \in \mathcal{W}(\mathbb{R}^n) \times \mathbb{R}$ is said to be a normalized solution of (1.1), if

$$\begin{aligned} & \iint_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)|^{\frac{n}{\alpha}-2} (u(x) - u(y))(v(x) - v(y))}{|x - y|^{2n}} dx dy + \mu \int_{\mathbb{R}^n} |\nabla u|^{p-2} \nabla u \cdot \nabla v dx \\ & = \lambda \int_{\mathbb{R}^n} |u|^{\frac{n}{\alpha}-2} u v dx + \int_{\mathbb{R}^n} \frac{f(u)v}{|x|^\beta} dx \end{aligned}$$

for all $v \in \mathcal{W}(\mathbb{R}^n)$. Our results are as follows.

Theorem 1.1. Assume that f satisfies (f_1) – (f_6) . Problem (1.1) admits at least one normalized solution of mountain pass type in $\mathcal{S}_m \times (-\infty, 0)$.

Theorem 1.2. Assume that f satisfies (f_1) – (f_6) . The mountain pass type solution obtained by theorem 1.1 is a normalized ground state solution of problem (1.1).

Remark 1.1. Our problem (1.1) covers several interesting cases in the literature, including: the $(2, p)$ -Laplacian when $\alpha = 1$ and $n = 2$, and the mixed local and nonlocal n/α -Laplacian when $p = n/\alpha$.

Since the parameter λ in (1.1) is unknown, the usual variational techniques can not be applied directly. Motivated by ideas in the literature, we can regard the parameter λ as a Lagrange multiplier. To achieve our goal, we define the energy functional $\mathcal{I}$ corresponding to (1.1) as

$$\mathcal{I}(u) = \frac{\alpha}{n} \iint_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)|^{\frac{n}{\alpha}}}{|x - y|^{2n}} dx dy + \frac{\mu}{p} \int_{\mathbb{R}^n} |\nabla u|^p dx - \int_{\mathbb{R}^n} \frac{F(u)}{|x|^\beta} dx, \quad (1.4)$$

for all $u \in \mathcal{W}(\mathbb{R}^n)$ and set

$$\mathcal{S}_m = \left\{ u \in \mathcal{W}(\mathbb{R}^n) \setminus \{0\} : \int_{\mathbb{R}^n} |u|^{\frac{n}{\alpha}} dx = m \right\}. \quad (1.5)$$

However, the functional $\mathcal{I}$ is not bounded from below on $\mathcal{S}_m$. To solve such type of difficulty, Bartsch and Soave [8] developed the natural constraint method in which a minimization problem was considered on the Pohozaev manifold. But, the Pohozaev identity of mixed local and nonlocal n -Laplacian is still unknown. To break through such restriction, our proof is based on the minimax technique developed by Jeanjean [22]. Due to the nonlinear term satisfying exponential critical growth, under the assumption (f_6) , we get the upper estimate of the mountain pass level. Then, we obtain that the Palais–Smale sequence is bounded. Moreover, since the absence of radial symmetry assumption, the main difficulty is the lack of compactness of the associated energy functional, we will overcome it by employing the idea of [31], and using Lions' compactness result theorem 2.3 and the lemma 2.7 that, the embedding $\mathcal{W}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n, |x|^{-\beta})$ is compact for $p \geq n/\alpha$. The final difficulty lies in, if there exists λ and u_0 (see lemma 3.4) such that

$$(-\Delta)_{n/\alpha}^\alpha u_0 - \mu \Delta_p u_0 = \lambda |u_0|^{\frac{n}{\alpha}-2} u_0 + |x|^{-\beta} f(u_0),$$

without the compactness of the embedding $\mathcal{W}(\mathbb{R}^n) \hookrightarrow L^{n/\alpha}(\mathbb{R}^n)$, it is challenging to prove that $u_0 \in \mathcal{S}_m$. In order to overcome this challenge, we introduce a new strategy that consists of two key steps: we first show that the parameter λ is strictly negative ($\lambda < 0$), and subsequently establish the strong convergence property of the Palais–Smale sequences by exploiting the monotonicity of operators.

Another key objective of this work is to establish the existence of normalized ground state solutions. Typically, such analysis is conducted on the Pohozaev manifold. However, as previously mentioned, the Pohozaev identity for the mixed local and nonlocal n -Laplacian remains unknown. To address this, our

strategy consists of proving that the mountain pass solution obtained in theorem 1.1 is indeed a ground state. Based on theorem 1.1, we show first that the set

$$\mathcal{P}_m = \{u \in \mathcal{S}_m : P(u) = 0\}$$

is nonempty, where

$$P(u) := \iint_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)|^{\frac{n}{\alpha}}}{|x - y|^{2n}} dx dy + \mu \frac{(\alpha + 1)p - n}{\alpha p} \int_{\mathbb{R}^n} |\nabla u|^p dx - \int_{\mathbb{R}^n} \frac{f(u)u - \frac{n-\beta}{\alpha} F(u)}{|x|^\beta} dx. \quad (1.6)$$

Then we consider the following problem

$$c^*(m) = \inf_{u \in \mathcal{P}_m} \mathcal{I}(u). \quad (1.7)$$

Finally, we show that the mountain pass type solution obtained by theorem 1.1 solves (1.7).

Our results can be seen as an extension and development of the existing results in the literature. However, without the radial symmetry assumption, we mainly discussed the effects of mixed local and nonlocal operators and Trudinger–Moser nonlinearity on the existence of normalized solutions of problem (1.1), which no results appeared in the literature. Indeed, the impaction of radial symmetry is reflected in proving that the solution is nontrivial and the strong convergence of Palais–Smale sequences. Specially, the singular weights disrupts translation invariance. Thus, some arguments relied on translation invariance, such as Lion’s lemma in [42], will no longer be applicable. Consequently, some interesting methods are developed in this paper to address these challenges, including: (1) Replacing symmetry-dependent arguments with techniques grounded in weighted Sobolev embeddings; (2) Addressing the lack of Pohozaev identity via the minimax technique developed by Jeanjean [22].

The structure of the paper is organized as follows: in section 2, we introduce necessary lemmas and prove the functional have the mountain pass geometry. In section 3, we will obtain the upper estimate and the monotonicity behaviour of the mountain pass level $\gamma(m)$. Moreover, by using mountain pass lemma, we will provide the proof of theorem 1.1. In section 4, we show that the mountain pass type solution obtained in theorem 1.1 is a ground state solution.

Notation.

- Throughout this paper, $C, C_1, C_2, \dots$, are indiscriminately used to denote various absolutely positive constants.
- Let $n \geq 2$, $p > 1$, $L^p(\mathbb{R}^n)$ denotes the Lebesgue space with the norm

$$\|u\|_p := \left(\int_{\mathbb{R}^n} |u|^p dx \right)^{\frac{1}{p}}.$$

2. Preliminaries

In this section, we recall the singular Trudinger–Moser inequality in $\mathbb{R}^n$ and then we list some basic results which will be used in the next section.

Lemma 2.1. ([45, lemma 2.1]) *Let $0 < \alpha < 1 < n/\alpha$ and define $\Psi(t) = e^t - \sum_{k=0}^{j_{n,\alpha}-2} \frac{t^k}{k!}$ for all $t \geq 0$ and $j_{n,\alpha} = \min\{j \in \mathbb{N}^+ : j \geq n/\alpha\}$. Then the function Ψ has the following property: for any $r \geq 1$ there holds $(\Psi(t))^r \leq \Psi(rt)$ for all $t \geq 0$.*

Theorem 2.1. ([38, theorem 1.7]) *If $0 < \alpha < 1 < n/\alpha$, then there exists $\vartheta_{\alpha,n}^* > 0$ such that*

$$\sup_{u \in W^{\alpha,n/\alpha}(\mathbb{R}^n) \setminus \{0\}, [u]_{\alpha,n/\alpha} \leq 1} \frac{1}{\|u\|_{n/\alpha}^{n/\alpha}} \int_{\mathbb{R}^n} \Psi(\vartheta |u|^{\frac{n}{n-\alpha}}) dx \begin{cases} < \infty, & \text{if } \vartheta < \vartheta_{\alpha,n}^*, \\ = \infty, & \text{if } \vartheta \geq \vartheta_{\alpha,n}^*. \end{cases}$$

where Ψ is given in lemma 2.1.

Theorem 2.2. *If $0 < \alpha < 1 < n/\alpha$, $\vartheta > 0$, $\beta \in [0, n)$ and $u \in W^{\alpha,n/\alpha}(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} \frac{\Psi(\vartheta |u|^{\frac{n}{n-\alpha}})}{|x|^\beta} dx < \infty,$$

where Ψ is given in lemma 2.1. Moreover,

$$\begin{aligned} & \sup_{u \in W^{\alpha, n/\alpha}(\mathbb{R}^n), [u]_{\alpha, n/\alpha} \leq 1, \|u\|_{n/\alpha}^{n/\alpha} = m > 0} \int_{\mathbb{R}^n} \frac{\Psi(\vartheta |u|^{n/(n-\alpha)})}{|x|^\beta} dx \\ &= \sup_{u \in W^{\alpha, n/\alpha}(\mathbb{R}^n) \setminus \{0\}, [u]_{\alpha, n/\alpha} \leq 1} \frac{m^{\frac{n-\beta}{n}}}{\|u\|_{n/\alpha}^{(n-\beta)/\alpha}} \int_{\mathbb{R}^n} \frac{\Psi(\vartheta |u|^{\frac{n}{n-\alpha}})}{|x|^\beta} dx \begin{cases} < \infty, & \text{if } \vartheta < \left(1 - \frac{\beta}{n}\right) \vartheta_{\alpha, n}^*, \\ = \infty, & \text{if } \vartheta \geq \left(1 - \frac{\beta}{n}\right) \vartheta_{\alpha, n}^*. \end{cases} \end{aligned}$$

Proof. If $\beta = 0$, the results hold by theorem 2.1 and [25]. For $\beta \in (0, n)$, we choose $\tau > 1$ close to 1 such that $\tau\beta \in (0, n)$. Then by the Hölder inequality and lemma 2.1, we have

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{\Psi(\vartheta |u|^{\frac{n}{n-\alpha}})}{|x|^\beta} dx &= \int_{|x| \leq 1} \frac{\Psi(\vartheta |u|^{\frac{n}{n-\alpha}})}{|x|^\beta} dx + \int_{|x| > 1} \frac{\Psi(\vartheta |u|^{\frac{n}{n-\alpha}})}{|x|^\beta} dx \\ &\leq \left(\int_{|x| \leq 1} \frac{1}{|x|^{\tau\beta}} dx \right)^{1/\tau} \left(\int_{\mathbb{R}^n} \Psi\left(\vartheta \frac{\tau}{\tau-1} |u|^{\frac{n}{n-\alpha}}\right) dx \right)^{\frac{\tau-1}{\tau}} + \int_{\mathbb{R}^n} \Psi(\vartheta |u|^{\frac{n}{n-\alpha}}) dx \\ &\leq C \left(\int_{\mathbb{R}^n} \Psi\left(\vartheta \frac{\tau}{\tau-1} |u|^{\frac{n}{n-\alpha}}\right) dx \right)^{\frac{\tau-1}{\tau}} + \int_{\mathbb{R}^n} \Psi(\vartheta |u|^{\frac{n}{n-\alpha}}) dx < \infty. \end{aligned}$$

If $\vartheta < (1 - \frac{\beta}{n})\vartheta_{\alpha, n}^*$, we can choose $\tau > 1$ is sufficiently close to 1 such that $\tau(\tau-1)^{-1}\vartheta < \vartheta_{\alpha, n}^*$ and $\tau < \frac{n}{\beta}$. Then by the Hölder inequality and lemma 2.1, we deduce that

$$\begin{aligned} & \sup_{u \in W^{\alpha, n/\alpha}(\mathbb{R}^n) \setminus \{0\}, [u]_{\alpha, n/\alpha} \leq 1, \|u\|_{n/\alpha}^{n/\alpha} = m} \int_{\mathbb{R}^n} \frac{\Psi(\vartheta |u|^{\frac{n}{n-\alpha}})}{|x|^\beta} dx \\ &\leq \left(\int_{|x| \leq 1} \frac{1}{|x|^{\tau\beta}} dx \right)^{1/\tau} \sup_{u \in W^{\alpha, n/\alpha}(\mathbb{R}^n) \setminus \{0\}, [u]_{\alpha, n/\alpha} \leq 1, \|u\|_{n/\alpha}^{n/\alpha} = m} \left(\int_{\mathbb{R}^n} \Psi\left(\vartheta \frac{\tau}{\tau-1} |u|^{\frac{n}{n-\alpha}}\right) dx \right)^{\frac{\tau-1}{\tau}} \\ &+ \sup_{u \in W^{\alpha, n/\alpha}(\mathbb{R}^n) \setminus \{0\}, [u]_{\alpha, n/\alpha} \leq 1, \|u\|_{n/\alpha}^{n/\alpha} = m} \int_{\mathbb{R}^n} \Psi(\vartheta |u|^{\frac{n}{n-\alpha}}) dx < \infty. \end{aligned}$$

Next we show that

$$\begin{aligned} & \sup_{u \in W^{\alpha, n/\alpha}(\mathbb{R}^n) \setminus \{0\}, [u]_{\alpha, n/\alpha} \leq 1, \|u\|_{n/\alpha}^{n/\alpha} = m} \int_{\mathbb{R}^n} \frac{\Psi(\vartheta |u|^{\frac{n}{n-\alpha}})}{|x|^\beta} dx \\ &= \sup_{u \in W^{\alpha, n/\alpha}(\mathbb{R}^n) \setminus \{0\}, [u]_{\alpha, n/\alpha} \leq 1} \frac{m^{\frac{n-\beta}{n}}}{\|u\|_{n/\alpha}^{(n-\beta)/\alpha}} \int_{\mathbb{R}^n} \frac{\Psi(\vartheta |u|^{\frac{n}{n-\alpha}})}{|x|^\beta} dx. \end{aligned} \quad (2.1)$$

Let $u \in W^{\alpha, n/\alpha}(\mathbb{R}^n) \setminus \{0\}$ with $[u]_{\alpha, n/\alpha} \leq 1$. For $\tau > 0$, we define $u_\tau(x) = u(\tau x)$ for all $x \in \mathbb{R}^n$. Clearly, $[u_\tau]_{\alpha, n/\alpha} = [u]_{\alpha, n/\alpha}$ and $\|u_\tau\|_{n/\alpha}^{n/\alpha} = \tau^{-n} \|u\|_{n/\alpha}^{n/\alpha}$. Choose $\tau^n = \frac{1}{m} \|u\|_{n/\alpha}^{n/\alpha}$. Then $\|u_\tau\|_{n/\alpha}^{n/\alpha} = m$ and $[u_\tau]_{\alpha, n/\alpha} \leq 1$. Consequently,

$$\begin{aligned} & \frac{m^{\frac{n-\beta}{n}}}{\|u\|_{n/\alpha}^{(n-\beta)/\alpha}} \int_{\mathbb{R}^n} \frac{\Psi(\vartheta |u|^{\frac{n}{n-\alpha}})}{|x|^\beta} dx = \int_{\mathbb{R}^n} \frac{\Psi(\vartheta |u_\tau|^{\frac{n}{n-\alpha}})}{|x|^\beta} dx \\ &\leq \sup_{u \in W^{\alpha, n/\alpha}(\mathbb{R}^n) \setminus \{0\}, [u]_{\alpha, n/\alpha} \leq 1, \|u\|_{n/\alpha}^{n/\alpha} = m} \int_{\mathbb{R}^n} \frac{\Psi(\vartheta |u|^{\frac{n}{n-\alpha}})}{|x|^\beta} dx. \end{aligned}$$

Thus,

$$\begin{aligned} & \sup_{u \in W^{\alpha, n/\alpha}(\mathbb{R}^n) \setminus \{0\}, [u]_{\alpha, n/\alpha} \leq 1} \frac{m^{\frac{n-\beta}{n}}}{\|u\|_{n/\alpha}^{(n-\beta)/\alpha}} \int_{\mathbb{R}^n} \frac{\Psi(\vartheta |u|^{\frac{n}{n-\alpha}})}{|x|^\beta} dx \\ &\leq \sup_{u \in W^{\alpha, n/\alpha}(\mathbb{R}^n) \setminus \{0\}, [u]_{\alpha, n/\alpha} \leq 1, \|u\|_{n/\alpha}^{n/\alpha} = m} \int_{\mathbb{R}^n} \frac{\Psi(\vartheta |u|^{\frac{n}{n-\alpha}})}{|x|^\beta} dx. \end{aligned} \quad (2.2)$$

The inverse inequality of (2.2) is obvious. Therefore, (2.1) holds.

For the case $\vartheta \geq (1 - \frac{\beta}{n})\vartheta_{\alpha,n}^*$, we define the Moser type functions which have been used in [35]:

$$\Phi_k(x) = \frac{1}{\gamma_{\alpha,n}^{\alpha/n}} \begin{cases} |\ln k|^{\frac{n-\alpha}{n}} & \text{if } |x| \leq \frac{1}{k}, \\ \frac{|\ln |x||}{|\ln k|^{\frac{\alpha}{n}}} & \text{if } \frac{1}{k} < |x| < 1, \\ 0 & \text{if } |x| \geq 1, \end{cases}$$

where

$$\gamma_{\alpha,n} := \frac{2(n\omega_{n-1})^2 \Gamma(\frac{n}{\alpha} + 1)}{n!} \sum_{j=0}^{\infty} \frac{n+j-1}{j!} \frac{1}{(n+2j)^{n/\alpha}}.$$

ω_{n-1} is the surface area of the unit sphere in $\mathbb{R}^n$. By the result in [35], we get

$$\lim_{k \rightarrow \infty} \|\Phi_k\|_{n/\alpha} = 0 \quad \text{and} \quad \lim_{k \rightarrow +\infty} [\Phi_k]_{\alpha,n/\alpha} = 1. \quad (2.3)$$

Consider $U_k = \frac{\Phi_k}{[\Phi_k]_{\alpha,n/\alpha}}$, then we can write

$$U_k^{n/(n-\alpha)}(x) = \gamma_{\alpha,n}^{-\alpha/(n-\alpha)} \ln k + d_k \quad \text{for } |x| \leq \frac{1}{k},$$

where $d_k \rightarrow 0$. Thus, for $\vartheta \geq (1 - \frac{\beta}{n})\vartheta_{\alpha,n}^* = (n - \beta)\gamma_{\alpha,n}^{\frac{\alpha}{n-\alpha}}$, we conclude that

$$\begin{aligned} \frac{m^{\frac{n-\beta}{n}}}{\|U_k\|_{n/\alpha}^{n/\alpha}} \int_{\mathbb{R}^n} \frac{\Psi(\vartheta |U_k|^{n/(n-\alpha)})}{|x|^\beta} dx &\geq \frac{m^{\frac{n-\beta}{n}}}{\|U_k\|_{n/\alpha}^{n/\alpha}} \int_{B_{1/k}(0)} \frac{\Psi(\vartheta |U_k|^{n/(n-\alpha)})}{|x|^\beta} dx \\ &= \frac{m^{\frac{n-\beta}{n}}}{\|U_k\|_{n/\alpha}^{n/\alpha}} \Psi[\vartheta(\gamma_{\alpha,n}^{-\alpha/(n-\alpha)} \ln k + d_k)] \frac{(\frac{1}{k})^{n-\beta} \omega_{n-1}}{n-\beta} \\ &\geq \frac{m^{\frac{n-\beta}{n}}}{\|U_k\|_{n/\alpha}^{n/\alpha}} \frac{1}{2} \exp[\vartheta(\gamma_{\alpha,n}^{-\alpha/(n-\alpha)} \ln k + d_k)] \frac{(\frac{1}{k})^{n-\beta} \omega_{n-1}}{n-\beta} \\ &= \frac{m^{\frac{n-\beta}{n}}}{\|U_k\|_{n/\alpha}^{n/\alpha}} \frac{1}{2} \exp[(\vartheta \gamma_{\alpha,n}^{-\alpha/(n-\alpha)} - n + \beta) \ln k + \vartheta d_k] \frac{\omega_{n-1}}{n-\beta} \\ &\geq \frac{m^{\frac{n-\beta}{n}}}{2\|U_k\|_{n/\alpha}^{n/\alpha}} \exp(\vartheta d_k) \frac{\omega_{n-1}}{n-\beta} \rightarrow \infty \quad \text{as } k \rightarrow \infty. \end{aligned}$$

Here we have used the inequality that $\Psi(t) \geq \frac{1}{2}e^t$ when t is large enough. This implies that

$$\sup_{u \in W^{\alpha,n/\alpha}(\mathbb{R}^n) \setminus \{0\}, [u]_{\alpha,n/\alpha} \leq 1} \frac{m^{\frac{n-\beta}{n}}}{\|u\|_{n/\alpha}^{(n-\beta)/\alpha}} \int_{\mathbb{R}^n} \frac{\Psi(\vartheta |u|^{n/(n-\alpha)})}{|x|^\beta} dx = \infty,$$

as $\vartheta \geq (1 - \frac{\beta}{n})\vartheta_{\alpha,n}^*$. In conclusion, the proof is complete. $\square$

Moreover, the following Lions' type result holds in the fractional Sobolev space $W^{\alpha,n/\alpha}(\mathbb{R}^n)$.

Theorem 2.3. *Let $\{u_k\}_k$ be a bounded sequence in $W^{\alpha,n/\alpha}(\mathbb{R}^n)$ satisfying $[u_k]_{\alpha,n/\alpha} = 1$ and converging weakly to a nonzero function u . Then for any $\vartheta < (1 - \frac{\beta}{n})\vartheta_{\alpha,n}^* (1 - [u]_{\alpha,n/\alpha}^{n/\alpha})^{-\alpha/(n-\alpha)}$ and $0 \leq \beta < n$,*

$$\sup_k \int_{\mathbb{R}^n} \frac{\Psi(\vartheta |u_k|^{n/(n-\alpha)})}{|x|^\beta} dx < \infty.$$

Proof. By Hölder's inequality, we obtain

$$\int_{\mathbb{R}^n \cap \{|x| \leq 1\}} \frac{\Psi(\vartheta |u_k|^{n/(n-\alpha)})}{|x|^\beta} dx \leq \left(\int_{\mathbb{R}^n} \Psi(\kappa \vartheta |u_k|^{n/(n-\alpha)}) \right)^{\frac{1}{\kappa}} \left(\int_{\mathbb{R}^n \cap \{|x| \leq 1\}} \frac{1}{|x|^{\beta \frac{\kappa}{\kappa-1}}} dx \right)^{\frac{\kappa-1}{\kappa}}$$

where $\kappa > \frac{n}{n-\beta}$ and is sufficiently close to $\frac{n}{n-\beta}$ such that $\vartheta \kappa < (1 - \frac{\beta}{n})\vartheta_{\alpha,n}^* (1 - [u]_{\alpha,n/\alpha}^{n/\alpha})^{-\alpha/n}$.

Observe that $\kappa > \frac{n}{n-\beta}$. Then $\beta \frac{\kappa}{\kappa-1} < n$. It follows that

$$\int_{\mathbb{R}^n \cap \{|x| \leq 1\}} \frac{1}{|x|^{\beta \frac{\kappa}{\kappa-1}}} dx < \infty.$$

On the other hand, we have

$$\int_{\mathbb{R}^n \cap \{|x| \geq 1\}} \frac{\Psi(\vartheta |u_k|^{n/(n-\alpha)})}{|x|^\beta} dx \leq \int_{\mathbb{R}^n \cap \{|x| \geq 1\}} \Psi(\vartheta |u_k|^{n/(n-\alpha)}) dx \leq \int_{\mathbb{R}^n} \Psi(\vartheta |u_k|^{n/(n-\alpha)}) dx$$

Thus,

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{\Psi(\vartheta |u_k|^{n/(n-\alpha)})}{|x|^\beta} dx &\leq \int_{\mathbb{R}^n} \Psi(\vartheta |u_k|^{n/(n-\alpha)}) dx \\ &+ \left(\int_{\mathbb{R}^n} \Psi(\kappa \vartheta |u_k|^{n/(n-\alpha)}) \right)^{\frac{1}{\kappa}} \left(\int_{\mathbb{R}^n \cap \{|x| \leq 1\}} \frac{1}{|x|^{\beta \frac{\kappa}{\kappa-1}}} dx \right)^{\frac{\kappa-1}{\kappa}}. \end{aligned}$$

To get our desired result, it suffices to show that for any $0 < \theta < (1 - \frac{\beta}{n})\vartheta_{\alpha,n}^*(1 - [u]_{\alpha,n/\alpha}^{n/\alpha})^{-\alpha/(n-\alpha)}$, there holds

$$\sup_k \int_{\mathbb{R}^n} \Psi(\vartheta |u_k|^{n/(n-\alpha)}) dx < \infty.$$

Arguing by contradiction, we assume that there exists some $0 < \vartheta_* < (1 - \frac{\beta}{n})\vartheta_{\alpha,n}^*(1 - [u]_{\alpha,n/\alpha}^{n/\alpha})^{-\alpha/(n-\alpha)}$ such that

$$\sup_k \int_{\mathbb{R}^n} \Psi(\vartheta_* |u_k|^{n/(n-\alpha)}) dx = \infty. \quad (2.4)$$

Motivated by [29], for $L \in (0, \infty)$ and $v \in W^{\alpha, n/\alpha}(\mathbb{R}^n)$, we define

$$\mathcal{T}_L(v) = \begin{cases} -L, & \text{if } v < -L, \\ v, & \text{if } |v| \leq L, \\ L, & \text{if } v > L \end{cases}$$

and

$$\mathcal{R}_L(v) = v - \mathcal{T}_L(v) = \begin{cases} v + L, & \text{if } v < -L, \\ 0, & \text{if } |v| \leq L, \\ v - L, & \text{if } v > L. \end{cases}$$

Since $0 < \vartheta_* < (1 - \frac{\beta}{n})\vartheta_{\alpha,n}^*(1 - [u]_{\alpha,n/\alpha}^{n/\alpha})^{-\alpha/(n-\alpha)}$, there exists $\varepsilon_0 > 0$ such that $\vartheta_*(1 + \varepsilon_0) < (1 - \frac{\beta}{n})\vartheta_{\alpha,n}^*(1 - [u]_{\alpha,n/\alpha}^{n/\alpha})^{-\alpha/(n-\alpha)}$. Clearly, $[\mathcal{T}_L(u)]_{\alpha,n/\alpha} \rightarrow [u]_{\alpha,n/\alpha}$ as $L \rightarrow \infty$. Further, we can choose L large enough such that

$$\vartheta_*(1 + \varepsilon_0) < \left(1 - \frac{\beta}{n}\right) \vartheta_{\alpha,n}^*(1 - [\mathcal{T}_L(u)]_{\alpha,n/\alpha}^{n/\alpha})^{-\alpha/(n-\alpha)}. \quad (2.5)$$

Next we show that

$$\limsup_{k \rightarrow \infty} [\mathcal{R}_L u_k]_{\alpha,n/\alpha}^{n/\alpha} < \left(\frac{\left(1 - \frac{\beta}{n}\right) \vartheta_{\alpha,n}^*}{\vartheta_*(1 + \varepsilon_0)} \right)^{\frac{n-\alpha}{\alpha}}. \quad (2.6)$$

If (2.6) does not true, there exists a subsequence of $\{u_k\}$ still denoted by itself such that

$$[\mathcal{R}_L u_k]_{\alpha,n/\alpha}^{n/\alpha} \geq \left(\frac{\left(1 - \frac{\beta}{n}\right) \vartheta_{\alpha,n}^*}{\vartheta_*(1 + \varepsilon_0)} \right)^{\frac{n-\alpha}{\alpha}}. \quad (2.7)$$

It follows from (2.7) that

$$\begin{aligned} 1 &= [\mathcal{u}_k]_{\alpha, n/\alpha}^{n/\alpha} \\ &\geq [\mathcal{R}_L(u_k)]_{\alpha, n/\alpha}^{n/\alpha} + [\mathcal{T}_L(u_k)]_{\alpha, n/\alpha}^{n/\alpha} \\ &\geq \left(\frac{\left(1 - \frac{\beta}{n}\right) \vartheta_{\alpha, n}^*}{\vartheta_*(1 + \varepsilon_0)} \right)^{\frac{n-\alpha}{\alpha}} + [\mathcal{T}_L(u_k)]_{\alpha, n/\alpha}^{n/\alpha}. \end{aligned}$$

Observe that $\{\mathcal{T}_L(u_k)\}_k \subset W^{\alpha, n/\alpha}(\mathbb{R}^n)$ is bounded and $\mathcal{T}_L(u_k) \rightharpoonup \mathcal{T}_L(u)$ weakly in $W^{\alpha, n/\alpha}(\mathbb{R}^n)$. By the weak semicontinuity of the norm, it yields that

$$1 \geq \left(\frac{\left(1 - \frac{\beta}{n}\right) \vartheta_{\alpha, n}^*}{\vartheta_*(1 + \varepsilon_0)} \right)^{\frac{n-\alpha}{\alpha}} + [\mathcal{T}_L(u)]_{\alpha, n/\alpha}^{n/\alpha}.$$

This contradicts with (2.5). Thus, (2.6) holds.

Set

$$D_{k,L} := \left\{ x \in \mathbb{R}^n : |u_k(x)| \geq \frac{1 + \varepsilon_0}{\varepsilon_0} L \right\}.$$

By (2.6), up to a subsequence, it leads to

$$[\mathcal{R}_L(u_k)]_{\alpha, n/\alpha}^{n/\alpha} < \left(\frac{\left(1 - \frac{\beta}{n}\right) \vartheta_{\alpha, n}^*}{\vartheta_*(1 + \varepsilon_0)} \right)^{\frac{n-\alpha}{\alpha}}.$$

Then

$$(\vartheta_*(1 + \varepsilon_0)) [\mathcal{R}_L(u_k)]_{\alpha, n/\alpha}^{\frac{n}{n-\alpha}} < \left(1 - \frac{\beta}{n}\right) \vartheta_{\alpha, n}^* \text{ for all } k \in \mathbb{N}.$$

Further, applying theorem 2.1, we deduce

$$\begin{aligned} &\sup_k \int_{D_{k,L}} \Psi \left(\vartheta_*(1 + \varepsilon_0) |u_k - L|^{n/(n-\alpha)} \right) dx \\ &\leq \sup_k \int_{\mathbb{R}^n} \Psi \left(\vartheta_*(1 + \varepsilon_0) |\mathcal{R}_L(u_k)|^{n/(n-\alpha)} \right) dx < \infty. \end{aligned} \quad (2.8)$$

Then

$$\int_{D_{k,L}} \Psi \left(\vartheta_* |u_k|^{n/(n-\alpha)} \right) dx \leq \int_{D_{k,L}} \Psi \left(\vartheta_*(1 + \varepsilon_0) |u_k - L|^{n/(n-\alpha)} \right) dx. \quad (2.9)$$

Here we have used the facts that Ψ is a nondecreasing function with respect to $t \geq 0$ and $|u_k| \leq (1 + \varepsilon_0) |u_k - L|$ in $D_{k,L}$. Gathering (2.8) and (2.9), we conclude that

$$\sup_k \int_{D_{k,L}} \Psi \left(\vartheta_* |u_k|^{n/(n-\alpha)} \right) dx < \infty. \quad (2.10)$$

To complete the proof, we need to estimate the following term

$$\begin{aligned} &\int_{\mathbb{R}^n \setminus D_{k,L}} \Psi \left(\vartheta_* |u_k|^{n/(n-\alpha)} \right) dx \\ &= \int_{|u_k| < \frac{(1+\varepsilon_0)L}{\varepsilon_0}} \sum_{j=j_{n,\alpha}-1}^{\infty} \frac{\left(\vartheta_* \left| \frac{\varepsilon_0 u_k}{(1+\varepsilon_0)L} \right|^{n/(n-\alpha)} \right)^j}{j!} \left(\frac{(1+\varepsilon_0)L}{\varepsilon_0} \right)^{\frac{nj}{n-\alpha}} dx \\ &\leq \int_{|u_k| < \frac{(1+\varepsilon_0)L}{\varepsilon_0}} \left| \frac{\varepsilon_0 u_k}{(1+\varepsilon_0)L} \right|^{n/\alpha} \sum_{j=j_{n,\alpha}-1}^{\infty} \frac{\vartheta_*^j \left(\frac{(1+\varepsilon_0)L}{\varepsilon_0} \right)^{\frac{nj}{n-\alpha}}}{j!} dx \end{aligned}$$

$$\begin{aligned}
&= \left(\frac{\varepsilon_0}{(1+\varepsilon_0)L} \right)^{n/\alpha} \int_{|u_k| < \frac{(1+\varepsilon_0)L}{\varepsilon_0}} |u_k|^{n/\alpha} dx \sum_{j=j_{n,\alpha}-1}^{\infty} \frac{\vartheta_*^j \left(\frac{(1+\varepsilon_0)L}{\varepsilon_0} \right)^{\frac{nj}{n-\alpha}}}{j!} \\
&\leq C \left(\frac{\varepsilon_0}{(1+\varepsilon_0)L} \right)^{n/\alpha} \exp \left(\vartheta_* \left(\frac{(1+\varepsilon_0)L}{\varepsilon_0} \right)^{\frac{n}{n-\alpha}} \right).
\end{aligned}$$

This together with (2.10) yields

$$\sup_k \int_{\mathbb{R}^n} \Psi \left(\vartheta_* |u_k|^{n/(n-\alpha)} \right) dx < \infty,$$

which contradicts with (2.4). In conclusion, we get the desired result. $\square$

We need the following Gagliardo-Nirenberg inequality in $W^{\alpha, n/\alpha}(\mathbb{R}^n)$.

Lemma 2.2. ([47]) Assume $\alpha \in (0, 1)$ and $1 < n/\alpha$. For any $\kappa > n/\alpha$, there exists a constant $C_{\kappa, n} > 0$ such that

$$\|u\|_{\kappa}^{\kappa} \leq C_{\kappa, n} \|u\|_{n/\alpha}^{n/\alpha} [u]_{\alpha, n}^{\kappa - n/\alpha}, \forall u \in W^{\alpha, n/\alpha}(\mathbb{R}^n).$$

Lemma 2.3. Let $n/\alpha \leq r < \infty$ and $0 < \beta < n$. Then the embedding $W^{\alpha, n/\alpha}(\mathbb{R}^n) \hookrightarrow L^r(\mathbb{R}^n, |x|^{-\beta})$ is compact. Moreover, $\mathcal{W}(\mathbb{R}^n) \hookrightarrow L^r(\mathbb{R}^n, |x|^{-\beta})$ is compact.

Proof. We first show that the embedding $W^{\alpha, n/\alpha}(\mathbb{R}^n) \hookrightarrow L^r(\mathbb{R}^n, |x|^{-\beta})$ is continuous. From Lemma 2.2 of [27], the embedding $W^{\alpha, n/\alpha}(\mathbb{R}^n) \hookrightarrow L^r(\mathbb{R}^n)$ is continuous for any $r \geq n/\alpha$; $W^{\alpha, n/\alpha}(\mathbb{R}^n) \hookrightarrow L^r(B_R(0))$ is compact for any $r \geq 1$ and $R > 0$. Then, we obtain that

$$\int_{\mathbb{R}^n \setminus B_1(0)} \frac{|u|^r}{|x|^\beta} dx \leq \int_{\mathbb{R}^n \setminus B_1(0)} |u|^r dx \leq \|u\|_{\alpha, n/\alpha}^r,$$

and

$$\int_{B_1(0)} \frac{|u|^r}{|x|^\beta} dx \leq \left(\int_{B_1(0)} |u|^{rt} dx \right)^{\frac{1}{t}} \left(\int_{B_1(0)} \frac{1}{|x|^{\beta t'}} dx \right)^{\frac{1}{t'}} \leq C \|u\|_{\alpha, n/\alpha}^r,$$

where $t > 1$, $t' = \frac{t}{t-1}$ and $\beta t' < n$. Then, the embedding $W^{\alpha, n/\alpha}(\mathbb{R}^n) \hookrightarrow L^r(\mathbb{R}^n, |x|^{-\beta})$ is continuous.

Next, using a similar argument to theorem 1.2 of [46], we show that the embedding $W^{\alpha, n/\alpha}(\mathbb{R}^n) \hookrightarrow L^r(\mathbb{R}^n, |x|^{-\beta})$ is compact for any $n/\alpha \leq r < \infty$. For any bounded sequence $\{u_k\}_k$ in $W^{\alpha, n/\alpha}(\mathbb{R}^n)$, up to a subsequence of $\{u_k\}_k$, there exists $u \in W^{\alpha, n/\alpha}(\mathbb{R}^n)$ such that

$$\begin{aligned}
u_k(x) &\rightarrow u(x) \quad \text{in } L^r(B_R(0)) \text{ for all } R > 0, \\
u_k(x) &\rightarrow u(x) \quad \text{a.e. in } \mathbb{R}^n.
\end{aligned}$$

For any $R > 0$, by applying the Egoroff theorem, we obtain that for any $\delta > 0$ there exists $E_\delta \subset B_R(0)$ with $|E_\delta| < \delta$ such that u_k uniformly converges to u in $B_R(0) \setminus E_\delta$. Now we will divide the integral into three parts.

$$\begin{aligned}
&\lim_{k \rightarrow +\infty} \int_{\mathbb{R}^n} \frac{|u_k - u|^r}{|x|^\beta} dx \\
&= \lim_{R \rightarrow +\infty} \lim_{\delta \rightarrow 0} \lim_{k \rightarrow +\infty} \int_{E_\delta} \frac{|u_k - u|^r}{|x|^\beta} dx + \lim_{R \rightarrow +\infty} \lim_{\delta \rightarrow 0} \lim_{k \rightarrow +\infty} \int_{B_R(0) \setminus E_\delta} \frac{|u_k - u|^r}{|x|^\beta} dx \\
&\quad + \lim_{R \rightarrow +\infty} \lim_{\delta \rightarrow 0} \lim_{k \rightarrow +\infty} \int_{\mathbb{R}^n \setminus B_R(0)} \frac{|u_k - u|^r}{|x|^\beta} dx \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

For I_1 , choosing $t > 1$, $t' = \frac{t}{t-1}$ and $\beta t' < n$, it follows from the Hölder inequality and the continuous embedding $W^{\alpha, n/\alpha}(\mathbb{R}^n) \hookrightarrow L^r(\mathbb{R}^n, |x|^{-\beta})$ that

$$\begin{aligned}
I_1 &\leq \lim_{\delta \rightarrow 0} \lim_{k \rightarrow +\infty} \left(\int_{E_\delta} 1 dx \right)^{\frac{1}{t}} \left(\int_{E_\delta} \frac{|u_k - u|^{rt'}}{|x|^{\beta t'}} dx \right)^{\frac{1}{t'}} \\
&\leq C \lim_{\delta \rightarrow 0} (m(E_\delta))^{\frac{1}{t}} \\
&= 0.
\end{aligned}$$

For I_2 , the uniform convergence of u_k in $B_R(0) \setminus E_\delta$ leads to

$$\begin{aligned} I_2 &= \lim_{R \rightarrow +\infty} \lim_{\delta \rightarrow 0} \lim_{k \rightarrow +\infty} \int_{B_R(0) \setminus E_\delta} \frac{|u_k - u|^r}{|x|^\beta} dx \\ &= \lim_{R \rightarrow +\infty} \lim_{\delta \rightarrow 0} \int_{B_R(0) \setminus E_\delta} \lim_{k \rightarrow +\infty} \frac{|u_k - u|^r}{|x|^\beta} dx \\ &= 0. \end{aligned}$$

For I_3 , by the continuous imbedding $W^{\alpha, n/\alpha}(\mathbb{R}^n) \hookrightarrow L^r(\mathbb{R}^n)$ for $r \geq n/\alpha$, we obtain that

$$\begin{aligned} I_3 &\leq \lim_{R \rightarrow +\infty} \lim_{\delta \rightarrow 0} \lim_{k \rightarrow +\infty} \frac{1}{R^\beta} \int_{\mathbb{R}^n \setminus B_R(0)} |u_k - u|^r dx \\ &\leq C \lim_{R \rightarrow +\infty} \frac{1}{R^\beta} \\ &= 0. \end{aligned}$$

Therefore,

$$\lim_{k \rightarrow +\infty} \int_{\mathbb{R}^n} \frac{|u_k - u|^r}{|x|^\beta} dx = 0,$$

Thus, for any $r \geq n/\alpha$, the embedding $W^{\alpha, n/\alpha}(\mathbb{R}^n) \hookrightarrow L^r(\mathbb{R}^n, |x|^{-\beta})$ is compact.

Moreover, by the definition of $\mathcal{W}(\mathbb{R}^n)$, we also have that $\mathcal{W}(\mathbb{R}^n) \hookrightarrow L^r(\mathbb{R}^n, |x|^{-\beta})$ is compact for any $n/\alpha \leq r < \infty$. □

Lemma 2.4. Assume (f_5) holds. Then, $\frac{f(t)t - \mathcal{K}_1 F(t)}{|t|^{\frac{n-\beta}{\alpha}}}$ is nonincreasing on $(-\infty, 0)$ and nondecreasing on $(0, \infty)$.

Proof. Recall that $\mathcal{H}(t) = f(t)t - \frac{n-\beta}{\alpha} F(t)$. Set $h(t) = \frac{\mathcal{H}(t)}{|t|^{\mathcal{K}_1}}$. By (f_5) and a direct calculation, we obtain that

$$k'(t) = \frac{\mathcal{H}(t)}{t^{\frac{n-\beta}{\alpha} + 1}} = h(t)t^{\mathcal{K}_1 - \frac{n-\beta}{\alpha} - 1}, \quad k(t) = \frac{F(t)}{|t|^{\frac{n-\beta}{\alpha}}}$$

for all $t \in \mathbb{R}$. Define $g(t) = \frac{f(t)t - \mathcal{K}_1 F(t)}{|t|^{\frac{n-\beta}{\alpha}}}$. If $0 < t_1 < t_2$, by a direct calculation,

$$\begin{aligned} g(t_2) - g(t_1) &= t_2^{\mathcal{K}_1 - \frac{n-\beta}{\alpha}} h(t_2) - t_1^{\mathcal{K}_1 - \frac{n-\beta}{\alpha}} h(t_1) + (\mathcal{K}_1 - \frac{n-\beta}{\alpha})(k(t_2) - k(t_1)) \\ &= t_2^{\mathcal{K}_1 - \frac{n-\beta}{\alpha}} h(t_2) - t_1^{\mathcal{K}_1 - \frac{n-\beta}{\alpha}} h(t_1) - (\mathcal{K}_1 - \frac{n-\beta}{\alpha}) \int_{t_1}^{t_2} h(t)t^{\mathcal{K}_1 - \frac{n-\beta}{\alpha} - 1} dt \\ &\geq q t_2^{\mathcal{K}_1 - \frac{n-\beta}{\alpha}} h(t_2) - t_1^{\mathcal{K}_1 - \frac{n-\beta}{\alpha}} h(t_1) + h(t_2)(t_1^{\mathcal{K}_1 - \frac{n-\beta}{\alpha}} - t_2^{\mathcal{K}_1 - \frac{n-\beta}{\alpha}}) \\ &= t_1^{\mathcal{K}_1 - \frac{n-\beta}{\alpha}} (h(t_2) - h(t_1)) \geq 0. \end{aligned}$$

The case $t < 0$ can be proved similarly. Thus, we complete the proof. □

Now, we show that the functional $\mathcal{I}$ possesses the mountain pass geometry property.

Lemma 2.5. Assume $(f_1) - (f_6)$ hold. Set $t \star u(\cdot) = t^\alpha u(t \cdot)$ for $t \in (0, +\infty)$. Then the functional $\mathcal{I}$ defined by (1.4) possesses the following properties:

- (i) for any $u \in \mathcal{W}(\mathbb{R}^n) \setminus \{0\}$, $\mathcal{I}(t \star u) \rightarrow 0^+$ as $t \rightarrow 0^+$;
- (ii) for any $u \in \mathcal{W}(\mathbb{R}^n) \setminus \{0\}$, $\mathcal{I}(t \star u) \rightarrow -\infty$ as $t \rightarrow \infty$.

Proof. By $(f_1) - (f_2)$, there exist $r > s + \frac{n}{\alpha}$ and $\vartheta > \vartheta_0$ such that, for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$|F(\xi)| \leq \epsilon |\xi|^{2n/\alpha} + C_\epsilon |\xi|^r \Psi(\vartheta |\xi|^{n/(n-1)}), \quad \forall \xi \in \mathbb{R},$$

where Ψ is given in theorem 2.1. Then, by lemma 2.1 and the Hölder inequality, we get

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{|F(u)|}{|x|^\beta} dx &\leq \epsilon \int_{\mathbb{R}^n} \frac{|u|^{2n/\alpha}}{|x|^\beta} + C_\epsilon \int_{\mathbb{R}^n} \frac{|u|^r \Psi(|u|^{\frac{n}{n-\alpha}})}{|x|^\beta} dx \\ &\leq \epsilon \int_{\mathbb{R}^n} \frac{|u|^{2n/\alpha}}{|x|^\beta} dx + C_\epsilon \left(\int_{\mathbb{R}^n} \frac{|u|^m}{|x|^\beta} dx \right)^{\frac{1}{n}} \left(\int_{\mathbb{R}^n} \frac{\Psi(\frac{n\vartheta}{n-1} |u|^{\frac{n}{n-\alpha}})}{|x|^\beta} dx \right)^{\frac{n-1}{n}}. \end{aligned}$$

Define $v(x) := \left(\frac{2n^2\vartheta}{(n-\beta)(n-1)\vartheta_{\alpha,n}^*} \right)^{\frac{n-\alpha}{n}} t^\alpha u(tx)$ for $t > 0$. Then

$$\int_{\mathbb{R}^n} |v|^{\frac{n}{\alpha}} dx = \left(\frac{2n^2\vartheta}{(n-\beta)(n-1)\vartheta_{\alpha,n}^*} \right)^{\frac{n-\alpha}{\alpha}} \int_{\mathbb{R}^n} |u|^{\frac{n}{\alpha}} dx$$

and

$$\int_{\mathbb{R}^n} |\nabla v|^n dx = \left(\frac{2n^2\vartheta}{(n-\beta)(n-1)\vartheta_{\alpha,n}^*} \right)^{\frac{p(n-\alpha)}{n}} t^{p\alpha+p-n} \int_{\mathbb{R}^n} |\nabla u|^p dx \rightarrow 0 \text{ as } t \rightarrow 0^+.$$

Moreover,

$$[v]_{\alpha,n/\alpha}^{n/\alpha} = \left(\frac{2n^2\vartheta}{(n-\beta)(n-1)\vartheta_{\alpha,n}^*} \right)^{\frac{n-\alpha}{\alpha}} t^n [u]_{\alpha,n/\alpha}^{n/\alpha} \rightarrow 0 \text{ as } t \rightarrow 0^+.$$

Thus, using theorem 2.1, there exists $t_0 > 0$ small enough such that, for all $t \leq t_0$,

$$\int_{\mathbb{R}^n} \frac{\Psi(\frac{n\vartheta}{n-1} |t^\alpha u(tx)|^{\frac{n}{n-\alpha}})}{|x|^\beta} dx = \int_{\mathbb{R}^n} \frac{\Psi(\frac{(n-\beta)}{2n} \vartheta_n |v|^{\frac{n}{n-\alpha}})}{|x|^\beta} dx \leq C_1$$

for some $C_1 > 0$. Further, for $t \leq t_0$,

$$\int_{\mathbb{R}^n} \frac{|F(t^\alpha u(tx))|}{|x|^\beta} dx \leq \epsilon t^{n+\beta} \int_{\mathbb{R}^n} \frac{|u|^{2n/\alpha}}{|x|^\beta} dx + C_\epsilon C_1^{\frac{n-1}{n}} t^{(r\alpha-1+\frac{\beta}{n})} \left(\int_{\mathbb{R}^n} \frac{|u(x)|^m}{|x|^\beta} dx \right)^{\frac{1}{n}}. \quad (2.11)$$

Consequently,

$$\begin{aligned} \mathcal{I}(t \star u) &= \frac{\mu}{p} \int_{\mathbb{R}^n} |\nabla(t^\alpha u(tx))|^p dx + \frac{\alpha}{n} \iint_{\mathbb{R}^{2n}} \frac{|t^\alpha u(tx) - t^\alpha u(ty)|^{\frac{n}{\alpha}}}{|x-y|^{2n}} dx dy - \int_{\mathbb{R}^n} \frac{F(t^\alpha u(tx))}{|x|^\beta} dx \\ &\geq \frac{\mu}{p} t^{p\alpha+p-n} \int_{\mathbb{R}^n} |\nabla u|^p dx + \frac{\alpha}{n} t^n \iint_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)|^{\frac{n}{\alpha}}}{|x-y|^{2n}} dx dy \\ &\quad - \epsilon t^{n+\beta} \int_{\mathbb{R}^n} \frac{|u(x)|^{2n/\alpha}}{|x|^\beta} dx - C_\epsilon C_1^{\frac{n-1}{n}} t^{r\alpha-1+\frac{\beta}{n}} \left(\int_{\mathbb{R}^n} \frac{|u(x)|^m}{|x|^\beta} dx \right)^{\frac{1}{n}}. \end{aligned}$$

By $r > s + \frac{n}{\alpha}$, we can obtain that there exists $t_1 \leq t_0$ such that $\mathcal{I}(t \star u) \geq 0$ for $t \leq t_1$. Furthermore, by (2.11) it follows that

$$\int_{\mathbb{R}^n} \frac{F(t^\alpha u(tx))}{|x|^\beta} dx \rightarrow 0 \text{ as } t \rightarrow 0^+,$$

which implies that $\mathcal{I}(t \star u) \rightarrow 0^+$ as $t \rightarrow 0^+$. Therefore, (i) holds.

For (ii), by (f_6) , there exists $C_2 > 0$ such that

$$F(t) \geq C_2 t^q, \quad \forall t > 0.$$

By $q > s + \frac{n}{\alpha}$, it follows that

$$\begin{aligned} \mathcal{I}(t \star u) &= \frac{\mu}{p} \int_{\mathbb{R}^n} |\nabla(t^\alpha u(tx))|^p dx + \frac{\alpha}{n} \iint_{\mathbb{R}^{2n}} \frac{|t^\alpha u(tx) - t^\alpha u(ty)|^{\frac{n}{\alpha}}}{|x-y|^{2n}} dx dy - \int_{\mathbb{R}^n} \frac{F(t^\alpha u(tx))}{|x|^\beta} dx \\ &\leq \frac{\mu}{p} t^{p\alpha+p-n} \int_{\mathbb{R}^n} |\nabla u|^p dx + \frac{\alpha}{n} t^n \iint_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)|^{\frac{n}{\alpha}}}{|x-y|^{2n}} dx dy - C_2 t^{q\alpha-n+\beta} \int_{\mathbb{R}^n} \frac{|u|^q}{|x|^\beta} dx \\ &\rightarrow -\infty \text{ as } t \rightarrow \infty. \end{aligned}$$

Thus, (ii) holds. □

Lemma 2.6. Assume (f_1) – (f_6) hold. Then

(i) for any $u \in \mathcal{W}(\mathbb{R}^n) \setminus \{0\}$, there exists a number $t_0 \in \mathbb{R}^+$ such that $P(t_0 * u) = 0$ and if $u \in \mathcal{P}_m$, then $\mathcal{I}(u) = \max_{t \in \mathbb{R}^+} \mathcal{I}(t * u)$;

(ii) there exists $\delta_0 > 0$ such that $\inf_{u \in \mathcal{P}_m} \left([u_k]_{\alpha, n/\alpha}^{n/\alpha} + \mu \|\nabla u\|_p^p \right) \geq \delta_0$.

Proof. For any $u \in \mathcal{W}(\mathbb{R}^n) \setminus \{0\}$, we deduce from lemma 2.5 that $\phi_u(t) := \mathcal{I}(t * u)$ has a global maximum attained at some $t_0 \in \mathbb{R}^+$. Observe that $\frac{d}{dt} \mathcal{I}(t * u) = \alpha t^{-1} P(t * u)$. Then it follows that $P(t_0 * u) = 0$. By lemma 2.4, we get that for any $\xi \neq 0$, it holds

$$\begin{aligned} & \frac{d}{dt} \left[\frac{\alpha(1-t^{p\alpha+p-n})}{p\alpha+p-n} f(\xi)\xi - \frac{(n-\beta)(1-t^{p\alpha+p-n}) + p\alpha + p - n}{p\alpha+p-n} F(\xi) + t^{-n+\beta} F(t^\alpha \xi) \right] \\ &= -\alpha t^{p\alpha+p-n-1} |\xi|^{\mathcal{K}_1} \left[\frac{f(\xi)\xi - \frac{n-\beta}{\alpha} F(\xi)}{|\xi|^{\mathcal{K}_1}} - \frac{f(t^\alpha \xi)t^\alpha \xi - \frac{n-\beta}{\alpha} F(t^\alpha \xi)}{|t^\alpha \xi|^{\mathcal{K}_1}} \right] \\ & \quad \times \begin{cases} \geq 0, & t \geq 1, \\ \leq 0, & 0 < t \leq 1. \end{cases} \end{aligned}$$

It follows that

$$\frac{\alpha(1-t^{p\alpha+p-n})}{p\alpha+p-n} f(\xi)\xi - \frac{(n-\beta)(1-t^{p\alpha+p-n}) + p\alpha + p - n}{p\alpha+p-n} F(\xi) + t^{-n+\beta} F(t^\alpha \xi) \geq 0$$

for all $t > 0$. Similarly, one can show that

$$\frac{\alpha(1-t^n)}{n} - \frac{\alpha(1-t^{p\alpha+p-n})}{p\alpha+p-n} \geq 0$$

for all $t > 0$. Therefore, for each fixed $u \in \mathcal{S}_m$, we have

$$\begin{aligned} & \mathcal{I}(u) - \mathcal{I}(t * u) \\ &= \frac{\mu}{p} \|\nabla u\|_p^p + \frac{\alpha}{n} [u]_{\alpha, n/\alpha}^{n/\alpha} - \int_{\mathbb{R}^n} \frac{F(u)}{|x|^\beta} dx \\ & \quad - \frac{\mu t^{p\alpha+p-n}}{p} \|\nabla u\|_p^p - \frac{\alpha}{n} t^n [u]_{\alpha, n/\alpha}^{n/\alpha} + t^{-n+\beta} \int_{\mathbb{R}^n} \frac{F(t^\alpha u)}{|x|^\beta} dx \\ &= \frac{\alpha(1-t^{p\alpha+p-n})}{(\alpha+1)p-n} P(u) + \left[\frac{\alpha(1-t^n)}{n} - \frac{\alpha(1-t^{p\alpha+p-n})}{p\alpha+p-n} \right] [u]_{\alpha, n/\alpha}^{n/\alpha} \\ & \quad + \int_{\mathbb{R}^n} \left[\frac{\alpha(1-t^{p\alpha+p-n})}{p\alpha+p-n} \frac{f(u)u}{|x|^\beta} - \frac{(n-\beta)(1-t^{p\alpha+p-n}) + p\alpha + p - n}{p\alpha+p-n} \frac{F(u)}{|x|^\beta} + t^{-n+\beta} \frac{F(t^\alpha u)}{|x|^\beta} \right] dx \\ &\geq \frac{\alpha(1-t^{p\alpha+p-n})}{(\alpha+1)p-n} P(u), \end{aligned}$$

for all $t > 0$, which implies that, if $u \in \mathcal{P}_m$, $\mathcal{I}(u) \geq \mathcal{I}(t * u)$ for all $t \in (0, +\infty)$. Thus, (i) holds.

For (ii), we prove it by contradiction. Assume that there exists a sequence $\{u_k\}_k \subset \mathcal{P}_m$ such that $\mu \|\nabla u_k\|_p^p + [u_k]_{\alpha, n/\alpha}^{n/\alpha} \rightarrow 0$ as $k \rightarrow +\infty$. Choose $t > 1$ very close to 1 such that $t\beta < n$, $t' = \frac{t}{t-1}$. Then, by theorem 2.1, there exists $C_1 > 0$ such that

$$\int_{\mathbb{R}^n} \frac{\Psi\left(t\vartheta |u_k|^{\frac{n-\alpha}{n}}\right)}{|x|^{t\beta}} dx \leq C_1.$$

Using similar arguments as in lemma 2.5, we deduce from lemma 2.2 that for any $\epsilon > 0$, there exist $C_\epsilon > 0$ and $r > s + \frac{n}{\alpha}$ such that

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{|f(u_k)u_k|}{|x|^\beta} dx &\leq \epsilon \int_{|x| \geq 1} |u_k|^{2n/\alpha} dx + \epsilon C \left(\int_{|x| \leq 1} |u_k|^{2n'/\alpha} dx \right)^{\frac{1}{t'}} \\
&\quad + C_\epsilon \left(\int_{\mathbb{R}^n} |u_k|^{rt'} dx \right)^{\frac{1}{t'}} \left(\int_{\mathbb{R}^n} \frac{\Psi(t\vartheta|u_k|^{\frac{n}{n-1}})}{|x|^{t\beta}} dx \right)^{\frac{1}{t}} \\
&\leq \epsilon C(n)m[u_k]_{\alpha,n/\alpha}^{n/\alpha} + \epsilon C(n,t)m^{\frac{1}{t'}} \left([u_k]_{\alpha,n/\alpha}^{n/\alpha} \right)^{\frac{2}{\alpha} - \frac{1}{t'}} \\
&\quad + C_\epsilon C_1^{\frac{1}{t}} C(n,t)m^{\frac{1}{t'}} \left([u_k]_{\alpha,n/\alpha}^{n/\alpha} \right)^{\frac{r}{n} - \frac{1}{t'}}
\end{aligned}$$

where $C(n), C(n, t)$ are constants depending on n, t . Then, in view of $F(t) \geq 0, \forall t \in \mathbb{R}$, we deduce from $\{u_k\} \subset \mathcal{P}_m$ that

$$\begin{aligned}
[u_k]_{\alpha,n/\alpha}^{n/\alpha} + \mu \frac{p\alpha + p - n}{p\alpha} \|\nabla u_k\|_p^p &= \int_{\mathbb{R}^n} \frac{f(u_k)u_k - \frac{(n-\beta)}{\alpha} F(u_k)}{|x|^\beta} dx \leq \int_{\mathbb{R}^n} \frac{f(u_k)u_k}{|x|^\beta} dx \\
&\leq \epsilon C(n)m[u_k]_{\alpha,n/\alpha}^{n/\alpha} + \epsilon C(n,t)m^{\frac{1}{t'}} \left([u_k]_{\alpha,n/\alpha}^{n/\alpha} \right)^{2/\alpha - \frac{1}{t'}} \\
&\quad + C_\epsilon C_1^{\frac{1}{t}} C(n,t)m^{\frac{1}{t'}} \left([u_k]_{\alpha,n/\alpha}^{n/\alpha} \right)^{\frac{r}{n} - \frac{1}{t'}}.
\end{aligned}$$

Choosing $\epsilon = \frac{1}{2C(n)m}$, we arrive at

$$\frac{1}{2} \leq \frac{C(n,t)}{2C(n)} m^{\frac{1}{t'}-1} \left([u_k]_{\alpha,n/\alpha}^{n/\alpha} + \mu \|\nabla u\|_p^p \right)^{\frac{2}{\alpha} - \frac{1}{t'}-1} + C_\epsilon C_1^{\frac{1}{t}} C(n,t)m^{\frac{1}{t'}-1} \left([u_k]_{\alpha,n/\alpha}^{n/\alpha} + \mu \|\nabla u\|_p^p \right)^{\frac{r}{n} - \frac{1}{t'}-1}.$$

By $t' > 1 > \alpha$ and $r > s + \frac{n}{\alpha}$, we obtain a contradiction. Hence, there exists $\delta_0 > 0$ such that

$$\inf_{u \in \mathcal{P}_m} \left([u_k]_{\alpha,n/\alpha}^{n/\alpha} + \mu \|\nabla u\|_n^n \right) \geq \delta_0.$$

This ends the proof. □

Lemma 2.7. Assume $(f_1)-(f_6)$ hold. For any $m > 0$, there exists $K(m) > 0$ small enough such that

$$0 < \sup_{u \in A_m} \mathcal{I}(u) < \inf_{u \in B_m} \mathcal{I}(u),$$

where

$$A_m := \left\{ u \in \mathcal{S}_m : \frac{\mu}{p} \|\nabla u\|_p^p + \frac{\alpha}{n} [u]_{\alpha,n/\alpha}^{n/\alpha} \leq K(m) \right\}, \quad B_m := \left\{ u \in \mathcal{S}_m : \frac{\mu}{p} \|\nabla u\|_p^p + \frac{\alpha}{n} [u]_{\alpha,n/\alpha}^{n/\alpha} = 2K(m) \right\}.$$

Moreover, $K(m)$ is nonincreasing with respect to m , and $\mathcal{I}(u) > 0$ for all $u \in A_m$.

Proof. Since F is nonnegative, for any $u \in A_m, v \in B_m$, we have

$$\mathcal{I}(v) - \mathcal{I}(u) \geq \frac{\mu}{p} \|\nabla v\|_p^p + \frac{\alpha}{n} [v]_{\alpha,n/\alpha}^{n/\alpha} - \frac{\mu}{p} \|\nabla u\|_p^p - \frac{\alpha}{n} [u]_{\alpha,n/\alpha}^{n/\alpha} - \int_{\mathbb{R}^n} \frac{F(v)}{|x|^\beta} dx.$$

Let $t > 1$ close to 1, $t' = \frac{t}{t-1} > 1$, $\vartheta > \vartheta_0$ close to ϑ_0 . Suppose that $K(m) \leq \min \left\{ 1, \alpha \left(\frac{(n-\beta)\vartheta_n}{n\vartheta_0} \right)^{\frac{n-\alpha}{\alpha}} \right\}$ is chosen sufficiently small to ensure $t\vartheta[v]_{\alpha,n/\alpha}^{\frac{n-1}{n}} < \frac{n-\beta}{n} \vartheta_{\alpha,n}^*$ for all $v \in B_m$. Similar to lemma 2.5, for any $\epsilon > 0$ and $r > s + \frac{n}{\alpha}$, there exists $C_\epsilon > 0$ such that

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{|F(v)|}{|x|^\beta} dx &\leq \epsilon C(n)m[v]_{\alpha,n/\alpha}^{n/\alpha} + \epsilon C(n,\kappa)m^{\frac{1}{\kappa'}} \left([v]_{\alpha,n/\alpha}^{n/\alpha} \right)^{\frac{2}{\alpha} - \frac{1}{\kappa'}} + C_\epsilon C_1^{\frac{1}{\kappa}} C(n,\kappa)m^{\frac{1}{\kappa'}} \left([v]_{\alpha,n/\alpha}^{n/\alpha} \right)^{\frac{r}{n} - \frac{1}{\kappa'}} \\
&\leq \epsilon C(n)mK(m) + \epsilon C(n,\kappa)m^{\frac{1}{\kappa'}} K(m)^{\frac{2}{\alpha} - \frac{1}{\kappa'}} + C_\epsilon C_1^{\frac{1}{\kappa}} C(n,\kappa)m^{\frac{1}{\kappa'}} K(m)^{\frac{r}{n} - \frac{1}{\kappa'}}.
\end{aligned} \tag{2.12}$$

Then, we get

$$\begin{aligned} \mathcal{I}(v) - \mathcal{I}(u) &> K(m) - \epsilon C(n)mK(m) - \epsilon C(n, \kappa)m^{\frac{1}{\kappa'}} (K(m))^{\frac{2}{\alpha} - \frac{1}{\kappa'}} - C_\epsilon C_1^{\frac{1}{\alpha}} C(n, t)m^{\frac{1}{\kappa'}} (K(m))^{\frac{r}{n} - \frac{1}{\kappa'}} \\ &> K(m) - \epsilon \left(C(n)m + C(n, \kappa)m^{\frac{1}{\kappa'}} \right) K(m) - C_\epsilon C_1^{\frac{1}{\alpha}} C(n, \kappa)m^{\frac{1}{\kappa'}} (K(m))^{\frac{r}{n} - \frac{1}{\kappa'}}. \end{aligned}$$

Choose

$$\epsilon = \frac{1}{2(C(n)m + C(n, \kappa)m^{\frac{1}{\kappa'}})}, \quad K(m) = \min \left\{ 1, \alpha \left(\frac{(n-\beta)\vartheta_n}{n\vartheta_0} \right)^{\frac{n-\alpha}{\alpha}}, \left(\frac{1}{2nC_\epsilon C_1^{\frac{1}{\alpha}} C(n, \kappa)m^{\frac{1}{\kappa'}}} \right)^{\frac{r}{n} - 1 - \frac{1}{\kappa'}} \right\}$$

such that $\mathcal{I}(v) - \mathcal{I}(u) > 0$. Moreover, it is clear that $K(m)$ is nonincreasing with respect to m . In particular, (2.12) also holds after replacing v by any $u \in A_m$. Then one can see that $\mathcal{I}(u) > 0$ for all $u \in A_m$. $\square$

3. Mountain pass type solutions

In this section, we investigate the existence of normalized solutions of mountain pass type. Motivated by [22], we define $\mathcal{I}^0 = \{u \in \mathcal{S}_m : \mathcal{I}(u) < 0\}$ and the mountain pass level

$$\gamma(m) = \inf_{h \in \Gamma_m} \max_{t \in [0,1]} \mathcal{I}(h(t)) > 0,$$

where

$$\Gamma_m = \{h \in C([0,1], \mathcal{S}_m) : h(0) \in A_m, h(1) \in \mathcal{I}^0\}$$

and A_m is given in lemma 2.7. We also define

$$\tilde{\gamma}(m) = \inf_{\tilde{h} \in \tilde{\Gamma}_m} \max_{t \in [0,1]} \tilde{\mathcal{I}}(\tilde{h}(t)) > 0,$$

where

$$\tilde{\Gamma}_m = \{\tilde{h}(t) \in C([0,1], \mathbb{R}^+ \times \mathcal{S}_m) : \tilde{h}(0) \in \{1\} \times A_m, \tilde{h}(1) \in \{1\} \times \mathcal{I}^0\},$$

and set $\tilde{\mathcal{I}}(t, u) = \mathcal{I}(t \star u)$. Obviously, it holds that $\gamma(m) = \tilde{\gamma}(m)$.

Motivated by [22, 44], we have the following lemma that $\mathcal{I}$ has a (PS) sequence at the level $\gamma(m)$.

Lemma 3.1. Suppose that (f_1) – (f_6) hold. Then, there exists a sequence $\{u_k\}_k \subset \mathcal{S}_m$ such that

$$\mathcal{I}(u_k) \rightarrow \gamma(m), \quad (\mathcal{I}|_{\mathcal{S}_m})'(u_k) \rightarrow 0 \quad \text{and} \quad P(u_k) \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty,$$

where the Pohozaev functional P is given by (1.6).

Proof. By the definition of $\gamma(m)$, for any $k \in \mathbb{N}$, there exists a sequence $\{h_k\}_k \subset \Gamma_m$ such that $\max_{t \in [0,1]} \mathcal{I}(h_k(t)) < \gamma(m) + \frac{1}{k}$. Let $\tilde{h}_k := \{1\} \times h_k \in \tilde{\Gamma}_m$. Then we have

$$\max_{t \in [0,1]} \tilde{\mathcal{I}}(\tilde{h}_k(t)) < \tilde{\gamma}(m) + \frac{1}{k}.$$

Then, from theorem 5.2 of [20], there exists a sequence $\{(t_k, v_k)\}_k$ such that

$$\begin{cases} |t_k - 1| + \text{dist}_{\mathcal{W}(\mathbb{R}^n)}(v_k, \tilde{h}_k([0,1])) \rightarrow 0, \\ \mathcal{I}(t_k \star v_k) = \tilde{\mathcal{I}}(t_k, v_k) \rightarrow \tilde{\gamma}(m), \\ (\tilde{\mathcal{I}}|_{\mathbb{R} \times \mathcal{S}_m})'(t_k, v_k) \rightarrow 0, \end{cases}$$

as $k \rightarrow \infty$. Especially, we have

$$\partial_t \tilde{\mathcal{I}}(t_k, v_k) \rightarrow 0. \quad (3.1)$$

Let $u_k = t_k \star v_k$. Then $\{u_k\}_{f_k}$ is a (PS) sequence for $\mathcal{I}|_{\mathcal{S}_m}$ at the level $\gamma(m)$. Observe that

$$\begin{aligned}\partial_t \tilde{\mathcal{I}}(t, u_k) &= \partial_t \mathcal{I}(t \star u_k) \\ &= \mu \frac{(\alpha+1)p-n}{p} t^{(\alpha+1)p-n-1} \|\nabla u_k\|_p^p + \alpha t^{n-1} [u_k]_{\alpha, n/\alpha}^{n/\alpha} \\ &\quad - (\beta-n) t^{\beta-n-1} \int_{\mathbb{R}^n} \frac{F(t^\alpha u_k)}{|x|^\beta} dx - \alpha t^{\beta-n-1} \int_{\mathbb{R}^n} \frac{f(t^\alpha u_k) t^\alpha u_k}{|x|^\beta} dx \\ &= \alpha t^{-1} P(t \star u_k).\end{aligned}$$

It follows from (3.1) that $P(u_k) \rightarrow 0$ as $k \rightarrow \infty$. This completes the proof. $\square$

As follow, we introduce some estimates on the mountain pass level and the upper bound of Palais–Smale sequence of $\mathcal{I}$ on $\mathcal{S}_m$.

Lemma 3.2. Assume (f_1) – (f_6) hold. Then there holds

$$\gamma(m) < \frac{\alpha}{n} \left(\frac{(n-\beta) \vartheta_{\alpha, n}^*}{n \vartheta_0} \right)^{\frac{n-\alpha}{\alpha}}.$$

Proof. From lemma 2.3 and $q > \frac{2n}{\alpha}$, we obtain that the embedding $\mathcal{W}(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n, |x|^{-\beta})$ is compact, which means that the optimal constant S_* defined in (f_6) can be attained and further, there exists a function u_* such that

$$\int_{\mathbb{R}^N} \frac{|u_*|^q}{|x|^\beta} dx = 1 \quad \text{and} \quad \|\nabla u_*\|_p + \|u_*\|_{\alpha, n/\alpha} = S_*. \quad (3.2)$$

Then,

$$\|\nabla u_*\|_p^p < (\|\nabla u_*\|_p + \|u_*\|_{\alpha, n/\alpha})^p = S_*^p \quad \text{and} \quad [u_*]_{\alpha, n/\alpha}^{n/\alpha} < (\|\nabla u_*\|_n + \|u_*\|_{\alpha, n/\alpha})^{\frac{n}{\alpha}} = S_*^{n/\alpha}. \quad (3.3)$$

Set $C_0 = \int_{\mathbb{R}^N} |u_*|^{\frac{n}{\alpha}} dx$ and $u_0 = \left(\frac{m}{C_0}\right)^{\frac{\alpha}{n}} u_*$. Clearly, $u_0 \in \mathcal{S}_m$. It follows from (3.2), (3.3) and (f_6) that

$$\begin{aligned}\gamma(m) &\leq \max_{t>0} \mathcal{I}(t \star u_0) \\ &\leq \max_{t>0} \left\{ \frac{\mu}{p} t^{p(\alpha+1)-n} \|\nabla u_0\|_p^p + \frac{\alpha}{n} t^n [u_0]_{\alpha, n/\alpha}^{n/\alpha} - C_q t^{(q\alpha-n+\beta)} \int_{\mathbb{R}^n} \frac{|u_0|^q}{|x|^\beta} dx \right\} \\ &= \max_{t>0} \left\{ \frac{\mu}{p} t^{p(\alpha+1)-n} \left(\frac{m}{C_0}\right)^{\frac{\alpha p}{n}} \|\nabla u_*\|_p^p + \frac{\alpha}{n} t^n \frac{m}{C_0} [u_*]_{\alpha, n/\alpha}^{n/\alpha} - C_q t^{(q\alpha-n+\beta)} \left(\frac{m}{C_0}\right)^{\frac{q\alpha}{n}} \int_{\mathbb{R}^n} \frac{|u_*|^q}{|x|^\beta} dx \right\} \\ &< \max_{t>0} \left\{ \frac{\mu}{p} \left(\frac{m}{C_0}\right)^{\frac{\alpha p}{n}} S_*^p t^{p(\alpha+1)-n} + \frac{\alpha}{n} \frac{m}{C_0} S_*^{n/\alpha} t^n - C_q t^{(q\alpha-n+\beta)} \left(\frac{m}{C_0}\right)^{\frac{q\alpha}{n}} \right\}. \quad (3.4)\end{aligned}$$

Set

$$\mathcal{G}(t) = \frac{\mu((\alpha+1)p-n)}{p} \left(\frac{m}{C_0}\right)^{\frac{\alpha p}{n}} S_*^p t^{p(\alpha+1)-n} + \frac{\alpha}{n} \frac{m}{C_0} S_*^{n/\alpha} t^n - C_q \left(\frac{m}{C_0}\right)^{\frac{q\alpha}{n}} t^{q\alpha-n+\beta} \quad \text{for } t \geq 0.$$

We write $\mathcal{G}$ as

$$\mathcal{G}(t) = \underbrace{\frac{\mu}{p} \left(\frac{m}{C_0}\right)^{\frac{\alpha p}{n}} S_*^p t^{p(\alpha+1)-n} - \frac{C_q}{2} \left(\frac{m}{C_0}\right)^{\frac{q\alpha}{n}} t^{q\alpha-n+\beta}}_{I_1} + \underbrace{\frac{\alpha}{n} \frac{m}{C_0} S_*^{n/\alpha} t^n - \frac{C_q}{2} \left(\frac{m}{C_0}\right)^{\frac{q\alpha}{n}} t^{q\alpha-n+\beta}}_{I_2}.$$

It follows from (f_6) that

$$\begin{aligned} \max_{t>0} \mathcal{G}(t) &\leq \max_{t>0} I_1 + \max_{t>0} I_2 \\ &< \frac{\mu}{p} \left(\frac{m}{C_0} \right)^{\frac{\alpha p}{n}} S_*^p \left(\frac{2\mu(p\alpha + p - n) \left(\frac{m}{C_0} \right)^{\frac{(p-q)\alpha}{n}} S_*^p}{pC_q(q\alpha - n + \beta)} \right)^{\frac{p\alpha + p - n}{(q-p)\alpha - p + \beta}} \\ &\quad + \frac{\alpha}{n} \frac{m}{C_0} S_*^{n/\alpha} \left(\frac{2\alpha \left(\frac{m}{C_0} \right)^{\frac{n-q\alpha}{n}} S_*^{\frac{n}{\alpha}}}{C_q(q\alpha - n + \beta)} \right)^{\frac{n}{q\alpha - n + \beta}} \\ &< \frac{\alpha}{2n} \left(\frac{(n-\beta)\vartheta_{\alpha,n}^*}{n\vartheta_0} \right)^{\frac{n-\alpha}{\alpha}} + \frac{\alpha}{2n} \left(\frac{(n-\beta)\vartheta_{\alpha,n}^*}{n\vartheta_0} \right)^{\frac{n-\alpha}{\alpha}} \\ &= \frac{\alpha}{n} \left(\frac{(n-\beta)\vartheta_{\alpha,n}^*}{n\vartheta_0} \right)^{\frac{n-\alpha}{\alpha}}. \end{aligned}$$

Then,

$$\gamma(m) < \frac{\alpha}{n} \left(\frac{(n-\beta)\vartheta_{\alpha,n}^*}{n\vartheta_0} \right)^{\frac{n-\alpha}{\alpha}}.$$

The proof is complete. $\square$

Lemma 3.3. Suppose that (f_1) – (f_6) hold. Let $\{u_k\}_k \subset \mathcal{S}_m$ be a (PS) sequence of $\mathcal{I}|_{\mathcal{S}_m}$ at the level $\gamma(m)$ such that $P(u_k) = o_k(1)$. Then $\{u_k\}_k$ is bounded in $\mathcal{W}(\mathbb{R}^n)$.

Proof. Since $\{u_k\}_k$ is a (PS) sequence at the level $\gamma(m)$ for $\mathcal{I}|_{\mathcal{S}_m}$ with $P(u_k) = o_k(1)$, we have

$$\gamma(m) + o_k(1) = \mathcal{I}(u_k) = \frac{\mu}{p} \|\nabla u_k\|_p^p + \frac{\alpha}{n} [u_k]_{\alpha, n/\alpha}^{n/\alpha} - \int_{\mathbb{R}^n} \frac{F(u_k)}{|x|^\beta} dx,$$

and

$$P(u_k) = \mu \frac{p\alpha + p - n}{\alpha p} \|\nabla u_k\|_p^p + [u_k]_{\alpha, n/\alpha}^{n/\alpha} - \int_{\mathbb{R}^n} \frac{f(u_k)u_k - \frac{n-\beta}{\alpha} F(u_k)}{|x|^\beta} dx = o_k(1).$$

Then for $r > 0$ there holds

$$\begin{aligned} \gamma(m) + o_k(1) &\geq \frac{\mu}{p} \|\nabla u_k\|_p^p + \frac{\alpha}{n} [u_k]_{\alpha, n/\alpha}^{n/\alpha} - \frac{1}{r} \left(\mu \frac{p\alpha + p - n}{\alpha p} \|\nabla u_k\|_p^p + [u_k]_{\alpha, n/\alpha}^{n/\alpha} \right) \\ &\quad + \frac{1}{r} \int_{\mathbb{R}^n} \frac{f(u_k)u_k - (\frac{n-\beta}{\alpha} + r)F(u_k)}{|x|^\beta} dx. \end{aligned}$$

Choosing $\mathcal{K}_0 < r < \mathcal{K} - \frac{n-\beta}{\alpha}$, using assumption (f_3) and lemma 3.2, we can prove that $\{u_k\}_k$ is a bounded sequence in $\mathcal{W}(\mathbb{R}^n)$. $\square$

Lemma 3.4. Suppose that (f_1) – (f_6) hold. Let $\{u_k\} \subset \mathcal{S}_m$ be a (PS) sequence at the level $\gamma(m)$ for $\mathcal{I}|_{\mathcal{S}_m}$ with $P(u_k) = o_k(1)$. Then there exist $0 \neq u_0 \in W_\mu(\mathbb{R}^N)$ and $\lambda < 0$ such that

$$(-\Delta)_{n/\alpha}^\alpha u_0 - \mu \Delta_p u_0 = \lambda |u_0|^{\frac{n}{\alpha}-2} u_0 + \frac{f(u_0)}{|x|^\beta}, \quad \forall x \in \mathbb{R}^n.$$

Proof. Since $\{u_k\}_k$ is a (PS) sequence at the level $\gamma(m)$ for $\mathcal{I}|_{\mathcal{S}_m}$ with $P(u_k) = o_k(1)$, we have

$$\begin{aligned} \mu \frac{p\alpha + p - n}{\alpha p} \|\nabla u_k\|_p^p + [u_k]_{\alpha, n/\alpha}^{n/\alpha} - \int_{\mathbb{R}^n} \frac{f(u_k)u_k - \frac{n-\beta}{\alpha} F(u_k)}{|x|^\beta} dx &= o_k(1), \\ \gamma(m) + o_k(1) &= \frac{\mu}{p} \|\nabla u_k\|_p^p + \frac{\alpha}{n} [u_k]_{\alpha, n/\alpha}^{n/\alpha} - \int_{\mathbb{R}^n} \frac{F(u_k)}{|x|^\beta} dx. \end{aligned} \quad (3.5)$$

By lemma 3.3, we know that $\{u_k\}_k$ is bounded in $\mathcal{S}_m$. Furthermore, it follows from (3.3.5) that

$$\int_{\mathbb{R}^n} \frac{F(u_k)}{|x|^\beta} dx \leq C_2, \quad \int_{\mathbb{R}^n} \frac{f(u_k)u_k}{|x|^\beta} dx \leq C_3.$$

Then up to a subsequence of $\{u_k\}_k$, there exists $u_0 \in \mathcal{W}(\mathbb{R}^n)$ such that

$$\begin{cases} u_k \rightharpoonup u_0 \text{ weakly in } \mathcal{W}(\mathbb{R}^n), \\ u_k \rightarrow u_0 \text{ strongly in } L^r(|x|^\beta, \mathbb{R}^n) \text{ for } r \in [n/\alpha, +\infty), \\ u_k \rightarrow u_0 \text{ a.e. in } \mathbb{R}^n. \end{cases}.$$

We divide the rest proof in four steps.

Step 1. We claim that

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} \frac{F(u_k)}{|x|^\beta} dx = \int_{\mathbb{R}^n} \frac{F(u_0)}{|x|^\beta} dx.$$

This limit can be proved by using a similar discussion as [17, 28]. Indeed, one can show first that

$$\frac{f(u_k)}{|x|^\beta} \rightarrow \frac{f(u_0)}{|x|^\beta} \text{ strongly in } L^1_{\text{loc}}(\mathbb{R}^N).$$

Then using (f_4) and a similar discussion as in [28, proposition 5.2], one can prove that the claim is true.

Step 2. We prove that $u_0 \neq 0$. If not, then we have

$$\int_{\mathbb{R}^n} \frac{F(u_k)}{|x|^\beta} dx = o(1).$$

It follows from (3.5) that

$$\limsup_{k \rightarrow \infty} \left(\frac{\mu}{p} \|\nabla u_k\|_p^p + \frac{\alpha}{n} [u_k]_{\alpha, n/\alpha}^{n/\alpha} \right) \leq \gamma(m) < \frac{\alpha}{n} \left(\frac{(n-\beta)\vartheta_{\alpha, n}^*}{n\vartheta_0} \right)^{\frac{n-\alpha}{\alpha}},$$

then,

$$\limsup_{k \rightarrow \infty} [u_k]_{\alpha, n/\alpha}^{n/\alpha} < \left(\frac{(n-\beta)\vartheta_{\alpha, n}^*}{n\vartheta_0} \right)^{\frac{n-\alpha}{\alpha}}.$$

Similar to lemma 2.5, using the Trudinger–Moser inequality in theorem 2.2, we can derive that

$$\int_{\mathbb{R}^n} \frac{f(u_k)u_k}{|x|^\beta} dx \leq \epsilon \int_{\mathbb{R}^n} \frac{|u_k|^{\frac{2n}{\alpha}}}{|x|^\beta} dx + C_\epsilon C \left(\int_{\mathbb{R}^n} \frac{|u_k|^{\kappa t'}}{|x|^\beta} dx \right)^{\frac{1}{\kappa t'}},$$

where $t' > 1$, $\kappa > \max \left\{ 2p, \frac{2n}{\alpha} \right\}$. It follows from the lemma 2.7 that

$$\int_{\mathbb{R}^n} \frac{|f(u_k)u_k|}{|x|^\beta} dx = o_k(1),$$

which together with (3.5) implies that $[u_k]_{\alpha, n/\alpha}^{n/\alpha} + \mu \|\nabla u_k\|_p^p = o_k(1)$. Furthermore, it immediately follows from (3.5) that $\gamma(m) = 0$, which is absurd. Therefore, it must hold that $u_0 \neq 0$.

Step 3. $\mathcal{I}(u_0) \geq 0$.

Arguing by contradiction, we assume that $\mathcal{I}(u_0) < 0$. By lemma 2.6 (i), there exists $t_0 \in (0, 1)$ such that $\max_{t \in [0, 1]} \mathcal{I}(t \star u_0) = \mathcal{I}(t_0 \star u_0)$ and $P(t_0 \star u_0) = 0$. Set $v_0 = t_0 \star u_0$, $\mathcal{I}(v_0) = \max_{t \geq 0} \mathcal{I}(t \star v_0)$. Therefore, we get

$$\begin{aligned} \gamma(m) &\leq \max_{t > 0} \mathcal{I}(t \star v_0) \\ &= \mathcal{I}(v_0) = \mathcal{I}(v_0) - \frac{1}{\mathcal{K}_0} P(v_0) \end{aligned}$$

$$\begin{aligned}
&= \mu \left(\frac{1}{p} - \frac{p\alpha + p - n}{\mathcal{K}_0 p \alpha} \right) \|\nabla v_0\|_p^p + \left(\frac{\alpha}{n} - \frac{1}{\mathcal{K}_0} \right) [v_0]_{\alpha, n/\alpha}^{n/\alpha} \\
&\quad + \frac{1}{\mathcal{K}_0} \int_{\mathbb{R}^n} \frac{1}{|x|^\beta} \left[f(v_0) v_0 - \left(\mathcal{K}_0 + \frac{n-\beta}{\alpha} \right) F(v_0) \right] dx \\
&= \mu \left(\frac{1}{p} - \frac{p\alpha + p - n}{\mathcal{K}_0 p \alpha} \right) t_0^{p\alpha + p - n} \|\nabla u_0\|_p^p + \left(\frac{\alpha}{n} - \frac{1}{\mathcal{K}_0} \right) t_0^n [u_0]_{\alpha, n/\alpha}^{n/\alpha} \\
&\quad + \frac{1}{\mathcal{K}_0} \int_{\mathbb{R}^n} \frac{1}{|x|^\beta} \frac{\left[f(t_0^\alpha u_0) t_0^\alpha u_0 - \left(\mathcal{K}_0 + \frac{n-\beta}{\alpha} \right) F(t_0^\alpha u_0) \right]}{t_0^{n-\beta}} dx \\
&= \mu \left(\frac{1}{p} - \frac{p\alpha + p - n}{\mathcal{K}_0 p \alpha} \right) t_0^{p\alpha + p - n} \|\nabla u_0\|_p^p + \left(\frac{\alpha}{n} - \frac{1}{\mathcal{K}_0} \right) t_0^n [u_0]_{\alpha, n/\alpha}^{n/\alpha} \\
&\quad + \frac{1}{\mathcal{K}_0} \int_{\mathbb{R}^n} \frac{1}{|x|^\beta} \frac{\left[f(t_0^\alpha u_0) t_0^\alpha u_0 - \left(\mathcal{K}_0 + \frac{n-\beta}{\alpha} \right) F(t_0^\alpha u_0) \right]}{|t_0^\alpha u_0|^{\frac{n-\beta}{\alpha}}} |u_0|^{\frac{n-\beta}{\alpha}} dx \\
&< \mu \left(\frac{1}{p} - \frac{p\alpha + p - n}{\mathcal{K}_0 p \alpha} \right) \|\nabla u_0\|_p^p + \left(\frac{\alpha}{n} - \frac{1}{\mathcal{K}_0} \right) [u_0]_{\alpha, n/\alpha}^{n/\alpha} \\
&\quad + \frac{1}{\mathcal{K}_0} \int_{\mathbb{R}^n} \frac{1}{|x|^\beta} \left[f(u_0) u_0 - \left(\mathcal{K}_0 + \frac{n-\beta}{\alpha} \right) F(u_0) \right] dx. \tag{3.6}
\end{aligned}$$

Here we have used the facts that $t_0 < 1$ and lemma 2.4. By the weak lower semicontinuity of norm, we have

$$\|\nabla u_0\|_p^p \leq \liminf_{k \rightarrow \infty} \|\nabla u_k\|_p^p, \quad [u_0]_{\alpha, n/\alpha}^{n/\alpha} \leq \liminf_{k \rightarrow \infty} [u_k]_{\alpha, n/\alpha}^{n/\alpha}.$$

By Fatou's lemma, we get

$$\int_{\mathbb{R}^n} \frac{1}{|x|^\beta} \left[f(u_0) u_0 - \left(\mathcal{K}_0 + \frac{n-\beta}{\alpha} \right) F(u_0) \right] dx \leq \liminf_{k \rightarrow \infty} \int_{\mathbb{R}^n} \frac{1}{|x|^\beta} \left[f(u_k) u_k - \left(\mathcal{K}_0 + \frac{n-\beta}{\alpha} \right) F(u_k) \right] dx.$$

Consequently, it follows from (3.6) that

$$\begin{aligned}
\gamma(m) &< \lim_{k \rightarrow \infty} \left[\mu \left(\frac{1}{p} - \frac{p\alpha + p - n}{\mathcal{K}_0 p \alpha} \right) \|\nabla u_k\|_p^p + \left(\frac{\alpha}{n} - \frac{1}{\mathcal{K}_0} \right) [u_k]_{\alpha, n/\alpha}^{n/\alpha} \right. \\
&\quad \left. + \frac{1}{\mathcal{K}_0} \int_{\mathbb{R}^n} \frac{1}{|x|^\beta} \left(f(u_k) u_k - \left(\mathcal{K}_0 + \frac{n-\beta}{\alpha} \right) F(u_k) \right) dx \right] \\
&\leq \lim_{k \rightarrow \infty} \left[\mathcal{I}(u_k) - \frac{1}{\mathcal{K}_0} P(u_k) \right] = \gamma(m),
\end{aligned}$$

which is a contradiction. Thus, $\mathcal{I}(u_0) \geq 0$.

Step 4. The strong convergence of $\{u_k\}_k$. To this aim, we first show that

$$\mathcal{I}(u_0) = \gamma(m). \tag{3.7}$$

Obviously, we have $\mathcal{I}(u_0) \leq \gamma(m)$ by (3.5). Next we prove that the case $\mathcal{I}(u_0) < \gamma(m)$ can not occur. If $\mathcal{I}(u_0) < \gamma(m)$, then

$$\frac{\mu}{p} \|\nabla u_0\|_p^p + \frac{\alpha}{n} [u_0]_{\alpha, n/\alpha}^{n/\alpha} < \lim_{k \rightarrow \infty} \left(\frac{\mu}{p} \|\nabla u_k\|_p^p + \frac{\alpha}{n} [u_k]_{\alpha, n/\alpha}^{n/\alpha} \right).$$

Without loss of generality, we may assume that

$$\lim_{k \rightarrow \infty} \|\nabla u_k\|_p = \xi, \quad \lim_{k \rightarrow \infty} [u_k]_{\alpha, n/\alpha} = \eta$$

and

$$[u_0]_{\alpha, n/\alpha}^{n/\alpha} < \lim_{k \rightarrow \infty} [u_k]_{\alpha, n/\alpha}^{n/\alpha} = \eta^{n/\alpha}.$$

Then (3.5) implies that

$$\frac{\mu}{p} \xi^p + \frac{\alpha}{n} \eta^{n/\alpha} = \gamma(m) + \int_{\Omega} \frac{F(u_0)}{|x|^\beta} dx.$$

Set $v_k = \frac{u_k}{[u_k]_{\alpha, n/\alpha}}$. Then $v_k \rightharpoonup v = u_0/\eta$ in $W^{\alpha, n/\alpha}(\mathbb{R}^n)$ and $[v]_{\alpha, n/\alpha} < 1$. Thus, it follows from theorem 2.2 that

$$\sup_k \int_{\mathbb{R}^n} \frac{1}{|x|^\beta} \Psi(\vartheta' v_k^{n/(n-\alpha)}) dx < \infty, \quad \forall \vartheta' < \frac{(1 - \frac{\beta}{n})\vartheta_{\alpha, n}^*}{(1 - [v]_{\alpha, n/\alpha}^{n/\alpha})^{\alpha/(n-\alpha)}}. \quad (3.8)$$

On the other hand, by (3.5), we have

$$\gamma(m) - \mathcal{I}(u_0) = \frac{\mu}{p} \xi^n + \frac{\alpha}{n} \eta^{n/\alpha} - \left(\frac{\mu}{p} \|\nabla u_0\|_p^p + \frac{\alpha}{n} [u_0]_{\alpha, n/\alpha}^{n/\alpha} \right).$$

In view of $\mathcal{I}(u_0) \geq 0$, it leads to

$$\begin{aligned} \frac{\alpha}{n} \eta^{n/\alpha} &\leq \gamma(m) + \frac{\alpha}{n} [u_0]_{\alpha, n/\alpha}^{n/\alpha} - \frac{\mu}{p} \xi^n + \frac{\mu}{p} \|\nabla u_0\|_p^p \\ &\leq \gamma(m) + \frac{\alpha}{n} [u_0]_{\alpha, n/\alpha}^{n/\alpha}. \end{aligned}$$

Thus,

$$\eta^{n/\alpha} \leq \frac{n}{\alpha} \gamma(m) + [u_0]_{\alpha, n/\alpha}^{n/\alpha}. \quad (3.9)$$

Observe that

$$\eta^{n/\alpha} = \frac{\eta^{n/\alpha} - [u_0]_{\alpha, n/\alpha}^{n/\alpha}}{1 - [v]_{\alpha, n/\alpha}^{n/\alpha}}.$$

Then by (3.9) and lemma 3.2, it yields

$$\eta^{n/\alpha} \leq \frac{n}{\alpha (1 - [v]_{\alpha, n/\alpha}^{n/\alpha})} \gamma(m) < \frac{((n - \beta)\vartheta_{\alpha, n}^*/n\vartheta_0)^{\frac{n-\alpha}{\alpha}}}{1 - [v]_{\alpha, n/\alpha}^{n/\alpha}}.$$

Thus, there exist $k_0 \in \mathbb{N}$ and $\vartheta'' > 0$ such that

$$\vartheta_0 [u_k]_{\alpha, n/\alpha}^{n/(n-\alpha)} < \vartheta'' < \frac{(n - \beta)\vartheta_{\alpha, n}^*/n\vartheta_0}{(1 - [v]_{\alpha, n/\alpha}^{n/\alpha})^{\frac{\alpha}{n-\alpha}}}.$$

for all $k \geq k_0$. We choose $\nu > 1$ close to 1 and $\vartheta' > \vartheta_0$ close to ϑ_0 such that

$$\nu \vartheta' [u_k]_{\alpha, n/\alpha}^{n/(n-\alpha)} \leq \vartheta'' < \frac{(n - \beta)\vartheta_n/n\vartheta_0}{(1 - [v]_{\alpha, n/\alpha}^{n/\alpha})^{\frac{\alpha}{n-\alpha}}}.$$

By virtue of theorem 2.1, there exists $C > 0$ such that

$$\int_{\mathbb{R}^n} \frac{1}{|x|^\beta} \Psi(\nu \vartheta' |u_k|^{n/(n-\alpha)}) dx \leq \int_{\mathbb{R}^n} \frac{1}{|x|^\beta} \Psi(\vartheta'' |u_k|^{n/(n-\alpha)}) dx \leq C.$$

Therefore, we deduce from (2.12) that

$$\begin{aligned} &\left| \int_{\mathbb{R}^n} \frac{1}{|x|^\beta} f(u_k)(u_k - u_0) dx \right| \\ &\leq C \left(\int_{\mathbb{R}^n} \frac{1}{|x|^\beta} |u_k - u_0|^{\frac{n}{\alpha}} dx + \int_{\mathbb{R}^n} \frac{1}{|x|^\beta} |u_k - u| \Psi(\vartheta' |u_k|^{n/(n-\alpha)}) dx \right) \\ &\leq C \left(\int_{\mathbb{R}^n} \frac{1}{|x|^\beta} |u_k - u_0|^{\frac{n}{\alpha}} dx + \int_{\mathbb{R}^n} \frac{1}{|x|^\beta} |u_k - u|^{\frac{\nu}{\nu-1}} dx \right) \rightarrow 0 \end{aligned}$$

as $k \rightarrow \infty$. Recall that $(\mathcal{I}|_{\mathcal{S}_m})'(u_k) \rightarrow 0$. Then by proposition 5.12 of [42], there exists $\{\lambda_k\}_k \subset \mathbb{R}$ such that

$$-\mu \Delta_p u_k + (-\Delta)_{n/\alpha}^\alpha u_k - |x|^{-\beta} f(u_k) = \lambda_k |u_k|^{\frac{n}{\alpha}-2} u_k + o_k(1) \text{ in } (\mathcal{W}(\mathbb{R}^n))^*. \quad (3.10)$$

Furthermore, we have

$$\lambda_k m = \mu \|\nabla u_k\|_p^p + [u_k]_{\alpha, n/\alpha}^{n/\alpha} - \int_{\mathbb{R}^n} \frac{f(u_k) u_k}{|x|^\beta} dx + o_k(1).$$

Then, $\{\lambda_k\}_k$ is bounded in $\mathbb{R}$. Thus, up to a subsequence, we may assume that $\lambda_k \rightarrow \lambda$ for some $\lambda \in \mathbb{R}$. Furthermore, it follows from $P(u_k) = o_k(1)$ that

$$\mu \frac{p\alpha + p - n}{p\alpha} \|\nabla u_k\|_n^n + [u_k]_{\alpha, n/\alpha}^{n/\alpha} = \int_{\mathbb{R}^n} \frac{f(u_k)u_k - \frac{n-\beta}{\alpha} F(u_k)}{|x|^\beta} dx.$$

Then

$$\lambda_k m = -\frac{n-\beta}{\alpha} \int_{\mathbb{R}^n} \frac{F(u_k)}{|x|^\beta} dx - \mu \frac{p-n}{p\alpha} \|\nabla u_k\|_p^p + o_k(1) < 0.$$

This together with $u_0 \neq 0$ implies that $\lambda < 0$.

It follows from (3.10) that

$$\begin{aligned} o_k(1) &= \langle \mathcal{I}'(u_k), u_k - u_0 \rangle - \lambda_k \int_{\mathbb{R}^n} |u_k|^{n/\alpha-2} u_k (u_k - u_0) dx \\ &= \langle u_k, u_k - u_0 \rangle_{\alpha, n} - \int_{\Omega} \frac{1}{|x|^\beta} f(u_k) (u_k - u_0) dx, \end{aligned} \quad (3.11)$$

where

$$\begin{aligned} \langle u_k, u_k - u_0 \rangle_{\alpha, n} &= \mu \int_{\mathbb{R}^n} |\nabla u_k|^{p-2} \nabla u_k \cdot (\nabla u_k - \nabla u_0) dx - \lambda_k \int_{\mathbb{R}^n} |u_k|^{n/\alpha-2} u_k (u_k - u_0) dx \\ &\quad + \iint_{\mathbb{R}^{2n}} \frac{|u_k(x) - u_k(y)|^{\frac{n}{\alpha}-2} (u_k(x) - u_k(y)) (u_k(x) - u_0(x) - u_k(y) + u_0(y))}{|x - y|^{2n}} dx dy. \end{aligned}$$

Define a functional L as follows:

$$\langle L(v), w \rangle = \langle v, w \rangle_{\alpha, n}$$

for all $v, w \in \mathcal{W}(\mathbb{R}^n)$. By the Hölder inequality, one can see that

$$|\langle L(v), w \rangle| \leq (\mu \|\nabla v\|_p^{p-1} + |\lambda_k| \|v\|_{n/\alpha}^{n/\alpha-1} + [v]_{\alpha, n/\alpha}^{\frac{n}{\alpha}-1}) \|w\|,$$

which together with the definition of L implies that for each v , $L(v)$ is a bounded linear functional on $\mathcal{W}(\mathbb{R}^n)$. Thus, $\langle L(u_0), u_k - u_0 \rangle = o_k(1)$, that is,

$$\langle u_0, u_k - u_0 \rangle_{\alpha, n} = o_k(1).$$

In conclusion, we can deduce from (3.11) that

$$[\langle u_k, u_k - u_0 \rangle_{\alpha, n} - \langle u_0, u_k - u_0 \rangle_{\alpha, n}] = o_k(1).$$

This together with Simon's inequality implies that $u_k \rightarrow u_0$ in $\mathcal{W}(\mathbb{R}^n)$. Furthermore, we have $\mathcal{I}(u_0) = \gamma(m)$, which is a contradiction. Thus, the assertion (3.7) holds.

Combining $\mathcal{I}(u_0) = \gamma(m)$ with $\mathcal{I}(u_k) \rightarrow \gamma(m)$ and $\lambda < 0$, we conclude that $\|u_k - u_0\| \rightarrow 0$ as $k \rightarrow \infty$. This finishes the proof. $\square$

Proof of theorem 1.1. The proof follows directly from lemma 3.4. $\square$

4. Normalized ground states

In this section, we discuss the existence of normalized ground state solutions of problem (1.1). Consider the Pohozaev manifold associated to the functional $\mathcal{I}$, that is,

$$\mathcal{P}_m = \{u \in \mathcal{S}_m : P(u) = 0\}.$$

Clearly, $\mathcal{P}_m$ is a nonempty subset of $\mathcal{S}_m$. Moreover, we have the following result.

Lemma 4.1. *The minimizing problem $c^*(m) := \inf_{u \in \mathcal{P}_m} \mathcal{I}(u)$ is well defined.*

Proof. For each $u \in \mathcal{P}_m$, we deduce from (f_3) that

$$\begin{aligned} \mathcal{I}(u) &= \mathcal{I}(u) - \frac{1}{r}P(u) \\ &= \frac{\mu}{p}\|\nabla u\|_p^p + \frac{\alpha}{n}[u]_{\alpha, n/\alpha}^{n/\alpha} - \frac{1}{r}\left(\mu\frac{p\alpha + p - n}{p\alpha}\|\nabla u\|_p^p + [u]_{\alpha, n/\alpha}^{n/\alpha}\right) \\ &\quad + \frac{1}{r}\int_{\mathbb{R}^n} \frac{f(u)u - (\frac{n-\beta}{\alpha} + r)F(u)}{|x|^\beta} dx \\ &\geq \mu\left(\frac{1}{p} - \frac{p\alpha + p - n}{rp\alpha}\right)\|\nabla u\|_p^p + \left(\frac{\alpha}{n} - \frac{1}{r}\right)[u]_{\alpha, n/\alpha}^{n/\alpha}, \end{aligned}$$

where $\mathcal{K}_0 < r < \mathcal{K} - \frac{n-\beta}{\alpha}$. It follows that $\mathcal{I}$ is coercive and bounded from below on $\mathcal{P}_m$. Thus, $c^* := \inf_{u \in \mathcal{P}_m} \mathcal{I}(u)$ is well defined. $\square$

The next result is crucial in our arguments to get the existence of a normalized ground state solution for (1.1).

Lemma 4.2. Assume that (f_1) – (f_6) are satisfied. Then $\gamma(m) = c^*(m)$.

Proof. For any $u \in \mathcal{P}_m$, by lemma 2.5, there exist $t_1 > 0$ small enough and t_2 large enough such that $t_1 \star u \in A_m$ and $\mathcal{I}(t_2 \star u) < 0$. Let $h(t) = (t_2 t + (1-t)t_1) \star u$ with $t \in [0, 1]$, it is clear that $h(t) \in \Gamma_m$, by lemma 2.6, it holds that

$$\mathcal{I}(u) = \max_{t \in (0, +\infty)} \mathcal{I}(t \star u) \geq \max_{t \in [t_1, t_2]} \mathcal{I}(t \star u) = \max_{t \in [0, 1]} \mathcal{I}(h(t)),$$

which means that $\gamma(m) \leq c^*(m)$. Clearly, by theorem 1.1 there exists $u_0 \in \mathcal{P}_m$ such that $\mathcal{I}(u_0) = \gamma(m)$. Thus, $\gamma(m) = \mathcal{I}(u_0) \geq c^*(m)$. This proves the lemma. $\square$

Proof of theorem 1.2. By theorem 1.1, we know that problem (1.1) has a normalized solution $(u_0, \lambda) \in \mathcal{W}(\mathbb{R}^n) \times (-\infty, 0)$. Moreover, $\mathcal{I}(u_0) = \gamma(m)$ and $u_0 \in \mathcal{P}_m$. By lemma 4.2, we have $\mathcal{I}(u_0) = c^*(m)$. Thus, $\mathcal{I}(u_0) = c^*(m) = \inf_{u \in \mathcal{P}_m} \mathcal{I}(u)$. Consequently, u_0 is a ground state of problem (1.1). $\square$

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Data availability statement

Since our paper only consists of mathematical proofs, no additional datasets have been used. No new data were created or analysed in this study.

Conflict of interest

The authors declare that there is no conflict of interest.

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