Nonlinear Differ. Equ. Appl.
(2016) 23:42

DOI 10.1007/s00030-016-0394-x

Nonlinear Differential Equations and Applications NoDEA

On noncoercive elliptic problems

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Abstract. We consider a nonlinear noncoercive elliptic equation driven by the p-Laplacian. We show that if the L^∞-perturbation has small norm, then the problem admits a positive solution. Moreover, if the L^∞-perturbation is nonzero and nonnegative, then we find two positive solutions. Also, we consider a class of semilinear equations with an indefinite and unbounded potential. Using critical groups, we show that there is a nontrivial solution and under a global sign condition, we show that this solutions is nodal. Our results extend and improve a recent work of Rădulescu (Discr. Cont. Dyn. Syst. Ser. S , 5:845–856, [14]).

Mathematics Subject Classification. 35J20, 35J60, 35J92, Secondary 58E05.

Keywords. Positive solution, Nonlinear regularity, Nonlinear maximum principle, Critical groups, Local linking, Nodal solution.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper first (see Sect. 3), we study the following nonlinear Dirichlet problem

$$\left\{ \begin{array}{ll} -\Delta_p u(z) = \beta(z)u(z)^{p-1} + f(z,u(z)) + g(z) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \ u > 0, \ 1 < p < \infty, \ g \in L^\infty(\Omega). & \end{array} \right\} \qquad (1)_g$$

Here Δ_p denotes the p-Laplacian differential operator defined by

$$\Delta_p u = \operatorname{div}(|Du|^{p-2}Du) \text{ for all } u \in W_0^{1,p}(\Omega).$$

Also $\beta \in L^\infty(\Omega)$ and $\beta(z) \leqslant \hat{\lambda}_1(p)$ for a.a. $z \in \Omega$ with strict inequality on a set of positive measure. Here $\hat{\lambda}_1(p) > 0$ denotes the principal eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$. The perturbation $f(z,x)$ is a Carathéodory function (that is, for all $x \in \mathbb{R}$, $z \longmapsto f(z,x)$ is measurable and for a.a. $z \in \Omega$, $x \longmapsto f(z,x)$ is continuous), which exhibits $(p-1)$-superlinear growth near $+\infty$, but without satisfying the usual Ambrosetti–Rabinowitz condition (AR-condition for short). We show that for $||g||_\infty$ sufficiently small, problem $(1)_g$ admits at least one positive solution. Moreover, we show that if g is nonzero and

Published online: 23 June 2016

nonnegative, then a second positive solution can be found. Problem $(1)_g$ was investigated recently by Rădulescu [14], when $p = 2$ (semilinear equation) and with a perturbation function $f(z, x) = f(x)$ which is in $C^1(\mathbb{R})$ and satisfies the AR-condition. Under these conditions, the author shows that the problem has a positive solution for $\|g\|_\infty$ small (see Theorem 2.1 of [14]). Our work here generalizes the result of Rădulescu [14] and provides additional information for problem $(1)_g$.

In Sect. 4, we deal with the following semilinear problem:

$$\left\{ \begin{array}{ll} -\Delta u(z) + \beta(z)u(z) = \lambda u(z) + f(z, u(z)) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. & \end{array} \right\} \qquad (2)_\lambda$$

In this problem, $\beta \in L^\tau(\Omega)$ with $\tau > \frac{N}{2}$ and is general indefinite. Also $\lambda \in \mathbb{R}$ is a parameter and $f(z, x)$ is a measurable function on $\Omega \times \mathbb{R}$ which is C^1 in the $x \in \mathbb{R}$ variable and $x \longmapsto f(z, x)$ exhibits $(p-1)$-superlinear growth near $\pm\infty$ again without satisfying the AR-condition. We show for all $\lambda \geqslant \hat{\lambda}_1(2, \beta)$, problem $(2)_\lambda$ admits a nontrivial solution (by $\hat{\lambda}_1(2, \beta)$ we denote the principal eigenvalue of $(-\Delta + \beta I,\ H_0^1(\Omega))$). In fact, under a global sign condition on $f(z, \cdot)$, we show that any nontrivial solution of $(2)_\lambda$ is necessarily nodal (sign changing), that is, the problem has no nontrivial constant sign solutions. Problem $(2)_\lambda$ was also studied by Rădulescu [14] under the hypotheses that $\beta \equiv 0$, $f(z, x) = f(x)$ and $f \in C^1(\mathbb{R})$ satisfies the AR-condition and it is strictly increasing and onto. In fact in [14] it was left as an open problem whether the strict monotonicity and surjectivity conditions on $f(\cdot)$ can be relaxed. Here we show that the answer to this open problem is affirmative and in fact we go even further establishing the existence of solutions for a broader class of equations with more general perturbations $f(z, x)$.

Our approach is variational based on the critical point theory, coupled with suitable truncation and comparison techniques. In Sect. 4 we also use critical groups. In the next section for the convenience of the reader, we review the main mathematical tools that we will use in this paper.

2. Mathematical background

Let X be a Banach space and X^* its topological dual. By $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair (X^*, X). Given $\varphi \in C^1(X)$, we say that it satisfies the Cerami condition (the C-condition for short), if the following holds:

"Every sequence $\{u_n\}_{n\geqslant1} \subseteq X$ such that $\{\varphi(u_n)\}_{n\geqslant1} \subseteq \mathbb{R}$ is bounded and

$$(1 + ||u_n||)\varphi'(u_n) \to 0 \text{ in } X^* \text{ as } n \to \infty,$$

admits a strongly convergent subsequence".

This is a compactness-type condition on the functional φ which is more general than the more common Palais-Smale condition. The C-condition leads to a deformation theorem, from which we can derive the minimax theory for the critical values of φ. One of the main results in this theory is the so-called "mountain pass theorem" due to Ambrosetti and Rabinowitz [4]. Here we state this in a slightly more general form (see Gasinski and Papageorgiou [8]).

Theorem 1. *Assume that X is a Banach space, $\varphi \in C^1(X)$ satisfies the C-condition, $u_0, u_1 \in X$ with $||u_1 - u_0|| > \rho > 0$*

$$\max\{\varphi(u_0), \varphi(u_1)\} < \inf\left[\varphi(u) : ||u - u_0|| = \rho\right] = \eta_\rho$$

and $c = \inf_{\gamma \in \Gamma} \max_{0 \leqslant t \leqslant 1} \varphi(\gamma(t))$ with $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = u_0,\ \gamma(1) = u_1\}$. Then $c \geqslant \eta_\rho$ and c is a critical value of φ.

In the analysis of problems $(1)_g$ and $(2)_\lambda$, we will use the Sobolev spaces $W_0^{1,p}(\Omega)$ and $H_0^1(\Omega)$ and the Banach space $C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0\}$. The latter is an ordered Banach space with positive cone

$$C_+ = \{u \in C_0^1(\overline{\Omega}) : u(z) \geqslant 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior

$$\operatorname{int} C_+ = \left\{u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega,\ \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} < 0\right\},$$

where $n(\cdot)$ denotes the outward unit normal on $\partial\Omega$.

We consider the following nonlinear eigenvalue problem:

$$-\Delta_p u(z) = \hat{\lambda}|u(z)|^{p-2}u(z) \text{ in } \Omega,\ u|_{\partial\Omega} = 0. \tag{3}$$

We say that $\hat{\lambda} \in \mathbb{R}$ is an eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$, if problem (3) admits a nontrivial solution $\hat{u} \in W_0^{1,p}(\Omega)$ known as an eigenfunction corresponding to $\hat{\lambda}$. The nonlinear regularity theory (see, for example, Gasinski and Papageorgiou [8, pp. 737–738]), implies that $\hat{u} \in C_0^1(\overline{\Omega})$. We know that $(-\Delta_p, W_0^{1,p}(\Omega))$ has a smallest eigenvalue $\hat{\lambda}_1(p)$ such that:

- (i) $\hat{\lambda}_1(p) > 0$ and it is isolated (that is, there exists $\epsilon > 0$ such that $\left[\hat{\lambda}_1(p), \hat{\lambda}_1(p) + \epsilon\right)$ does not contain any other eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$);
- (ii) $\hat{\lambda}_1(p)$ is simple (that is, if $\hat{u}, \hat{v} \in C_0^1(\overline{\Omega})$ are eigenfunctions corresponding to $\hat{\lambda}_1(p)$, then $\hat{u} = \xi\hat{v}$ with $\xi \neq 0$)

and

$$\hat{\lambda}_1(p) = \inf\left[\frac{||Du||_p^p}{||u||_p^p} : u \in W_0^{1,p}(\Omega),\ u \neq 0\right]. \tag{4}$$

The infimum in (4) is realized on the one-dimensional eigenspace corresponding to $\hat{\lambda}_1(p) > 0$. It is clear from (4) that the elements of this eigenspace do not change sign. Let $\hat{u}_1(p)$ be the L^p-normalized (that is, $||\hat{u}_1(p)||_p = 1$), positive eigenfunction corresponding to $\hat{\lambda}_1(p)$. The nonlinear maximum principle (see, for example, Gasinski and Papageorgiou [8, p. 738]), implies that $\hat{u}_1(p) \in \operatorname{int} C_+$. We mention that $\hat{\lambda}_1(p)$ is the only eigenvalue with eigenfunctions of constant sign. Every eigenvalue $\hat{\lambda} \neq \hat{\lambda}_1(p)$ has nodal eigenfunctions.

As a consequence of the above properties of $\hat{\lambda}_1(p) > 0$ and $\hat{u}_1(p) \in \operatorname{int} C_+$, we have the following lemma (see Papageorgiou and Kyritsi [10, p. 356]).

Lemma 2. *If $\beta \in L^{\infty}(\Omega)$ and $\beta(z) \leqslant \hat{\lambda}_1(p)$ a.e. in Ω with strict inequality on a set of positive measure, then there exists $\xi_0 > 0$ such that*

$$||Du||_p^p - \int_\Omega \beta(z)|u|^p dz \geqslant \xi_0 ||Du||_p^p \text{ for all } u \in W_0^{1,p}(\Omega).$$

To deal with problem $(2)_\lambda$, we will use the spectrum of $(-\Delta + \beta I, H_0^1(\Omega))$. So, we consider the following linear eigenvalue problem

$$-\Delta u(z) + \beta(z)u(z) = \lambda u(z) \text{ in } \Omega,\ u|_{\partial\Omega} = 0. \tag{5}$$

Recall that $\beta \in L^{\tau}(\Omega)$ with $\tau > \frac{N}{2}$ and in general is indefinite (that is, sign-changing). Problem (5) has a strictly increasing sequence $\{\hat{\lambda}_k(2,\beta)\}_{k\geqslant 1} \subseteq \mathbb{R}$ of eigenvalues such that $\hat{\lambda}_k(2,\beta) \to +\infty$ as $k \to +\infty$. By $E(\hat{\lambda}_k(2,\beta))$ we denote the eigenspace corresponding to the eigenvalue $\hat{\lambda}_k(2,\beta)$. We have $E(\hat{\lambda}_k(2,\beta)) \subseteq C_0^1(\overline{\Omega})$ and the eigenspace has the so-called unique continuation property (UCP for short), that is, if $\hat{u} \in E(\hat{\lambda}_k(2,\beta))$ and $\hat{u}$ vanishes on a set of positive measure, then $\hat{u} \equiv 0$. We have the following variational characterizations of these eigenvalues:

$$\hat{\lambda}_1(2,\beta) = \inf\left[\frac{||Du||_2^2 + \int_\Omega \beta(z)u^2 dz}{||u||_2^2} : u \in H_0^1(\Omega),\ u \neq 0\right] \tag{6}$$

and for $k \geqslant 2$, we have

$$\begin{aligned} \hat{\lambda}_k(2,\beta) &= \sup\left[\frac{||Du||_2^2 + \int_\Omega \beta(z)u^2 dz}{||u||_2^2} : u \in \overset{k}{\underset{\mathrm{i}=1}{\oplus}} E(\hat{\lambda}_i(2,\beta)), u \neq 0\right] \\ &= \inf\left[\frac{||Du||_2^2 + \int_\Omega \beta(z)u^2 dz}{||u||_2^2} : u \in \overline{\underset{\mathrm{i}\geqslant \mathrm{k}}{\oplus} E(\hat{\lambda}_i(2,\beta))}, u \neq 0\right]. \end{aligned} \tag{7}$$

In (6) and (7), the infimum and the supremum are realized on the corresponding eigenspace $E(\hat{\lambda}_k(2,\beta))$ (see Kyritsi and Papageorgiou [9]).

We have the following orthogonal direct sum decomposition

$$H_0^1(\Omega) = \overline{H}_k \oplus \hat{H}_k$$

with $\overline{H}_k = \oplus_{\mathrm{i}=1}^k E(\hat{\lambda}_i(2,\beta))$ and $\hat{H}_k = \overline{\oplus_{\mathrm{i}\geqslant \mathrm{k}+1} E(\hat{\lambda}_i(2,\beta))}$.

Next let X be a Banach space and $\varphi \in C^1(X)$, $c \in \mathbb{R}$. We introduce the following sets

$$\begin{aligned} \varphi^c &= \{u \in X : \varphi(u) \leqslant c\},\ K_\varphi = \{u \in X : \varphi'(u) = 0\}, \\ K_\varphi^c &= \{u \in K_\varphi : \varphi(u) = c\}. \end{aligned}$$

Let (Y_1, Y_2) be a topological pair such that $Y_2 \subseteq Y_1 \subseteq X$. For every integer $k \geqslant 0$, by $H_k(Y_1, Y_2)$ we denote the kth relative singular homology group for the pair (Y_1, Y_2) with integer coefficients. Recall that $H_k(Y_1, Y_2) = 0$ for all $k < 0$. The critical groups of φ at an isolated critical point $u \in K_\varphi^c$ are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U,\ \varphi^c \cap U \setminus \{u\}) \text{ for every } k \geqslant 0,$$

where U is a neighborhood of u such that $\varphi^c \cap K_\varphi \cap U = \{u\}$. The excision property of singular homology theory, implies that the above definition of critical groups is independent of the choice of the neighborhood U.

Suppose that $\varphi \in C^1(X)$ satisfies the C-condition and $\inf \varphi(K_\varphi) > -\infty$. Let $c < \inf \varphi(K_\varphi)$. The critical groups of φ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \text{ for all } k \geqslant 0.$$

The second deformation theorem (see, for example, Gasinski and Papageorgiou [8, p. 628]), implies that this definition is independent of the choice of level $c < \inf \varphi(K_\varphi)$. If for some $k \geqslant 0$, $C_k(\varphi, 0) \neq 0$, $C_k(\varphi, \infty) = 0$, then φ admits a nontrivial critical point.

We conclude this section by fixing our notation. By $||\cdot||$ we denote the norm of the Sobolev space $W_0^{1,p}(\Omega)$. By virtue of the Poincaré inequality, we have

$$||u|| = ||Du||_p \text{ for all } u \in W_0^{1,p}(\Omega).$$

For every $x \in \mathbb{R}$, we set $x^\pm = \max\{\pm x, 0\}$. Then given $u \in W_0^{1,p}(\Omega)$, we define $u^\pm(\cdot) = u(\cdot)^\pm$. We have

$$u^\pm \in W_0^{1,p}(\Omega),\ u = u^+ - u^-,\ |u| = u^+ + u^-.$$

By $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$. Finally, if $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function (for example, a Carathéodory function), then we define

$$N_h(u)(\cdot) = h(\cdot, u(\cdot)) \text{ for all } u \in W_0^{1,p}(\Omega)$$

(the Nemytskii operator corresponding to function $h(\cdot,\cdot)$). Note that $z \longmapsto N_f(u)(z)$ is measurable.

3. Solutions for problem $(1)_g$

In this section, we show that for $||g||_\infty$ small, problem $(1)_g$ has at least one positive solution and for nonzero and nonnegative g, it has two positive solutions. The hypotheses on the perturbation $f(z, x)$, are the following:

$H_1 : f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$ and

(i) $|f(z, x)| \leqslant a(z)(1 + x^{r-1})$ for a.a $z \in \Omega$, all $x \geqslant 0$, with $a \in L^\infty(\Omega)_+$, $p < r < p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N \\ +\infty & \text{if } p \leqslant N \end{cases}$;

(ii) if $F(z, x) = \int_0^x f(z, s)ds$, then

$$\lim_{x \to +\infty} \frac{F(z, x)}{x^p} = +\infty \text{ uniformly for a.a } z \in \Omega;$$

(iii) there exists $\eta_0 > 0$ and $\tau \in \left(\max\{1, (r-p)\frac{N}{p}\}, p^*\right)$ such that

$$0 < \eta_0 \leqslant \liminf_{x \to +\infty} \frac{f(z, x)x - pF(z, x)}{x^\tau} \text{ uniformly for a.a. } z \in \Omega;$$

(iv) $\lim_{x\to 0^+} \frac{f(z,x)}{x^{p-1}} = 0$ uniformly for a.a. $z \in \Omega$;

(v) for every $\rho > 0$, there exists $\xi_\rho > 0$ such that for a.a. $z \in \Omega$, the application $x \longmapsto f(z,x) + \xi_\rho x^{p-1}$ is nondecreasing on $[0,\rho]$.

Remark 1. Since we are interested on positive solutions and the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0,+\infty)$, without any loss of generality, we may assume that for a.a. $z \in \Omega$ $f(z,x) = 0$ for all $x \leqslant 0$. Hypotheses $H_1(ii),(iii)$ imply that $f(z,\cdot)$ is $(p-1)$-superlinear near $+\infty$, more precisely we have

$$\lim_{x\to+\infty} \frac{f(z,x)}{x^{p-1}} = +\infty \text{ uniformly for a.a. } z \in \Omega.$$

Note that we do not employ the usual in such cases AR-condition (see [4]). Instead we use a weaker condition (see hypothesis $H_1(iii)$) which incorporates in our framework $(p-1)$-superlinear perturbations with "slower" growth near $+\infty$. For example, the function

$$f(x) = x^{p-1}\left[\ln x + \frac{1}{p}\right] \text{ for all } x \geqslant 0$$

(for the sake of simplicity we have dropped the z-dependence), satisfies hypotheses H_1 but fails to satisfy the AR-condition. So, Theorem 1 of [14] does not apply to this function. If $f(z,\cdot) \in C^1(\mathbb{R})$ and $f'_x(z,\cdot)$ is bounded on bounded sets, then hypothesis $H_1(v)$ is satisfied.

First we show that we cannot have a positive solution for problem $(1)_g$ for every $g \in L^\infty(\Omega)$. To this end, let $u \in W_0^{1,p}(\Omega)$ be a positive solution for problem $(1)_g$. The nonlinear regularity theory and the nonlinear maximum principle (see, for example, Gasinski and Papageorgiou [8, pp. 737–738]), imply that $u \in \operatorname{int} C_+$. Recall that $\hat{u}_1(p) \in \operatorname{int} C_+$. So, invoking Lemma 3.3 of Filippakis, Kristaly and Papageorgiou [6], we can find $c_1, c_2 > 0$ such that

$$\begin{aligned} c_1 u \leqslant \hat{u}_1(p) &\leqslant c_2 u \\ &\Rightarrow c_1 \leqslant \frac{\hat{u}_1(p)}{u} \leqslant c_2 \text{ in } \Omega. \end{aligned} \tag{8}$$

Let $R(\hat{u}_1(p),u)(z) = |D\hat{u}_1(p)(z)|^p - |Du(z)|^{p-2}\left(Du(z), D\left(\frac{\hat{u}_1(p)^p}{u^{p-1}}\right)(z)\right)_{\mathbb{R}^N}$. From the nonlinear Picone identity of Allegretto and Huang [3], we have

$$\begin{aligned} 0 &\leqslant \int_\Omega R(\hat{u}_1(p),u)dz \\ &= ||D\hat{u}_1(p)||_p^p - \int_\Omega (-\Delta_p u)\frac{\hat{u}_1(p)^p}{u^{p-1}}dz \end{aligned}$$

(using the nonlinear Green's identity, see Gasinski and Papageorgiou[[8], p. 211]

$$\begin{aligned} &= \hat{\lambda}_1(p)||\hat{u}_1(p)||_p^p - \int_\Omega \left[\beta(z)u^{p-1} + f(z,u) + g(z)\right]\frac{\hat{u}_1(p)^p}{u^{p-1}}dz \\ &= \int_\Omega \left[\hat{\lambda}_1(p) - \beta(z)\right]\hat{u}_1(p)^p dz - \int_\Omega f(z,u)\frac{\hat{u}_1(p)^p}{u^{p-1}}dz - \int_\Omega g(z)\frac{\hat{u}_1(p)^p}{u^{p-1}}dz. \end{aligned} \tag{9}$$

We know that

$$\vartheta_0 = \int_\Omega \left[\hat{\lambda}_1(p) - \beta(z)\right] \hat{u}_1(p)^p dz > 0. \tag{10}$$

As we already observed, hypotheses $H_1(ii), (iii)$ imply that

$$\lim_{x\to+\infty} \frac{f(z,x)}{x^{p-1}} = +\infty \text{ uniformly for a.a } z \in \Omega. \tag{11}$$

From (11) and hypothesis $H_1(i)$, we see that given $\xi > \vartheta_0$ we can find $c_3 = c_3(\xi) > 0$ such that

$$f(z,x) \geqslant \xi x^{p-1} - c_3 \text{ for a.a. } z \in \Omega, \text{ all } x \geqslant 0. \tag{12}$$

Returning to (9) and using (10) and (12), we have

$$\int_\Omega (g(z) - c_3) \left(\frac{\hat{u}_1(p)}{u}\right)^{p-1} \hat{u}_1(p) dz \leqslant \vartheta_0 - \xi ||\hat{u}_1(p)||_p^p < 0$$

(recall that $\xi > \vartheta_0$ and $||\hat{u}_1(p)||_p = 1$). Since $\frac{\hat{u}_1(p)}{u} \in L^\infty(\Omega)_+$ (see (8)), if $g(z) > c_3$ for almost all $z \in \Omega$, we have a contradiction. This suggests that in order to guarantee a positive solution of $(1)_g$ we need to restrict $||g||_\infty$.

Let $g \in L^\infty(\Omega)$ and let $e_g : \Omega \times \mathbb{R} \to \mathbb{R}$ be the Carathéodory function defined by

$$e_g(z,x) = \begin{cases} g(z) & \text{if} \quad x \leqslant 0 \\ \beta(z)x^{p-1} + f(z,x) + g(z) & \text{if} \quad x > 0. \end{cases} \tag{13}$$

We set $E_g(z,x) = \int_0^x e_g(z,s)ds$ and consider the C^1-functional $\varphi_g : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_g(u) = \frac{1}{p}||Du||_p^p - \int_\Omega E_g(z,u)dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

From Papageorgiou and Smyrlis [13], we have:

Proposition 3. *If hypotheses H_1 hold then for every $g \in L^\infty(\Omega)$ the functional φ_g satisfies the C-condition.*

The next result is an immediate consequence of hypothesis, $H_1(ii)$ and (13).

Proposition 4. *If hypotheses H_1 hold, $u \in intC_+$ and $g \in L^\infty(\Omega)$, then $\varphi_g(tu) \to -\infty$ as $t \to +\infty$.*

The next proposition shows that the mountain pass geometry (see Theorem 1) is satisfied by the functional φ_g for $||g||_\infty$ small.

Proposition 5. *If hypotheses H_1 hold, then there exist $\delta_0 > 0$ and $\rho_0 = \rho_0(\delta_0) > 0$ such that*

$$||g||_\infty < \delta_0 \Rightarrow \varphi_g(u) \geqslant m_0 > 0 \quad \text{for all } u \in W_0^{1,p}(\Omega) \textit{ with } ||u|| = \rho_0.$$

Proof. Hypotheses $H_1(i)$ and (iv) imply that given $\epsilon > 0$, we can find $c_5 = c_5(\epsilon) > 0$ such that

$$F(z,x) \leqslant \frac{\epsilon}{p}x^p + c_5 x^r \text{ for a.a. } z \in \Omega, \text{ all } x \geqslant 0. \tag{14}$$

Then for every $u \in W_0^{1,p}(\Omega)$, we have

$$\begin{aligned}
\varphi_g(u) &= \frac{1}{p}||Du||_p^p - \int_\Omega E_g(z,u)dz \\
&\geqslant \frac{1}{p}||Du||_p^p - \frac{1}{p}\int_\Omega \beta(z)|u|^p dz - \int_\Omega F(z,u)dz - c_6||g||_\infty ||u|| \\
&\qquad \text{for some } c_6 > 0 \text{ (see (13))} \\
&\geqslant \frac{1}{p}\left[\xi_0 - \frac{\epsilon}{\hat{\lambda}_1(p)}\right]||u||^p - c_7||u||^r - c_6||g||_\infty||u|| \text{ for some } c_7 > 0 \\
&\qquad \text{(see Lemma 2 and (4))}
\end{aligned}$$

Choosing $\epsilon \in (0, \hat{\lambda}_1(p)\xi_0)$, we obtain

$$\begin{aligned}
\varphi_g(u) &\geqslant c_8||u||^p - c_7||u||^r - c_6||g||_\infty||u|| \text{ with } c_8 = \frac{\hat{\lambda}_1(p)\xi_0 - \epsilon}{p\hat{\lambda}_1(p)} > 0 \\
&= \left[c_8 - \left(c_7||u||^{r-p} + c_6||g||_\infty||u||^{1-p}\right)\right]||u||^p.
\end{aligned} \tag{15}$$

Let $\gamma(t) = c_7 t^{r-p} + c_6||g||_\infty t^{1-p}$ for all $t \geqslant 0$. Evidently $\gamma \in C^1(0,\infty)$ and since $1 < p < r$, we have

$$\gamma(t) \to +\infty \text{ as } t \to 0^+ \text{ and } t \to +\infty.$$

So, we can find $t_0 \in (0, +\infty)$ such that

$$\begin{aligned}
\gamma(t_0) &= \inf_{t \geqslant 0} \gamma \\
&\Rightarrow \gamma'(t_0) = 0 \\
&\Rightarrow (r-p)c_7 t_0^{r-p-1} = (p-1)c_6||g||_\infty t_0^{-p} \\
&\Rightarrow t_0 = \left[\frac{(p-1)c_6||g||_\infty}{(r-p)c_7}\right]^{\frac{1}{r-1}}.
\end{aligned}$$

Then $\gamma(t_0) \to 0^+$ as $||g||_\infty \to 0^+$. So, we can find $\delta_0 > 0$ such that

$$\begin{aligned}
||g||_\infty &< \delta_0 \Rightarrow \gamma(t_0) < c_8 \\
&\Rightarrow \varphi_g(u) \geqslant m_0 > 0 = \varphi_g(0) \text{ for all } ||u|| = t_0 = \rho_0.
\end{aligned}$$

This completes the proof. □

These propositions lead to the following existence theorem for problem $(1)_g$ when $||g||_\infty$ is small.

Theorem 6. *If hypotheses H_1 hold, then there exists $\delta_1 \in (0, \delta_0]$ such that if $||g||_\infty < \delta_1$, then problem $(1)_g$ has at least one positive solution $u_0 \in intC_+$.*

Proof. Propositions 3, 4 and 5 imply that when $||g||_\infty < \delta_0$, then the functional φ_g satisfies the mountain pass geometry and the C-condition. So, we can apply Theorem 1 (the mountain pass theorem) and find $u_0 \in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned}\varphi_g'(u_0) &= 0 \text{ and } \varphi_g(0) = 0 < m_0 \leqslant \varphi_g(u_0)\\ &\Rightarrow u_0 \neq 0.\end{aligned}$$

In particular, let $g \equiv 0$ and let $\bar{u}_0$ be the critical point of φ_0 obtained above. We have

$$A(\bar{u}_0) = N_{e_0}(\bar{u}_0). \tag{16}$$

On (16) we act with $-\bar{u}_0^- \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned}||D\bar{u}_0||_p^p &= 0 \text{ (see (13) with } g \equiv 0)\\ &\Rightarrow \bar{u}_0 \geqslant 0,\ \bar{u}_0 \neq 0.\end{aligned}$$

So, $\bar{u}_0$ is a positive solution of problem $(1)_0$ (with $g \equiv 0$). Nonlinear regularity theory, implies that $\bar{u}_0 \in C_+ \backslash \{0\}$. Let $\rho = ||\bar{u}_0||_\infty$ and let $\xi_\rho > 0$ be as postulated by hypothesis $H_1(v)$. We have

$$\begin{aligned}&-\Delta_p \bar{u}_0(z) + \xi_\rho \bar{u}_0(z)^{p-1}\\ &\quad = \beta(z)u_0(z)^{p-1} + f(z, u_0(z)) + \xi_\rho \bar{u}_0(z)^{p-1} \geqslant 0 \text{ a.e. in } \Omega\\ &\quad \Rightarrow \Delta_p \bar{u}_0(z) \leqslant \xi_\rho \bar{u}_0(z)^{p-1} \text{ a.e. in } \Omega\\ &\quad \Rightarrow \bar{u}_0 \in \operatorname{int} C_+ \text{ (by the nonlinear maximum principle, see [8, p. 738]).}\end{aligned}$$

So, every positive solution of $(1)_0$ (with $g \equiv 0$), belongs to $\operatorname{int} C_+$.

Now, let $\{g_n\}_{n\geqslant 1} \subseteq L^\infty(\Omega)$ with $||g_n||_\infty < \delta_0$ for all $n \geqslant 1$ and assume that $g_n \to 0$ in $L^\infty(\Omega)$. Let $\{u_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(\Omega)$ be the corresponding critical points of φ_{g_n} obtained in the beginning of the proof via the mountain pass theorem (see Theorem 1). We have

$$-\Delta_p u_n(z) = e_{g_n}(z, u_n(z)) \text{ a.e. in } \Omega,\ u_n|_{\partial\Omega} = 0,\ n \geqslant 1.$$

From Gasinski and Papageorgiou [8, p. 737], we can find $c_9 > 0$ such that

$$||u_n||_\infty \leqslant c_9 \text{ for all } n \geqslant 1.$$

So, there exist $\alpha \in (0, 1)$ and $c_{10} > 0$ such that

$$u_n \in C_0^{1,\alpha}(\overline{\Omega}) \text{ and } ||u_n||_{C_0^{1,\alpha}(\overline{\Omega})} \leqslant c_{10} \text{ for all } n \geqslant 1$$

(see Gasinski and Papageorgiou [8, p. 738]). Exploiting the compact embedding of $C_0^{1,\alpha}(\overline{\Omega})$ into $C^1(\overline{\Omega})$, we may assume that

$$u_n \to \tilde{u} \text{ in } C_0^1(\overline{\Omega}),\ \text{ with } \tilde{u} \text{ solution of } (1)_0. \tag{17}$$

Recall that for all $n \geqslant 1$ we have

$$\varphi_{g_n}(u_n) \geqslant m_0 > 0 = \varphi_{g_n}(0)$$

(note that, by Proposition 5, since $||g_n||_\infty < \delta_0$ for all $n \in \mathbb{N}$, m_0 does not depend on n)

$$\Rightarrow \varphi_0(\tilde{u}) \geqslant m_0 > 0 = \varphi_0(0) \text{ (see (17) and (13))}$$

$$\Rightarrow \tilde{u} \neq 0, \text{ hence } \tilde{u} \in \operatorname{int} C_+ \text{ as established earlier.}$$

From (17) it follows that

$$u_n \in \operatorname{int} C_+ \text{ for all } n \geqslant n_0.$$

Therefore, we can find $\delta_1 \in (0, \delta_0]$ such that for $||g||_\infty < \delta_1$ problem $(1)_g$ has at least one positive solution $u_0 \in \operatorname{int} C_+$. $\square$

We can improve the conclusion of the above theorem and produce a second positive solution, provided g is nonzero and nonnegative and as before has small $L^\infty(\Omega)$-norm.

Theorem 7. *If hypotheses H_1 hold then there exists $\delta_1 \in (0, \delta_0]$ such that if $0 < ||g||_\infty < \delta_1$ and $g \geqslant 0$, then problem $(1)_g$ has at least two positive solutions*

$$u_0, \hat{u} \in \operatorname{int} C_+,\ u_0 \leqslant \hat{u},\ u_0 \neq \hat{u}.$$

Proof. From Theorem 6 we know that there exists $\delta_1 \in (0, \delta_0]$ such that if $||g||_\infty < \delta_1$, then problem $(1)_g$ has at least one positive solution $u \in \operatorname{int} C_+$.

Now we assume that $0 < ||g||_\infty < \delta_1$ and $g \geqslant 0$. Let $\eta \in (0, \delta_1 - ||g||_\infty)$ and let $g^* = g + \eta$. Evidently $||g^*||_\infty < \delta_1$ and so problem $(1)_{g^*}$ has a positive solution $u^* \in \operatorname{int} C_+$.

Claim 1. *We can find a positive solution $u_0 \in \operatorname{int} C_+$ of $(1)_g$ such that $u_0 \leqslant u^*$.*

We have

$$A(u^*) = \beta(z)(u^*)^{p-1} + N_f(u^*) + g^* \geqslant \beta(z)(u^*)^{p-1} + N_f(u^*) + g \quad (18)$$
$$\text{in } W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^* \left(\frac{1}{p} + \frac{1}{p'} = 1\right).$$

We consider the following Carathéodory function

$$\gamma_g(z,x) = \begin{cases} g(z) & \text{if} \quad x < 0 \\ \beta(z)x^{p-1} + f(z,x) + g(z) & \text{if} \quad 0 \leqslant x \leqslant u^*(z) \\ \beta(z)u^*(z)^{p-1} + f(z,u^*(z)) + g(z) & \text{if} \quad u^*(z) < x. \end{cases} \quad (19)$$

We set $\Gamma_g(z,x) = \int_0^x \gamma_g(z,s)ds$ and consider the C^1-functional $\tau_g : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\tau_g(u) = \frac{1}{p}||Du||_p^p - \int_\Omega \Gamma_g(z,u)dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

It is clear from (19) that τ_g is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $u_0 \in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned} \tau_g(u_0) &= \inf[\tau_g(u) : u \in W_0^{1,p}(\Omega)] \\ &\Rightarrow \tau_g'(u_0) = 0 \\ &\Rightarrow A(u_0) = N_{\gamma_g}(u_0). \end{aligned} \quad (20)$$

In (20) first we act with $-u_0^- \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned} ||Du_0^-||_p^p &= \int_\Omega g(z)(-u_0^-)dz \leqslant 0 \text{ (see (19) and recall } g \geqslant 0) \\ &\Rightarrow u_0 \geqslant 0. \end{aligned}$$

Also, on (20) we act with $(u_0-u^*)^+\in W_0^{1,p}(\Omega)$. Then

$$
\begin{aligned}
&\left\langle A(u_0),(u_0-u^*)^+\right\rangle=\int_\Omega \gamma_g(z,u_0)(u_0-u^*)^+dz\\
&\quad=\int_\Omega\left[\beta(z)(u^*)^{p-1}+f(z,u^*)+g\right](u_0-u^*)^+dz \text{ (see (19))}\\
&\quad\leqslant\left\langle A(u^*),(u_0-u^*)^+\right\rangle \text{ (see (18))},\\
&\quad\Rightarrow\int_{\{u_0>u^*\}}\left(|Du_0|^{p-2}Du_0-|Du^*|^{p-2}Du^*,Du_0-Du^*\right)_{\mathbb{R}^N}dz\leqslant 0\\
&\quad\Rightarrow|\{u_0>u^*\}|_N=0,\ \text{hence } u_0\leqslant u^*.
\end{aligned}
$$

So, we have proved that

$$u_0\in[0,u^*]=\{u\in W_0^{1,p}(\Omega):0\leqslant u(z)\leqslant u^*(z)\text{ a.e. in }\Omega\}.$$

Then from (19) and (20) it follows that u_0 is a solution of $(1)_g$ and since $g\neq 0,\ u_0\neq 0$. The nonlinear regularity theory and the nonlinear maximum principle imply that $u_0\in\operatorname{int}C_+$.

Using $u_0\in\operatorname{int}C_+$ we introduce the following truncation of the reaction of problem $(1)_g$:

$$k_g(z,x)=\begin{cases}\beta(z)u_0(z)^{p-1}+f(z,u_0(z))+g(z) & \text{if}\quad x<u_0(z)\\ \beta(z)x^{p-1}+f(z,x)+g(z) & \text{if}\quad u_0(z)\leqslant x.\end{cases}\tag{21}$$

We set $K_g(z,x)=\int_0^x k_g(z,s)ds$ and consider the C^1-functional $\psi_g:W_0^{1,p}(\Omega)\to\mathbb{R}$ defined by

$$\psi_g(u)=\frac{1}{p}||Du||_p^p-\int_\Omega K_g(z,u)dz \text{ for all } u\in W_0^{1,p}(\Omega).$$

If $[u_0)=\{u\in W_0^{1,p}(\Omega):u_0(z)\leqslant u(z)\text{ for almost all }z\in\Omega\}$, then from (13)) we see that

$$\psi_g|_{[u_0)}=\varphi|_{[u_0)}+\xi^*\text{ for some }\xi^*\in\mathbb{R}.\tag{22}$$

From (22) and Proposition 3 it follows that

$$\psi_g\text{ satisfies the }C\text{-condition.}\tag{23}$$

Moreover, Proposition 4 implies that for any $u\in\operatorname{int}C_+$, we have

$$\psi_g(tu)\to-\infty\text{ as }t\to+\infty.\tag{24}$$

Claim 2. *We have $K_{\psi_g}\subseteq[u_0)=\{u\in W_0^{1,p}(\Omega):u_0(z)\leqslant u(z)$ a.e. in $\Omega\}$*

Indeed, let $u\in K_{\psi_g}$. Then

$$A(u)=N_{k_g}(u).\tag{25}$$

On (25) we act with $(u_0-u)^+ \in W_0^{1,p}(\Omega)$. We have

$$\begin{aligned}\left\langle A(u),(u_0-u)^+\right\rangle &= \int_\Omega k_g(z,u)(u_0-u)^+dz\\ &= \int_\Omega \left[\beta(z)u_0^{p-1}+f(z,u_0)+g(z)\right](u_0-u)^+dz\\ &= \left\langle A(u_0),(u_0-u)^+\right\rangle \text{ (since } u_0 \text{ is a solution of } (1)_g)\\ &\Rightarrow \int_{\{u_0>u\}}\left(|Du_0|^{p-2}Du_0-|Du|^{p-2}Du, Du_0-Du\right)_{\mathbb{R}^N}dz=0\\ &\Rightarrow |\{u_0>u\}|_N=0, \text{ hence } u_0\leqslant u.\end{aligned}$$

This proves Claim 2.

By virtue of Claim 2 every element of K_{ψ_g} is a positive solution of $(1)_g$. Arguing by contradiction, suppose $K_{\psi_g}=\{u_0\}$ (see (21)).

Claim 3. *$u_0 \in intC_+$ is a local minimizer of the functional ψ_g.*

Recall that $0\leqslant u_0\leqslant u^*$ and consider the following truncation of $k_g(z,\cdot)$:

$$\hat{k}_g(z,x)=\begin{cases} k_g(z,x) & \text{if } x<u^*(z)\\ k_g(z,u^*(z)) & \text{if } u^*(z)\leqslant x.\end{cases} \tag{26}$$

This is a Carathéodory function. We set $\hat{K}_g(z,x)=\int_0^x \hat{k}_g(z,s)ds$ and consider the C^1-functional $\hat{\psi}_g: W_0^{1,p}(\Omega)\to\mathbb{R}$ defined by

$$\hat{\psi}_g(u)=\frac{1}{p}||Du||_p^p-\int_\Omega \hat{K}_g(z,u)dz \text{ for all } u\in W_0^{1,p}(\Omega).$$

Note that $\hat{\psi}_g$ is coercive (see (26)) and sequentially weakly lower semicontinuous. So, we can find $\tilde{u}\in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned}\hat{\psi}_g(\tilde{u}) &= \inf\left[\hat{\psi}_g(u): u\in W_0^{1,p}(\Omega)\right]\\ &\Rightarrow \hat{\psi}_g'(\tilde{u})=0\\ &\Rightarrow A(\tilde{u})=N_{\hat{k}_g}(\tilde{u}).\end{aligned} \tag{27}$$

On (27), first we can act with $(u_0-\tilde{u})^+\in W_0^{1,p}(\Omega)$ and as before using (21) and (26), we obtain $u_0\leqslant\tilde{u}$. Then on (27) we act with $(\tilde{u}-u^*)^+\in W_0^{1,p}(\Omega)$ and using (18), (21), (26), we show that $\tilde{u}\leqslant u^*$. Therefore

$$\begin{aligned}\tilde{u}\in[u_0,u^*] &= \{u\in W_0^{1,p}(\Omega): u_0(z)\leqslant u(z)\leqslant u^*(z) \text{ a.e. in } \Omega\}\\ &\Rightarrow \tilde{u}=u_0 \text{ (see (21), (26) and recall } K_{\psi_g}=\{u_0\}).\end{aligned}$$

Let $\rho=||u_0||_\infty$ and let $\xi_\rho>0$ be as postulated by hypothesis $H_1(v)$. Then

$$\begin{aligned}&-\Delta_p u_0(z)+\xi_\rho u_0(z)^{p-1}\\ &\quad=\beta(z)u_0(z)^{p-1}+f(z,u_0(z))+g(z)+\xi_\rho u_0(z)^{p-1}\\ &\quad\leqslant\beta(z)u^*(z)^{p-1}+f(z,u^*(z))+g^*(z)+\xi_\rho u^*(z)^{p-1}\\ &\quad\text{(see } H_1(v) \text{ and recall that } u_0\leqslant u^*,\ g\leqslant g^*)\end{aligned}$$

$$= -\Delta_p u^*(z) + \xi_\rho u^*(z)^{p-1} \text{ a.e. in } \Omega$$
$$\Rightarrow u^* - u_0 \in \operatorname{int} C_+ \text{ (see Arcoya and Ruiz [5, Proposition 2.6]).}$$

Also, recall that $u_0 \in \operatorname{int} C_+$. Since $\psi_g|_{[0,u^*]} = \hat{\psi}_g|_{[0,u^*]}$ (see (21) and (26)) it follows that $u_0 \in \operatorname{int} C_+$ is a local $C_0^1(\overline{\Omega})$-minimizer of ψ_g. Then from Garcia Azorero, Manfredi and Peral Alonso [7, Theorem 1.1], it follows that $u_0 \in \operatorname{int} C_+$ is a local $W_0^{1,p}(\Omega)$-minimizer of ψ_g. This proves Claim 3.

By virtue of Claim 3, we can find $\rho \in (0,1)$ small such that

$$\psi_g(u_0) < \inf\left[\psi_g(u) : ||u - u_0|| = \rho\right] = \eta_\rho \tag{28}$$

(see Aizicovici, Papageorgiou and Staicu [1] (proof of Proposition 29)). From (23), (24) and (28), we see that we can apply Theorem 1 (the mountain pass theorem). So, there exists $\hat{u} \in W_0^{1,p}(\Omega)$ such that

$$\hat{u} \in K_{\psi_g} \text{ and } \eta_\rho \leqslant \psi_g(\hat{u}). \tag{29}$$

From Claim 2, (28) and (29) it follows that

$$u_0 \leqslant \hat{u},\ \hat{u} \neq u_0 \text{ and } \hat{u} \in \operatorname{int} C_+ \text{ solves problem } ((1)_g) \text{ (see (21)).}$$

This completes the proof. □

Remark 2. The results of this section can be extended to problems driven by a nonhomogeneous differential operator $\operatorname{div} a(Du)$ with $a : \mathbb{R}^N \to \mathbb{R}^N$ as in Papageorgiou and Rădulescu [12] (see also Papageorgiou and Rădulescu [11]). For the sake of simplicity in the presentation, we have chosen to work with the p-Laplacian.

4. Solutions for problem $(2)_\lambda$

In this section we deal with problem $(2)_\lambda$.

The hypotheses on the data of problem $(2)_\lambda$ are the following:

$H(\beta)$: $\beta \in L^\tau(\Omega)$ with $\tau > \frac{N}{2}$.

H_2 : $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that for a.a. $z \in \Omega$ $f(z,0) = 0$, $f(z,\cdot) \in C^1(\mathbb{R})$ and

(i) $|f'_x(z,x)| \leqslant a(z)(1 + |x|^{r-2})$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)_+$, $2 < r < 2^*$;

(ii) if $F(z,x) = \int_0^x f(z,s)ds$, then

$$\lim_{x \to \pm\infty} \frac{F(z,x)}{x^2} = +\infty \text{ uniformly for a.a. } z \in \Omega;$$

(iii) there exist $\eta_0 > 0$ and $\vartheta \in \left(\max\left\{1, (r-2)\frac{N}{2}\right\}, 2^*\right)$ such that

$$0 < \eta_0 \leqslant \liminf_{x \to \pm\infty} \frac{f(z,x)x - 2F(z,x)}{|x|^\vartheta} \text{ uniformly for a.a. } z \in \Omega;$$

(iv) $0 = f'_x(z,0) = \lim\limits_{x \to 0} \frac{f(z,x)}{x}$ uniformly for a.a. $z \in \Omega$;

(v) there exists $\delta > 0$ such that $f(z,x)x \geqslant 0$ for a.a. $z \in \Omega$, all $|x| \leqslant \delta$.

Theorem 8. *If hypotheses H_2 hold and $\lambda \geqslant \hat{\lambda}_1(2)$, then problem* $(2)_\lambda$ *admits at least one nontrivial solution* $u_0 \in C_0^1(\overline{\Omega})$.

Proof. Let $k \geqslant 1$ such that $\lambda \in \left[\hat{\lambda}_k(2), \hat{\lambda}_{k+1}(2)\right)$. We set

$$\bar{H}_k = \overset{k}{\underset{\mathrm{i}=1}{\oplus}} E(\hat{\lambda}_i(2)) \text{ and } \hat{H}_k = \bar{H}_k^{\perp} = \overline{\underset{i \geqslant k+1}{\oplus} E(\hat{\lambda}_i(2))}.$$

We have the following orthogonal direct sum decomposition

$$H_0^1(\Omega) = \bar{H}_k \oplus \hat{H}_k.$$

By virtue of hypotheses $H_2(iv), (v)$, given $\epsilon > 0$, we can find $\delta_1 \in (0, \delta]$ such that

$$0 \leqslant F(z,x) \leqslant \frac{\epsilon}{2} x^2 \text{ for a.a. } z \in \Omega, \text{ all } |x| \leqslant \delta_1. \tag{30}$$

Since $\bar{H}_k$ is finite dimensional, all norms are equivalent and so we can find $\rho_0 > 0$ such that

$$||u|| \leqslant \rho_0 \Rightarrow ||u||_\infty \leqslant \delta_1 \text{ for all } u \in \bar{H}_k. \tag{31}$$

Let $\varphi_\lambda : H_0^1(\Omega) \to \mathbb{R}$ be the energy functional for problem $(2)_\lambda$ defined by

$$\varphi_\lambda(u) = \frac{1}{2}\tau(u) - \frac{\lambda}{2}||u||_2^2 - \int_\Omega F(z,u)dz \text{ for all } u \in H_0^1(\Omega)$$

with $\tau(u) = ||Du||_2^2 + \int_\Omega \beta(z)u^2 dz$ for all $u \in H_0^1(\Omega)$. Evidently $\varphi_\lambda \in C^2(H_0^1(\Omega))$.

For $u \in \bar{H}_k$ with $||u|| \leqslant \rho_0$, we have

$$\begin{aligned} \varphi_\lambda(u) &\leqslant \frac{1}{2}\tau(u) - \frac{\lambda}{2}||u||_2^2 \text{ (see (31))} \\ &\leqslant 0 \text{ (see (7) and recall that } \lambda \leqslant \hat{\lambda}_k(2)). \end{aligned}$$

From (30) and hypothesis $H_2(i)$, we have

$$F(z,x) \leqslant \frac{\epsilon}{2}x^2 + c_{11}|x|^r \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ some } c_{11} = c_{11}(\epsilon) > 0. \tag{32}$$

For $u \in \hat{H}_k$, we have

$$\varphi_\lambda(u) \geqslant \frac{1}{2}\tau(u) - \frac{\lambda + \epsilon}{2}||u||_2^2 - c_{11}||u||_r^r \quad \text{(see (32))}.$$

Choose $\epsilon > 0$ small such that $\lambda + \epsilon < \hat{\lambda}_{k+1}(2)$ (recall $\lambda \in \left[\hat{\lambda}_k(2), \hat{\lambda}_{k+1}(2)\right)$). Then we have

$$\varphi_\lambda(u) \geqslant c_{12}||u||^2 - c_{13}||u||^r \text{ for some } c_{12}, c_{13} > 0 \text{ (see (7))}. \tag{33}$$

Since $r > 2$, from (33) it follows that we can find $\rho \in (0, \rho_0]$ small such that

$$\varphi_\lambda(u) \geqslant 0 \text{ for all } u \in \hat{H}_k \text{ with } ||u|| \leqslant \rho.$$

So, we have proved that φ_λ has a local linking at the origin with respect to the orthogonal direct sum decomposition $H_0^1(\Omega) = \overline{H}_k \oplus \hat{H}_k$. Since $\varphi_\lambda \in C^2(H_0^1(\Omega))$, from Su [16, Proposition 2.3], we have

$$C_k(\varphi_\lambda, 0) = \delta_{k,d_k}\mathbb{Z} \text{ with } d_k = \dim \overline{H}_k. \tag{34}$$

On the other hand, from Aizicovici, Papageorgiou and Staicu [2], we have

$$C_k(\varphi_\lambda, \infty) = 0 \text{ for all } k \geqslant 0. \tag{35}$$

From (34) and (35) it follows that we can find $u_0 \in K_{\varphi_\lambda}\backslash\{0\}$. Then u_0 solves problem $(2)_\lambda$ and from the regularity theory (see Struwe [15, p. 218]), we have that $u_0 \in H_0^1(\overline{\Omega})$. □

If we strengthen hypothesis $H_2(v)$, we can improve the conclusion of Theorem 8 and provide more information about the solution u_0.

The new hypotheses on the perturbation $f(z,x)$ are the following:

H_3 : $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that for a.a. $z \in \Omega$, $f(z,0) = 0$, $f(z,\cdot) \in C^1(\mathbb{R})$, hypotheses $H_3(i) \to (iv)$ are the same as the corresponding hypotheses $H_2(i) \to (iv)$ and
(v) $f(z,x)x \geqslant 0$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$ and the inequality is strict for all $(z,x) \in \Omega_0 \times \mathbb{R}$ with $|\Omega_0|_N > 0$ and $x \neq 0$.

Theorem 9. *If hypotheses H_3 hold and $\lambda \geqslant \hat{\lambda}_1(2)$, then problem $(2)_\lambda$ admits a nodal solution $u_0 \in C_0^1(\overline{\Omega})$.*

Proof. From Theorem 8 we know that problem $(2)_\lambda$ has a nontrivial solution $u_0 \in H_0^1(\overline{\Omega})$. Suppose that u_0 has constant sign and to fix things assume that $u_0 \geqslant 0$. We have

$$A(u) + \beta(z)u = \lambda u + N_f(u). \tag{36}$$

On (36) we act with $\hat{u}_1(2,\beta) \in \operatorname{int} C_+$. Then

$$\begin{aligned}\langle A(u) + \beta u, \hat{u}_1(2,\beta)\rangle &= \lambda \int_\Omega u\hat{u}_1(2,\beta)dz + \int_\Omega f(z,u)\hat{u}_1(2,\beta)dz \\ &\Rightarrow (\hat{\lambda}_1(2,\beta) - \lambda)\int_\Omega u\hat{u}_1(2,\beta)dz = \int_\Omega f(z,u)\hat{u}_1(2,\beta)dz.\end{aligned}$$

Note that $(\hat{\lambda}_1(2,\beta) - \lambda)\int_\Omega u\hat{u}_1(2,\beta)dz \leqslant 0$, while $\int_\Omega f(z,u)\hat{u}_1(2,\beta)dz > 0$ (see $H_3(v)$ and recall that we have assumed that $u \geqslant 0$). So, we have a contradiction and this proves that u_0 is nodal. □

Remark 3. Our results here answer the question posed in Rădulescu [14] and show that hypothesis (8) in [14] is not necessary. Finally we stress that our approach here differs from that of [8].

Acknowledgements

The authors wish to thank two knowledgeable referees for their comments and remarks that helped to improve the paper. V. Rădulescu has been partially supported by a grant of the Romanian National Authority for Scientific Research, CNCS-UEFISCDI, project number PN-II-ID-PCE-2011-3-0195.

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Received: 11 June 2015.
Accepted: 8 June 2016.