Nonlinear Partial Differential Equations of Elliptic Type

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Introduction

Nonlinear Analysis is one of the fields of Mathematics with the most spectacular development in the last decades of this end of century. The impressive number of results in this area is also a consequence of various problems raised by Physics, Optimization or Economy. In the modelling of natural phenomena a crucial role is played by the study of partial differential equations of elliptic type; they arise in every field of science. Consequently, the desire to understand the solutions of these equations has always a prominent place in the efforts of mathematicians; it has inspired such diverse fields as Functional Analysis, Variational Calculus or Algebraic Topology.

The present book is based on a one semester course at the University of Craiova. The goal of this textbook is to provide the background which is necessary to initiate work on a Ph.D. thesis in Applied Nonlinear Analysis. My purpose is to provide for the student a broad perspective in the subject, to illustrate the rich variety of phenomena encompassed by it and to impart a working knowledge of the most important techniques of analysis of the solutions of the equations. The level of this book is aimed at beginning graduate students. Prerequisites include a truly advanced Calculus course, basic knowledge on Functional Analysis and PDE, as well as the necessary tools on Sobolev spaces.

Throughout this work we have used intensively two classical results: the Mountain-Pass Lemma (in its C^1 statement!) of Ambrosetti and Rabinowitz (1973, [9]) and Ekeland's Variational Principle (1974, [35]). We recall in what follows these celebrated results.

Mountain Pass Lemma. *Let X be a real Banach space and let $F : X \to \mathbb{R}$ be a C^1-functional which satisfies the following assumptions:*

i) $F(0) < 0$ and there exists $e \in X \setminus \{0\}$ such that $F(e) < 0$.

ii) there is some $0 < R < \|e\|$ such that $F(u) \geq 0$, for all $u \in X$ with $\|u\| = R$.

Put

$$c = \inf_{p \in \mathcal{P}} \max_{t \in [0,1]} F(p(t)),$$

where $\mathcal{P}$ denotes the set of all continuous paths joining 0 and e.

Then the number c is an "almost critical value" of F, in the sense that there exists a sequence (x_n) in X such that

$$\lim_{n\to\infty} F(x_n) = c \quad \text{and} \quad \lim_{n\to\infty} \|F'(x_n)\|_{X^*} = 0\,.$$

Ekeland's Variational Principle. *Let (M, d) be a complete metric space and let $\psi : M \to (-\infty, +\infty]$, $\psi \not\equiv +\infty$ be a lower semicontinuous functional bounded from below.*

Then the following hold:
i) Let $\varepsilon > 0$ be arbitrary and let $z_0 \in M$ be such that

$$\psi(z_0) \leq \inf_{x \in M} \psi(x) + \varepsilon\,.$$

Then, for every $\lambda > 0$, there exists $z_\lambda \in M$ such that , for all $x \in M \setminus \{z_\lambda\}$,

$$\psi(z_\lambda) \leq \psi(z_0)\,,$$

$$\psi(x) > \psi(z_\lambda) - \frac{\varepsilon}{\lambda}\, d(x, z_\lambda)\,,$$

$$d(z_\lambda, z_0) \leq \lambda\,.$$

ii) For each $\varepsilon > 0$ and $z_0 \in M$, there is some $z \in M$ such that, for any $x \in M$,

$$\psi(x) \geq \psi(z) - \varepsilon d(x, z)\,,$$

$$\psi(z) \leq \psi(z_0) - \varepsilon d(z_0, z)\,.$$

In the first part of this book we present the method of sub and super solutions, which is one of the main tools in Nonlinear Analysis for finding solutions to a boundary value problem. The proofs are simple and we give several examples to illustrate the theory. We continue with another elementary method for finding solutions, namely the Implicit Function Theorem. The main application is a celebrated theorem due to H. Amann which is related to a bifurcation problem associated to a convex and positive function. This kind of equations arises frequently in physics, biology, combustion-diffusion etc. We mention only Brusselator type reactions, the combustion theory, dynamics of population, the Fitzhugh-Nagumo system, morphogenese, superconductivity, super-fluids etc. We give complete details in the case where the functional is asymptotically linear at infinity.

In the third chapter we give some basic results related to the Clarke generalized gradient of a locally Lipschitz functional (see Clarke [26], [27]). Then we develop a nonsmooth critical point theory which enables us to deduce the Brezis-Coron-Nirenberg Theorem [19], the "Saddle Point" Theorem of Rabinowitz [68] or the Ghoussoub-Preiss Theorem [41]. The motivation of this study is the following: some of the strongest tools for proving existence results in PDE are the "Mountain Pass" Lemma of Ambrosetti-Rabinowitz and the Lusternik-Schnirelmann Theorem. These results apply when the solutions of the given problem are critical points of a suitable "energetic" functional f, which is assumed to be of class C^1 and defined on a real Banach space. A natural question is what happens if the energy functional, associated to a given problem in a natural way, fails to be differentiable. The results we give here are based on the notion of Clarke generalized gradient of a locally Lipschitz functional which is very useful for the treatment of many problems arising in the calculus of variations, optimal control, hemivariational inequalities etc. Clarke's generalized gradient coincides with the usual one if f is differentiable or convex. In the classical framework the Fréchet differential of a C^1-functional is a linear and continuous operator. For the case of locally Lipschitz maps the property of linearity of the gradient does not remain valid. Thus, for fixed $x \in X$, the directional derivative $f^0(x, \cdot)$ is subadditive and positive homogeneous and its generalized gradient $\partial f(x)$ is a nonempty closed subset of the dual space.

In Chapter 4 the main results are two Lusternik-Schnirelmann type theorems. The first one uses the notion of critical point for a pairing of operators (see, e.g., Fucik-Necas-Soucek-Soucek [38]). The second theorem is related to locally Lipschitz functionals which are periodic with respect to a discrete subgroup and also uses the notion of Clarke subdifferential.

In the following part of this work we give several applications of the abstract results which appear in the first chapters. Here we recall a classical result of Chang [24] and prove multivalued variants of the Brezis-Nirenberg problem, as well as a solution of the forced pendulum problem, which was studied in the smooth case in Mawhin-Willem [56]. We also study multivalued problems at resonance of Landesman-Lazer type. The methods we develop here enable us to study several classes of discontinuous problems and all these techniques are based on Clarke's generalized gradient theory. This tool is very useful in the study of critical periodic orbits of hamiltonian systems (Clarke-Ekeland), the mathematical programming (Hiriart-Urruty), the duality theory (Rockafellar), optimal control (V. Barbu and F. Clarke), nonsmooth analysis (A.D. Ioffe and F. Clarke), hemivariational inequalities (P.D. Panagiotopoulos) etc.

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Chapter 1

Method of sub and super solutions

Let Ω be a smooth bounded domain in $\mathbb{R}^N$ and consider a Carathéodory function $f(x,u) : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ such that f is of class C^1 with respect to the variable u. Consider the problem

$$\begin{cases} -\Delta u = f(x,u)\,, & \text{in } \Omega \\ u = 0\,, & \text{on } \partial\Omega\,. \end{cases} \tag{1.1}$$

By solution of the problem (1.1) we mean a function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ which satisfies (1.1).

Definition 1. *A function $\underline{U} \in C^2(\Omega) \cap C(\overline{\Omega})$ is said to be subsolution of the problem (1.1) provided that*

$$\begin{cases} -\Delta \underline{U} \le f(x,\underline{U})\,, & \text{in } \Omega \\ \underline{U} \le 0\,, & \text{on } \partial\Omega\,. \end{cases}$$

Accordingly, if the signs are reversed in the above inequalities, we obtain the definition of a supersolution $\overline{U}$ for the problem (1.1).

Theorem 1. *Let $\underline{U}$ (resp., $\overline{U}$) be a subsolution (resp., a supersolution) to the problem (1.1) such that $\underline{U} \le \overline{U}$ in Ω. The following hold:*

(i) there exists a solution u of (1.1) which, moreover, satisfies $\underline{U} \le u \le \overline{U}$;

(ii) there exist a minimal and a maximal solution $\underline{u}$ and $\overline{u}$ of the problem (1.1) with respect to the interval $[\underline{U}, \overline{U}]$.

Remark 1. *The existence of the solution in this theorem, as well as the maximality (resp., minimality) of solutions given by (ii) have to be understood with respect to the given pairing of ordered sub and supersolutions. It is very possible that (1.1) has solutions which are* **not** *in the interval $[\underline{U}, \overline{U}]$. It may also happen that (1.1) has* **no** *maximal or minimal solution. Give such an exemple!*

Remark 2. *The hypothesis $\underline{U} \le \overline{U}$ is* **not** *automatically fulfilled for arbitrary sub and supersolution of (1.1). Moreover, it may occur that $\underline{U} > \overline{U}$ on the* **whole** *of Ω. An elementary example is the following:*

consider the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda_1 u\,, & \text{in } \Omega \\ u = 0\,, & \text{on } \partial\Omega\,. \end{cases}$$

We know that all solutions of this problem are of the form $u = Ce_1$, where C is a real constant and e_1 is not vanishing in Ω, say $e_1(x) > 0$, for any $x \in \Omega$. Choose $\underline{U} = e_1$ and $\overline{U} = -e_1$. Then $\underline{U}$ (resp., $\overline{U}$) is subsolution (resp., supersolution) to the problem (1.1), but $\underline{U} > \overline{U}$.

Proof of Theorem 1. (i) Let $g(x,u) := f(x,u) + au$, where a is a real constant. We can choose $a \geq 0$ sufficiently large so that the map $\mathbb{R} \ni u \longmapsto g(x,u)$ is increasing on $[\underline{U}(x), \overline{U}(x)]$, for every $x \in \Omega$. For this aim, it is enough to have $a \geq 0$ and

$$a \geq \max\left\{-f_u(x,u); x \in \overline{\Omega} \text{ and } u \in [\underline{U}(x), \overline{U}(x)]\right\}\,.$$

For this choice of a we define the sequence of functions $u_n \in C^2(\Omega) \cap C(\overline{\Omega})$ as follows: $u_0 = \overline{U}$ and, for every $n \geq 1$, u_n is the unique solution of the linear problem

$$\begin{cases} -\Delta u_n + a u_n = g(x, u_{n-1})\,, & \text{in } \Omega \\ u_n = 0\,, & \text{on } \partial\Omega\,. \end{cases} \tag{1.2}$$

Claim: $\underline{U} \leq \cdots \leq u_{n+1} \leq u_n \leq \cdots \leq u_0 = \overline{U}$.
Proof of Claim. Our arguments use in an essential manner the Weak Maximum Principle. So, in order to prove that $u_1 \leq \overline{U}$ we have, by the definition of u_1,

$$\begin{cases} -\Delta(\overline{U} - u_1) + a(\overline{U} - u_1) \geq g(x, \overline{U}) - g(x, \overline{U}) = 0\,, & \text{in } \Omega \\ \overline{U} - u_1 \geq 0\,, & \text{on } \partial\Omega\,. \end{cases}$$

Since the operator $-\Delta + aI$ is coercive, it follows that $\overline{U} \geq u_1$ in Ω. For the proof of $\underline{U} \leq u_1$ we observe that $\underline{U} \leq 0 = u_1$ on $\partial\Omega$ and, for every $x \in \Omega$,

$$-\Delta(\underline{U} - u_1) + a(\underline{U} - u_1) \leq f(x, \underline{U}) + a\underline{U} - g(x, \overline{U}) \leq 0\,,$$

by the monotonicity of g. The Maximum Principle implies $\underline{U} \leq u_1$.

Let us now assume that

$$\underline{U} \leq \cdots \leq u_n \leq u_{n-1} \leq \cdots \leq u_0 = \overline{U}\,.$$

It remains to prove that

$$\underline{U} \leq u_{n+1} \leq u_n\,.$$

Taking into account the equations satisfied by u_n and u_{n+1} we obtain

$$\begin{cases} -\Delta(u_n - u_{n+1}) + a(u_n - u_{n+1}) = g(x, u_{n-1}) - g(x, u_n) \geq 0\,, & \text{in } \Omega \\ u_n - u_{n+1} \geq 0\,, & \text{on } \partial\Omega\,, \end{cases}$$

which implies $u_n \geq u_{n+1}$ in Ω.

On the other hand, by

$$\begin{cases} -\Delta \underline{U} + a\underline{U} \leq g(x,\underline{U}) \,, & \text{in } \Omega \\ \underline{U} \leq 0 \,, & \text{on } \partial\Omega \end{cases}$$

and the definition of u_{n+1} we have

$$\begin{cases} -\Delta(u_{n+1} - \underline{U}) + a(u_{n+1} - \underline{U}) \geq g(x,u_n) - g(x,\underline{U}) \geq 0 \,, & \text{in } \Omega \\ u_{n+1} - \underline{U} \geq 0 \,, & \text{on } \partial\Omega \,. \end{cases}$$

Again, by the Maximum Principle, we deduce that $\underline{U} \leq u_{n+1}$ in Ω, which completes the proof of the Claim.

It follows that there exists a function u such that, for every fixed $x \in \Omega$,

$$u_n(x) \searrow u(x) \qquad \text{as } n \to \infty \,.$$

Our aim is to show that we can pass to the limit in (1.2). For this aim we use a standard bootstrap argument. Let $g_n(x) := g(x, u_n(x))$. We first observe that the sequence (g_n) is bounded in $L^\infty(\Omega)$, so in every $L^p(\Omega)$ with $1 < p < \infty$. It follows by (1.2) and standard Schauder estimates that the sequence (u_n) is bounded in $W^{2,p}(\Omega)$, for any $1 < p < \infty$. But the space $W^{2,p}(\Omega)$ is continuously embedded in $C^{1,\alpha}(\overline{\Omega})$, for $\alpha = 1 - \frac{N}{2p}$, provided that $p > \frac{N}{2}$. This implies that (u_n) is bounded in $C^{1,\alpha}(\overline{\Omega})$. Now, by standard estimates in Hölder spaces we deduce that (u_n) is bounded in $C^{2,\alpha}(\overline{\Omega})$. Since $C^{2,\alpha}(\overline{\Omega})$ is compactly embedded in $C^2(\overline{\Omega})$, it follows that, passing eventually at a subsequence,

$$u_n \to u \qquad \text{in } C^2(\overline{\Omega}) \,.$$

Since the sequence is monotone we obtain that the whole sequence converges to u in C^2. Now, passing at the limit in (1.2) as $n \to \infty$ we deduce that u is solution of the problem (1.1).

(ii) Let us denote by $\overline{u}$ the solution obtained with the above technique and choosing $u_0 = \overline{U}$. We justify that $\overline{u}$ is a maximal solution with respect to the given pairing $(\underline{U}, \overline{U})$. Indeed, let $u \in [\underline{U}, \overline{U}]$ be an arbitrary solution. With an argument similar to that given in the proof of (i) but with respect to the pairing of ordered sub-supersolutions $(u, \overline{U})$ we obtain that $u \leq u_n$, for any $n \geq 0$, which implies $u \leq \overline{u}$. $\square$

Let us now assume that f is a continuous function. We give in what follows an elementary variational proof for the existence of a solution to the problem (1.1), provided that sub and supersolution $\underline{U}$ and $\overline{U}$ exist, with $\underline{U} \leq \overline{U}$. For this aim, let

$$E(u) = \frac{1}{2}\int_\Omega \mid \nabla u \mid^2 - \int_\Omega F(x,u)$$

be the energy functional associated to the problem (1.1). Here, $F(x,u) = \int_0^u f(x,t)dt$.

Set

$$f_0(x,t) = \begin{cases} f(x,t)\,, & \text{if } \underline{U}(x) < t < \overline{U}(x) \\ f(x,\overline{U}(x))\,, & \text{if } t \geq \overline{U}(x) \\ f(x,\underline{U}(x))\,, & \text{if } t \leq \underline{U}(x)\,. \end{cases}$$

The associated energy functional is

$$E_0(u) = \frac{1}{2}\int_\Omega \mid \nabla u \mid^2 - \int_\Omega F_0(x,u)\,,$$

with an appropriate definition for F_0.

We observe the following:

- E_0 is well defined on $H^1(\Omega)$, since f_0 is uniformly bounded, so F_0 has a sublinear growth;
- E_0 is weak lower semicontinuous;
- the first term of E_0 is the dominating one at $+\infty$ and, moreover,

$$\lim_{\|u\|\to\infty} E_0(u) = +\infty\,.$$

Let

$$\alpha = \inf_{u\in H_0^1(\Omega)} E_0(u)\,.$$

We show in what follows that α is attained. Indeed, since E_0 is coercive, there exists a minimizing sequence $(u_n) \subset H_0^1(\Omega)$. We may assume without loss of generality that

$$u_n \rightharpoonup u\,, \qquad \text{weakly in } H_0^1(\Omega)\,.$$

So, by the lower semicontinuity of E_0 with respect to the weak topology,

$$\frac{1}{2}\int_\Omega \mid \nabla u_n \mid^2 - \int_\Omega F_0(x,u_n) \leq \alpha + o(1)\,.$$

This implies $E_0(u) = \alpha$. Now, since u is a critical point of E_0, it follows that it satisfies

$$-\Delta u = f_0(x,u)\,, \qquad \text{in } \mathcal{D}'(\Omega)\,.$$

The same bootstrap argument as in the above proof shows that u is smooth.

We prove in what follows that $\underline{U} \leq u \leq \overline{U}$. Indeed, we have

$$-\Delta\underline{U} \leq f(x,\underline{U})\,, \qquad \text{in } \Omega\,.$$

Therefore

$$-\Delta(\underline{U} - u) \leq f(x,\underline{U}) - f_0(x,u)\,.$$

After multiplication by $(\underline{U} - u)^+$ in this inequality and integration over Ω we find

$$\int_\Omega \mid \nabla(\underline{U} - u)^+ \mid^2 \leq \int_\Omega (\underline{U} - u)^+ \left(f(x,\underline{U}) - f_0(x,u)\right)\,.$$

Taking into account the definition of f_0 we obtain

$$\int_\Omega |\nabla(\underline{U}-u)^+|^2 = 0$$

which implies $\nabla(\underline{U}-u)^+ = 0$ in Ω. Therefore, $\underline{U} \leq u$ in Ω. □

We can interpret a solution u of the problem (1.1) as an equilibrium solution of the associated parabolic problem

$$\begin{cases} v_t - \Delta v = f(x,v)\,, & \text{in } \Omega \times (0,\infty) \\ v(x,t) = 0\,, & \text{on } \partial\Omega \times (0,\infty) \\ v(x,0) = u_0(x)\,, & \text{in } \Omega\,. \end{cases} \tag{1.3}$$

Suppose that the initial data $u_0(x)$ does not deviate too much from a stationary state $u(x)$. Does the solution of (1.3) return to $u(x)$ as $t \to \infty$? If this is the case then the solution u of the problem (1.1) is said to be stable. More precisely, a solution u of (1.1) is called **stable** if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|u(x) - v(x,t)\|_{L^\infty(\Omega\times(0,\infty))} < \varepsilon$, provided that $\|u(x) - u_0(x)\|_{L^\infty(\Omega)} < \delta$. Here, $v(x,t)$ is a solution of problem (1.3). We establish in what follows that the solutions given by the method of sub and supersolutions are, generally, stable, in the following sense.

Definition 2. *A solution u of the problem (1.1) is said to be stable provided that the first eigenvalue of the linearized operator at u is positive, that is, $\lambda_1(-\Delta - f_u(x,u)) > 0$. The solution u is called semistable if $\lambda_1(-\Delta - f_u(x,u)) \geq 0$.*

In the above definition we understand the first eigenvalue of the linearized operator with respect to homogeneous Dirichlet boundary condition.

Theorem 2. *Let $\underline{U}$ (resp., $\overline{U}$) be subsolution (resp., supersolution) of the problem (1.1) such that $\underline{U} \leq \overline{U}$ and let $\underline{u}$ (resp., $\overline{u}$) be the corresponding minimal (resp., maximal) solution of (1.1). Assume that $\underline{U}$ is not a solution of (1.1). Then $\underline{u}$ is semistable. Furthermore, if f is concave, then $\underline{u}$ is stable.*

Similarly, if $\overline{U}$ is not a solution then $\overline{u}$ is semistable and, if f is convex, then $\overline{u}$ is stable.

Proof. Let $\lambda_1 = \lambda_1(-\Delta - f_u(x,\underline{u}))$ and let φ_1 be the corresponding eigenfunction, that is,

$$\begin{cases} -\Delta\varphi_1 - f_u(x,\underline{u})\varphi_1 = \lambda_1\varphi_1\,, & \text{in } \Omega \\ \varphi_1 = 0\,, & \text{on } \partial\Omega\,. \end{cases}$$

We can suppose, without loss of generality, that $\varphi_1 > 0$ in Ω. Assume, by contradiction, that $\lambda_1 < 0$. Let us consider the function $v := \underline{u} - \varepsilon\varphi_1$, with $\varepsilon > 0$. We prove in what follows that the following hold:

(i) v is a supersolution to the problem (1.1), for ε small enough;

(ii) $v \geq \underline{U}$.

By (i), (ii) and Theorem 1 it follows that there exists a solution u such that $\underline{U} \leq u \leq v < \underline{u}$ in Ω, which contradicts the minimality of $\underline{u}$ and the hypothesis that $\underline{U}$ is not a solution.

In order to prove (i), it is enough to show that

$$-\Delta v \geq f(x,v) \quad \text{in } \Omega\,.$$

But

$$\begin{aligned}
&\Delta v + f(x,v) = \Delta \underline{u} - \varepsilon \Delta \varphi_1 + f(x, \underline{u} - \varepsilon\varphi_1) = \\
&-f(x,\underline{u}) + \varepsilon\lambda_1\varphi_1 + \varepsilon f_u(x,\underline{u})\varphi_1 + f(x,\underline{u}-\varepsilon\varphi_1) = \\
&-f(x,\underline{u}) + \varepsilon\left(\lambda_1\varphi_1 + f_u(x,\underline{u})\varphi_1\right) + f(x,\underline{u}) - \varepsilon f_u(x,\underline{u})\varphi_1 + o(\varepsilon\varphi_1) = \\
&\varepsilon\lambda_1\varphi_1 + o(\varepsilon)\varphi_1 = \varphi_1\left(\varepsilon\lambda_1 + o(\varepsilon)\right) = \varphi_1\varepsilon\left(\lambda_1 + o(1)\right) \leq 0\,,
\end{aligned}$$

provided that $\varepsilon > 0$ is sufficiently small.

Let us now prove (ii). We observe that $v \geq \underline{U}$ is equivalent to $\underline{u} - \underline{U} \geq \varepsilon\varphi_1$, for small ε. But $\underline{u} - \underline{U} \geq 0$ in Ω. Moreover, $\underline{u} - \underline{U} \not\equiv 0$, since $\underline{U}$ is not solution. Now we are in position to apply the Hopf Strong Maximum Principle in the following variant: assume v satisfies

$$\begin{cases} -\Delta w + aw = f(x) \geq 0\,, & \text{in } \Omega \\ w \geq 0\,, & \text{on } \partial\Omega\,, \end{cases}$$

where a is a nonnegative number. Then $w \geq 0$ in Ω and the following alternative holds: either
(i) $w \equiv 0$ in Ω
or
(ii) $w > 0$ in Ω and $\frac{\partial w}{\partial \nu} < 0$ on the set $\{x \in \partial\Omega; w(x) = 0\}$.

Let $w = \underline{u} - \underline{U} \geq 0$. We have

$$-\Delta w + aw = f(x,\underline{u}) + \Delta\underline{U} + a(\underline{u} - \underline{U}) \geq f(x,\underline{u}) - f(x,\underline{U}) + a(\underline{u} - \underline{U})\,.$$

So, in order to have $-\Delta w + aw \geq 0$ in Ω, it is sufficient to choose $a \geq 0$ so that the mapping $\mathbb{R} \ni u \longmapsto f(x,u) + au$ is increasing on $[\underline{U}(x), \overline{U}(x)]$, as already done in the proof of Theorem 1. Observing that $w \geq 0$ on $\partial\Omega$ and $w \not\equiv 0$ in Ω we deduce by the Strong Maximum Principle that

$$w > 0 \ \text{ in } \Omega \ \text{ and } \ \frac{\partial w}{\partial \nu} < 0 \ \text{ on } \{x \in \partial\Omega; \underline{u}(x) = \underline{U}(x) = 0\}\,.$$

We prove in what follows that we can choose $\varepsilon > 0$ sufficiently small so that $\varepsilon\varphi_1 \leq w$. This is an interesting consequence of the fact that the normal derivative is negative in the points of the boundary where the function vanishes. Arguing by contradiction, there exist a sequence $\varepsilon_n \to 0$ and $x_n \in \Omega$ such that

$$(w - \varepsilon_n\varphi_1)(x_n) < 0\,. \tag{1.4}$$

Moreover, we can choose the points x_n with the additional property

$$\nabla(w - \varepsilon_n\varphi_1)(x_n) = 0\,. \tag{1.5}$$

But, passing eventually at a subsequence, we can assume that $x_n \to x_0 \in \overline{\Omega}$. It follows now by (1.4) that $w(x_0) \le 0$ which implies $w(x_0) = 0$, that is, $x_0 \in \partial\Omega$. Furthermore, by (1.5), $\nabla w(x_0) = 0$, a contradiction, since $\frac{\partial w}{\partial \nu}(x_0) < 0$, by the Strong Maximum Principle.

Let us now assume that f is concave. We have to show that $\lambda_1 > 0$. Arguing again by contradiction, let us suppose that $\lambda_1 = 0$. With the same arguments as above we can show that $v \ge \underline{U}$. If we prove that v is a supersolution then we contradicts the minimality of $\underline{u}$. The above arguments do not apply since, in order to find a contradiction, the estimate

$$\Delta v + f(x,v) = \varepsilon\varphi_1\left(\lambda_1 + o(1)\right)$$

is not relevant in the case where $\lambda_1 = 0$. However

$$\Delta v + f(x,v) = -f(x,\underline{u}) + \varepsilon\left(\lambda_1\varphi_1 + f_u(x,\underline{u})\varphi_1\right) + f(x,\underline{u} - \varepsilon\varphi_1) \le$$
$$\varepsilon f_u(x,\underline{u})\varphi_1 + f_u(x,\underline{u})(-\varepsilon\varphi_1) = 0\,.$$

□

If neither $\underline{U}$ nor $\overline{U}$ are solutions to the problem (1.1) it is natural to ask if there exists a solution u such that $\underline{U} < u < \overline{U}$ and $\lambda_1(-\Delta - f_u(x,u)) > 0$. In general such a situation does not occur, as showed by the following example: consider the problem

$$\begin{cases} -\Delta u = \lambda_1 u - u^3\,, & \text{in } \Omega \\ u = 0\,, & \text{on } \partial\Omega\,, \end{cases}$$

where λ_1 denotes the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$. We remark that we can choose $\overline{U} = a$ and $\underline{U} = -a$, for every $a > \sqrt{\lambda_1}$. On the other hand, by Poincaré's Inequality,

$$\lambda_1 \int_\Omega u^2 \le \int_\Omega |\nabla u|^2 = \lambda_1 \int_\Omega u^2 - \int_\Omega u^4\,,$$

which shows that the unique solution is $u = 0$. However this solution is not stable, since $\lambda_1(-\Delta - f_u(0)) = 0$.

Another question which arises is under what hypotheses there exists a global maximal (resp., minimal) solution of (1.1), not only with respect to a prescribed pairing of sub and supersolutions. The following result shows that a sufficient condition is that the nonlinearity has a kind of sublinear growth. More precisely, let us consider the problem

$$\begin{cases} -\Delta u = f(x,u) + g(x)\,, & \text{in } \Omega \\ u = 0\,, & \text{on } \partial\Omega\,. \end{cases} \tag{1.6}$$

Theorem 3. *Assume $g \in C^\alpha(\overline{\Omega})$, for some $\alpha \in (0,1)$ and, for every $(x,u) \in \Omega \times \mathbb{R}$,*

$$f(x,u)\,\mathrm{sign}\, u \le a \mid u \mid + C \qquad \text{with } a < \lambda_1\,. \tag{1.7}$$

Then there exists a global minimal (resp., maximal) solution $\underline{u}$ (resp., $\overline{u}$) to the problem (1.6).

Proof. Assume without loss of generality that $C > 0$. We choose as supersolution of (1.6) the unique solution $\overline{U}$ of the problem

$$\begin{cases} -\Delta\overline{U} - a\overline{U} = C'\,, & \text{in } \Omega \\ \overline{U} = 0\,, & \text{on } \partial\Omega\,, \end{cases}$$

where C' is taken such that $C' \ge C + \sup_{\overline{\Omega}} \mid g \mid$. Since $a < \lambda_1$ it follows by the Maximum Principle that $\overline{U} \ge 0$.

Let $\underline{U} = -\overline{U}$ be a subsolution of (1.6). Thus, by Theorem 1, there exists $\underline{u}$ (resp., $\overline{u}$) minimal (resp., maximal) with respect to $(\underline{U}, \overline{U})$. We prove in what follows that $\underline{u} \le u \le \overline{u}$, for **every** solution u of the problem (1.6). For this aim, it is enough to show that $\underline{U} \le u \le \overline{U}$. Let us prove that $u \le \overline{U}$. Denote

$$\Omega_0 = \{x \in \Omega;\ u(x) > 0\}\,.$$

Consequently, it is sufficient to show that $u \le \overline{U}$ in Ω_0. The idea is to prove that

$$\begin{cases} -\Delta(\overline{U} - u) - a(\overline{U} - u) \ge 0\,, & \text{in } \Omega_0 \\ \overline{U} - u \ge 0\,, & \text{on } \partial\Omega_0 \end{cases}$$

and then to apply the Maximum Principle. On the one hand, it is obvious that

$$\overline{U} - u = \overline{U} \ge 0\,, \quad \text{on } \partial\Omega_0\,.$$

On the other hand,

$$\begin{aligned} &-\Delta(\overline{U} - u) - a(\overline{U} - u) = -\Delta\overline{U} - a\overline{U} - (-\Delta u - au) \ge \\ &C' - f(x,u) + au \ge 0\,, \qquad \text{in } \Omega_0\,, \end{aligned}$$

which ends our proof. □

Let us now consider the problem

$$\begin{cases} -\Delta u = f(u)\,, & \text{in } \Omega \\ u = 0\,, & \text{on } \partial\Omega \\ u > 0\,, & \text{in } \Omega\,, \end{cases} \tag{1.8}$$

where

$$f(0) = 0 \tag{1.9}$$

$$\limsup_{u \to +\infty} \frac{f(u)}{u} < \lambda_1 \, . \tag{1.10}$$

Observe that (1.10) implies

$$f(u) \leq au + C \, , \qquad \text{for every } u \geq 0 \, ,$$

with $a < \lambda_1$ and $C > 0$.

Clearly $\underline{U} = 0$ is a subsolution of (1.8). Choose $\overline{U}$ the unique solution of the problem

$$\begin{cases} -\Delta \overline{U} - a\overline{U} = C \, , & \text{in } \Omega \\ \overline{U} = 0 \, , & \text{on } \partial\Omega \, . \end{cases}$$

We then obtain a minimal solution $\underline{u}$ and a maximal solution $\overline{u} \geq 0$. However we can not state that $\overline{u} > 0$ (give an example!). A positive answer is given by

Theorem 4. *Assume f satisfies hypotheses (1.9), (1.10) and*

$$f'(0) > \lambda_1 \, . \tag{1.11}$$

Then there exists a maximal solution u to the problem (1.8) such that $u > 0$ in Ω.

Proof. The idea is to find another subsolution. Let $\underline{U} = \varepsilon\varphi_1$, where $\varphi_1 > 0$ is the first eigenfunction of $-\Delta$ in $H_0^1(\Omega)$. To obtain our conclusion it is sufficient to verify that for $\varepsilon > 0$ small enough we have
(i) $\varepsilon\varphi_1$ is a subsolution;
(ii) $\varepsilon\varphi_1 \leq \overline{U}$.

Let us verify (i). We observe that

$$f(\varepsilon\varphi_1) = f(0) + \varepsilon\varphi_1 f'(0) + o(\varepsilon\varphi_1) = \varepsilon\varphi_1 f'(0) + o(\varepsilon\varphi_1) \, .$$

So the inequality $-\Delta(\varepsilon\varphi_1) \leq f(\varepsilon\varphi_1)$ is equivalent to

$$\varepsilon\lambda_1\varphi_1 \leq \varepsilon\varphi_1 f'(0) + o(\varepsilon\varphi_1) \, ,$$

that is

$$\lambda_1 \leq f'(0) + o(1) \, .$$

This is true, by our hypothesis (1.11).

Let us now verify (ii). Recall that $\overline{U}$ satisfies

$$\begin{cases} -\Delta \overline{U} = a\overline{U} + C \, , & \text{in } \Omega \\ \overline{U} = 0 \, , & \text{on } \partial\Omega \, . \end{cases}$$

Thus, by the Maximum Principle, $\overline{U} > 0$ in Ω and $\frac{\partial \overline{U}}{\partial \nu} < 0$ on $\partial\Omega$. We observe that the other variant, namely $\overline{U} \equiv 0$, becomes impossible, since $C > 0$. Using the same trick as in the proof of Theorem 1 (more precisely, the fact that $\frac{\partial \overline{U}}{\partial \nu} < 0$ on $\partial\Omega$) we find $\varepsilon > 0$ small enough so that $\varepsilon\varphi_1 \leq \overline{U}$ in Ω. $\square$

Remark 3. *We observe that a necessary condition for the existence of a solution to the problem (1.8) is that the line $\lambda_1 u$ intersects the graph of the function $f = f(u)$ on the positive semi-axis.*

Indeed, if $f(u) < \lambda_1 u$ for any $u > 0$ then the unique solution is $u = 0$. After multiplication with φ_1 in (1.8) and integration we find

$$\int_\Omega (-\Delta u)\varphi_1 = -\int_\Omega u\Delta\varphi_1 = \lambda_1 \int_\Omega u\varphi_1 = \int_\Omega f(u)\varphi_1 < \lambda_1 \int_\Omega u\varphi_1 \,,$$

a contradiction. $\square$

Remark 4. *We can require instead of (1.11) that $f \in C^1(0,\infty)$ and $f'(0+) = +\infty$.*

Indeed, since $f(0) = 0$, there exists $c > 0$ such that, for all $0 < \varepsilon < c$,

$$\frac{f(\varepsilon\varphi_1)}{\varepsilon\varphi_1} = \frac{f(\varepsilon\varphi_1) - f(0)}{\varepsilon\varphi_1 - 0} > \lambda_1 \,.$$

It follows that $f(\varepsilon\varphi_1) > \varepsilon\lambda_1\varphi_1 = -\Delta(\varepsilon\varphi_1)$ and so, $\underline{U} = \varepsilon\varphi_1$ is a sub-solution. It is easy to check that $\overline{U} \geq \underline{U}$ in Ω and the proof continues with the same ideas as above. $\square$

The following result gives a sufficient condition for that the solution of (1.8) is unique.

Theorem 5. *Under hypotheses (1.9), (1.10), (1.11) assume furthermore that*

$$\textit{the mapping} \quad (0,+\infty) \ni u \longmapsto \frac{f(u)}{u} \quad \textit{is decreasing}. \tag{1.12}$$

Then the problem (1.8) has a unique solution.

Example. If f is concave then the mapping $\frac{f(u)}{u}$ is decreasing. Hence by the previous results the solution is unique and stable. For instance the problem

$$\begin{cases} -\Delta u = \lambda u - u^p \,, & \text{in } \Omega \\ u = 0 \,, & \text{on } \partial\Omega \\ u > 0 \,, & \text{in } \Omega \,, \end{cases}$$

with $p > 1$ and $\lambda > \lambda_1$ has a unique solution which is also stable.

Proof of Theorem 5. Let u_1, u_2 be arbitrary solutions of (1.8). We may assume that $u_1 \leq u_2$; indeed, if not, we choose u_1 as the minimal solution. Multiplying the equalities

$$-\Delta u_1 = f(u_1) \,, \quad \text{in } \Omega$$

and

$$-\Delta u_2 = f(u_2) \,, \quad \text{in } \Omega$$

by u_2, resp. u_1, and integrating on Ω we find

$$\int_\Omega (f(u_1)u_2 - f(u_2)u_1) = 0 \,,$$

or, equivalently,

$$\int_\Omega u_1 u_2 \left(\frac{f(u_1)}{u_1} - \frac{f(u_2)}{u_2} \right) = 0\,.$$

So, by $0 < u_1 \le u_2$ we deduce $\frac{f(u_1)}{u_1} = \frac{f(u_2)}{u_2}$ in Ω. Now, by (1.12) we conclude that $u_1 = u_2$. $\square$

Let us now consider the problem

$$\begin{cases} -\Delta u = f(u)\,, & \text{in } \Omega \\ u = 0\,, & \text{on } \partial\Omega \\ u > 0\,, & \text{in } \Omega\,. \end{cases} \tag{1.13}$$

Our aim is to establish in what follows a corresponding result in the case where the nonlinearity does not satisfy any growth assumptions, like (1.10) or (1.11). The following deeper result in this direction is due to Krasnoselski.

Theorem 6. *Assume that the nonlinearity f satisfies (1.12). Then the problem (1.13) has a unique solution.*

Proof. In order to probe the uniqueness it is enough to show that for any arbitrary solutions u_1 and u_2, we can suppose that $u_1 \le u_2$. Let

$$A = \{t \in [0,1]; tu_1 \le u_2\}\,.$$

We observe that $0 \in A$. Now we show that there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ we have $\varepsilon \in A$. This follows easily with arguments which are already done and using the crucial observations that

$$\begin{cases} u_2 > 0\,, & \text{in } \Omega \\ \dfrac{\partial u_2}{\partial \nu} < 0\,, & \text{on } \partial\Omega\,. \end{cases}$$

Let $t_0 = \max A$. Assume, by contradiction, that $t_0 < 1$. Hence $t_0 u_1 \le u_2$ in Ω. The idea is to show the existence of some $\varepsilon > 0$ such that $(t_0 + \varepsilon)u_1 \le u_2$, which contradicts the choice of t_0. For this aim we use the Maximum Principle. We have

$$-\Delta(u_2 - t_0 u_1) + a(u_2 - t_0 u_1) = f(u_2) + au_2 - t_0\left(f(u_1) + au_1\right)\,.$$

Now we choose $a > 0$ so that the mapping $u \longmapsto f(u) + au$ is increasing. Therefore

$$\begin{aligned} &-\Delta(u_2 - t_0 u_1) + a(u_2 - t_0 u_1) \ge \\ &f(t_0 u_1) + at_0 u_1 - t_0 f(u_1) - at_0 u_1 = f(t_0 u_1) - t_0 f(u_1) \ge 0\,. \end{aligned}$$

This implies either

(i) $u_2 - t_0 u_1 \equiv 0$

or

(ii) $u_2 - t_0 u_1 > 0$ in Ω and $\frac{\partial}{\partial \nu}(u_2 - t_0 u_1) < 0$ on $\partial\Omega$.

The first case is impossible since it would imply $t_0 f(u_1) = f(t_0 u_1)$, a contradiction. This reasoning is based on the elementary fact that if f is continuous and $f(Cx) = Cf(x)$ for any x in a nonempty interval then f is linear. $\square$

Chapter 2

Implicit Function Theorem and Applications to Boundary Value Problems

2.1 Abstract theorems

Let X, Y be Banach spaces. Our aim is to develop a general method which will enable us to solve equations of the type

$$F(u, \lambda) = v\,,$$

where $F : X \times \mathbb{R} \to Y$ is a prescribed sufficiently smooth function and $v \in Y$ is given.

Theorem 7. *Let X, Y be real Banach spaces and let $(u_0, \lambda_0) \in X \times \mathbb{R}$. Consider a C^1-mapping $F = F(u, \lambda) : X \times \mathbb{R} \to Y$ such that the following conditions hold:*

(i) $F(u_0, \lambda_0) = 0$;

(ii) the linear mapping $F_u(u_0, \lambda_0) : X \to Y$ is bijective.

Then there exists a neighbourhood U_0 of u_0 and a neighbourhood V_0 of λ_0 such that for every $\lambda \in V_0$ there is a unique element $u(\lambda) \in U_0$ so that $F(u(\lambda), \lambda) = 0$.

Moreover, the mapping $V_0 \ni \lambda \longmapsto u(\lambda)$ is of class C^1.

Proof. Consider the mapping $\Phi(u, \lambda) : X \times \mathbb{R} \to Y \times \mathbb{R}$ defined by $\Phi(u, \lambda) = (F(u, \lambda), \lambda)$. It is obvious that $\Phi \in C^1$. We apply to Φ the Inverse Function Theorem. For this aim, it remains to verify that the mapping $\Phi'(u_0, \lambda_0) : X \times \mathbb{R} \to Y \times \mathbb{R}$ is bijective. Indeed, we have

$$\Phi(u_0 + tu, \lambda_0 + t\lambda) = (F(u_0 + tu, \lambda_0 + t\lambda), \lambda_0 + t\lambda) =$$

$$(F(u_0, \lambda_0) + F_u(u_0, \lambda_0) \cdot (tu) + F_\lambda(u_0, \lambda_0) \cdot (t\lambda) + o(1), \lambda_0 + t\lambda)\ .$$

It follows that

$$F'(u_0, \lambda_0) = \begin{pmatrix} F_u(u_0, \lambda_0) & F_\lambda(u_0, \lambda_0) \\ 0 & \mathrm{I} \end{pmatrix}$$

which is a bijective operator, by our hypotheses. Thus, by the Inverse Function Theorem, there exist a neighbourhood $\mathcal{U}$ of the point (u_0, λ_0) and a neighbourhood $\mathcal{V}$ of $(0, \lambda)$ such that the equation

$$\Phi(u, \lambda) = (f, \lambda_0)$$

has a unique solution, for every $(f, \lambda) \in \mathcal{V}$. Now it is sufficient to take here $f = 0$ and our conclusion follows. $\Box$

With a similar proof one can justify the following **global** version of the Implicit Function Theorem.

Theorem 8. *Assume $F : X \times \mathbb{R} \to Y$ is a C^1-function on $X \times \mathbb{R}$ satisfying*

(i) $F(0, 0) = 0$;

(ii) the linear mapping $F_u(0, 0) : X \to Y$ is bijective.

Then there exist an open neighbourhood I of 0 and a C^1 mapping $I \ni \lambda \longmapsto u(\lambda)$ such that $u(0) = 0$ and $F(u(\lambda), \lambda) = 0$.

The following result will be of particular importance in the next applications.

Theorem 9. *Assume the same hypotheses on F as in Theorem 8. Then there exists an open and maximal interval I containing the origin and there exists a unique C^1-mapping $I \ni \lambda \longmapsto u(\lambda)$ such that the following hold:*

a) $F(u(\lambda), \lambda) = 0$, for every $\lambda \in I$;

b) the linear mapping $F_u(u(\lambda), \lambda)$ is bijective, for any $\lambda \in I$;

c) $u(0) = 0$.

Proof. Let u_1, u_2 be solutions and consider the corresponding open intervals I_1 and I_2 on which these solutions exist, respectively. It follows that $u_1(0) = u_2(0) = 0$ and

$$F(u_1(\lambda), \lambda) = 0\,, \quad \text{for every } \lambda \in I_1\,,$$

$$F(u_2(\lambda), \lambda) = 0\,, \quad \text{for every } \lambda \in I_2\,.$$

Moreover, the mappings $F_u(u_1(\lambda), \lambda)$ and $F_u(u_2(\lambda), \lambda)$ are one-to-one and onto on I_1, resp. I_2. But, for λ sufficiently close to 0 we have $u_1(\lambda) = u_2(\lambda)$. We wish to show that we have global uniqueness. For this aim, let

$$I = \{\lambda \in I_1 \cap I_2;\ u_1(\lambda) = u_2(\lambda)\}\,.$$

Our aim is to show that $I = I_1 \cap I_2$. We first observe that $0 \in I$, so $I \neq \emptyset$. A standard argument then shows that I is closed in $I_1 \cap I_2$. In order to show that $I = I_1 \cap I_2$, it is sufficient now to prove that I is an open set in $I_1 \cap I_2$. The proof of this statement follows by applying Theorem 7 for λ instead of 0. Thus, $I = I_1 \cap I_2$.

Now, in order to justify the existence of a maximal interval I, we consider the C^1-curves $u_n(\lambda)$ defined on the corresponding open intervals I_n, such that $0 \in I_n$, $u_n(0) = u_0$, $F(u_n(\lambda), \lambda) = 0$ and $F_u(u_n(\lambda), \lambda)$ is an isomorphism, for any $\lambda \in I_n$. Now a standard argument enables us to construct a maximal solution on the set $\cup_n I_n$. $\Box$

Corollary 1. *Let X, Y be Banach spaces and let $F : X \to Y$ be a C^1-function. Assume that the linear mapping $F_u(u) : X \to Y$ is bijective, for every $u \in X$ and there exists $C > 0$ such that $\| (F_u(u))^{-1} \| \leq C$, for any $u \in X$.*

Then F is onto.

Proof. Assume, without loss of generality, that $F(0) = 0$ and fix arbitrarily $f \in Y$. Consider the operator $F(u, \lambda) = F(u) - \lambda f$, defined on $X \times \mathbb{R}$. Then, by Theorem 9, there exists a C^1-function $u(\lambda)$ which is defined on a maximal interval I such that $F(u(\lambda)) = \lambda f$. In particular, $u := u(1)$ is a solution of the equation $F(u) = f$. We assert that $I = \mathbb{R}$. Indeed, we have

$$u_\lambda(\lambda) = (F_u(u))^{-1} f \,,$$

so u is a Lipschitz map on I, which implies $I = \mathbb{R}$.

The Implicit Function Theorem is used to solve equations of the type $F(u) = f$, where $F \in C^1(X, Y)$. A simple method for proving that F is onto, that is, $\operatorname{Im} F = Y$ is to prove the following:

(i) $\operatorname{Im} F$ is open;

(ii) $\operatorname{Im} F$ is closed.

For showing (i), usually we use the Inverse Function Theorem, more exactly, if $F_u(u)$ is one-to-one, for every $u \in X$, then (i) holds. A sufficient condition for that (ii) holds is that F is a proper map.

Another variant of the Implicit Function Theorem is given by

Theorem 10. *Let $F(u, \lambda)$ be a C^1-mapping in a neighbourhood of $(0,0)$ and such that $F(0,0) = 0$. assume that*

(i) $\operatorname{Im} F_u(0,0) = Y$;

(ii) the space $X_1 := \operatorname{Ker} F_u(0,0)$ has a closed complement X_2.

Then there exist $B_1 = \{u_1 \in X; \|u_1\| < \delta\}$, $B_2 = \{\lambda \in \mathbb{R}; |\lambda| < r\}$, $B_3 = \{g \in Y; \|g\| < R\}$ and a neighbourhood U of the origin in X_2 such that, for any $u_1 \in B_1$, $\lambda \in B_2$ and $g \in B_3$, there exists a unique solution $u_2 = \varphi(u_1, \lambda, g) \in U$ of the equation

$$F(u_1 + \varphi(u_1, \lambda, g), \lambda) = g \,.$$

Proof. Let $\Gamma = X \times \mathbb{R} \times Y$, that is, every element $\nu \in \Gamma$ has the form $\nu = (u_1, \lambda, g)$. It remains to apply then Implicit Function Theorem to the mapping $G : X \times \Gamma \to Y$ which is defined by $G(u_2, \nu) = F(u_1 + u_2, \lambda) - g$. □

We conclude this paragraph with the following elementary example: let Ω be a smooth bounded domain in $\mathbb{R}^N$ and let g be a C^1 real function defined on a neighbourhood of 0 and such that $g(0) = 0$. Consider the problem

$$\begin{cases} -\Delta u = g(u) + f(x) \,, & \text{in } \Omega \\ u = 0 \,, & \text{on } \partial\Omega \,. \end{cases} \tag{2.1}$$

Assume that $g'(0)$ is not a real number of $-\Delta$ in $H_0^1(\Omega)$, say $g'(0) \leq 0$. If f is sufficiently small, then the problem (2.1) has a unique solution, by the Implicit Function Theorem. Indeed, it is enough to

apply Theorem 7 to the operator $F(u) = -\Delta u - g(u)$ and after observing that $F_u(0) = -\Delta - g'(0)$. There are at least two distinct possibilities for defining F:
(i) $F : C_0^{2,\alpha}(\overline{\Omega}) \to C_0^{\alpha}(\overline{\Omega})$, for some $\alpha \in (0,1)$.
or
(ii) $F : W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \to L^p(\Omega)$. In order to obtain classical solutions (by a standard bootstrap argument that we will describe later), it is sufficient to choose $p > \frac{N}{2}$.

2.2 A basic bifurcation theorem

Consider a C^2 map $f : \mathbb{R} \to \mathbb{R}$ which is convex, positive and such that $f'(0) > 0$. Our aim is to study the problem

$$\begin{cases} -\Delta u = \lambda f(u), & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega\,, \end{cases} \tag{2.2}$$

where λ is a positive parameter. We are looking for classical solutions of this problem, that is, $u \in C^2(\Omega) \cap C(\overline{\Omega})$.

Trying to apply the Implicit Function Theorem to our problem (2.2), set

$$X = \{u \in C^{2,\alpha}(\overline{\Omega}); u = 0 \text{ on } \partial\Omega\}$$

and $Y = C^{0,\alpha}(\overline{\Omega})$, for some $0 < \alpha < 1$. Define $F(u, \lambda) = -\Delta u - \lambda f(u)$. It is clear that F verifies all the assumptions of the Implicit Function Theorem. Hence, there exist a maximal neighbourhood of the origin I and a unique map $u = u(\lambda)$ which is solution of the problem (P) and such that the linearized operator $-\Delta - \lambda f'(u(\lambda))$ is bijective. In other words, for every $\lambda \in I$, the problem (2.2) admits a stable solution which is given by the Implicit Function Theorem. Let $\lambda^\star := \sup I \le +\infty$. We shall denote from now on by $\lambda_1(-\Delta - a)$ the first eigenvalue in $H_0^1(\Omega)$ of the operator $-\Delta - a$, where $a \in L^\infty(\Omega)$.

Our aim is to prove in what follows the following celebrated result.

Amann's Theorem. *Assume that $f : \mathbb{R} \to \mathbb{R}$ is a C^2 which is convex, positive and such that $f'(0) > 0$. Then the following hold:*
i) $\lambda^\star < +\infty$;
ii) $\lambda_1(-\Delta - \lambda f'(u(\lambda))) > 0$;
iii) the mapping $I \ni \lambda \longmapsto u(x, \lambda)$ is increasing, for every $x \in \Omega$;
iv) for every $\lambda \in I$ and $x \in \Omega$, we have $u(x, \lambda) > 0$;
v) there is no solution of the problem (2.2), provided that $\lambda > \lambda^\star$;
vi) $u(\lambda)$ is a minimal solution of the problem (2.2);
vii) $u(\lambda)$ is the unique stable solution of the problem (2.2).

Proof. i) It is obvious, by the variational characterization of the first eigenvalue, that if $a, b \in L^\infty(\Omega)$ and $a \le b$ then

$$\lambda_1(-\Delta - a(x)) \ge \lambda_1(-\Delta - b(x))\,.$$

Assume iv) is already proved. Thus, by the convexity of f, $f'(u(\lambda)) \geq f'(0) > 0$, which implies that

$$\lambda_1(-\Delta - \lambda f'(0)) \geq \lambda_1(-\Delta - \lambda f'(u(\lambda))) > 0,$$

for every $\lambda \in I$. This implies $\lambda_1 - \lambda f'(0) > 0$, for any $\lambda < \lambda^\star$, that is, $\lambda^\star \leq \lambda_1/f'(0) < +\infty$.

ii) Set $\varphi(\lambda) = \lambda_1(-\Delta - \lambda f'(u(\lambda)))$. So, $\varphi(0) = \lambda_1 > 0$ and, for every $\lambda < \lambda^\star$, $\varphi(\lambda) \neq 0$, since the linearized operator $-\Delta - \lambda f'(u(\lambda))$ is bijective, by the Implicit Function Theorem. Now, by the continuity of the mapping $\lambda \longmapsto \lambda f'(u(\lambda))$, it follows that φ is continuous, which implies, by the above remarks, $\varphi > 0$ on $[0, \lambda^\star)$.

iii) We differentiate in (2.2) with respect to λ. Thus

$$-\Delta u_\lambda = f(u(\lambda)) + \lambda f'(u(\lambda)) \cdot u_\lambda \qquad \text{in } \ \Omega$$

and $u_\lambda = 0$ on $\partial\Omega$. Hence

$$(-\Delta - \lambda f'(u(\lambda)))u_\lambda = f(u(\lambda)) \qquad \text{in } \ \Omega.$$

But the operator $(-\Delta - \lambda f'(u(\lambda)))$ is coercive. So, by Stampacchia's Maximum Principle, either $u_\lambda \equiv 0$ in Ω, or $u_\lambda > 0$ in Ω. The first variant is not convenient, since it would imply that $f(u(\lambda)) = 0$, which is impossible, by our initial hypotheses. It remains that $u_\lambda > 0$ in Ω.

iv) follows from iii).

v) Assume that there exists some $\nu > \lambda^\star$ and there exists a corresponding solution v to our problem (2.2).

Claim. $u(\lambda) < v$ in Ω, for every $\lambda < \lambda^\star$.

Proof of the claim. By the convexity of f it follows that

$$-\Delta(v - u(\lambda)) = \nu f(v) - \lambda f(u(\lambda)) \geq$$

$$\lambda(f(v) - f(u(\lambda))) \geq \lambda f'(u(\lambda))(v - u(\lambda)).$$

Hence

$$-\Delta(v - u(\lambda)) - \lambda f'(u(\lambda))(v - u(\lambda)) =$$

$$(-\Delta - \lambda f'(u(\lambda)))(v - u(\lambda)) \geq 0 \quad \text{in } \ \Omega,$$

since the operator $-\Delta - \lambda f'(u(\lambda))$ is coercive. Thus, by Stampacchia's Maximum Principle, $v \geq u(\lambda)$ in Ω, for every $\lambda < \lambda^\star$.

Hence, $u(\lambda)$ is bounded in L^∞ by v. Passing to the limit as $\lambda \to \lambda^\star$ we find that $u(\lambda) \to u^\star < +\infty$ and $u^\star = 0$ on $\partial\Omega$.

We prove in what follows that

$$\lambda_1(-\Delta - \lambda^\star f'(u^\star))) = 0.$$

We already know that $\lambda_1(-\Delta - \lambda^\star f'(u^\star))) \geq 0$. Assume that $\lambda_1(-\Delta - \lambda^\star f'(u^\star))) > 0$, so, this operator is coercive. We apply the Implicit Function Theorem to $F(u, \lambda) = -\Delta u - \lambda f(u)$ at the point $(u^\star, \lambda^\star)$. We obtain that there is a curve of solutions of the problem (2.2) passing through $(u^\star, \lambda^\star)$, which contradicts the maximality of $\lambda^\star$. We have obtained that $\lambda_1(-\Delta - \lambda^\star f'(u^\star))) = 0$. So, there exists $\varphi_1 > 0$ in Ω, $\varphi_1 = 0$ on $\partial\Omega$, so that

$$-\Delta\varphi_1 - \lambda^\star f'(u^\star)\varphi_1 = 0 \qquad \text{in } \Omega. \tag{2.3}$$

Passing to the limit as $\lambda \to \lambda^\star$ in the relation

$$(-\Delta - \lambda f'(u(\lambda)))(v - u(\lambda)) \geq 0$$

we find

$$(-\Delta - \lambda^\star f'(u^\star)))(v - u^\star) \geq 0.$$

Multiplying this inequality by φ_1 and integrating, we obtain

$$-\int_\Omega (v - u^\star)\Delta\varphi_1 dx - \lambda^\star \int_\Omega f'(u^\star)(v - u^\star)\varphi_1 dx \geq 0.$$

In fact, by (2.3), the above relation is an equality, which implies that

$$-\Delta(v - u^\star) = \lambda^\star f'(u^\star)(v - u^\star) \qquad \text{in } \Omega.$$

It follows that $\nu f(v) = \lambda^\star f(u^\star)$ in Ω. But $\nu > \lambda^\star$ and $f(v) \geq f(u^\star)$. So, $f(v) = 0$ which is impossible.

vi) Fix an arbitrary $\lambda < \lambda^\star$. Assume that v is another solution of the problem (2.2). We have

$$-\Delta(v - u(\lambda)) = \lambda f(v) - \lambda f(u(\lambda)) \geq \lambda f'(u(\lambda))(v - u(\lambda)) \quad \text{in } \Omega.$$

Again, by Stampacchia's Maximum Principle applied to the coercive operator $-\Delta - \lambda f'(u(\lambda))$ we find that $v \geq u(\lambda)$ in Ω.

vii) Let v be another stable solution, for some $\lambda < \lambda^\star$. With the same reasoning as in vi), but applied to the coercive operator $-\Delta - \lambda f'(v)$, we get that $u(\lambda) \geq v$. Finally, $u(\lambda) = v$. □

2.3 Qualitative properties of the minimal solution in a neighbourhood of the bifurcation point

We assume, throughout this section, that

$$\lim_{t\to\infty} \frac{f(t)}{t} = a > 0\,. \tag{2.4}$$

We impose in this section a supplementary hypothesis to f, which essentially means that as $t \to \infty$, then f increases faster than t^a, for some $a > 1$. In this case we shall prove that the problem (2.2) has a weak solution $u^* \in H_0^1(\Omega)$ provided $\lambda = \lambda^*$. However, we can not obtain in this case a supplementary regularity of u^*.

Theorem 11. *Assume that f satisfies the following additional condition: there is some $a > 0$ and $\mu > 1$ such that, for every $t \geq a$,*

$$tf'(t) \geq \mu f(t).$$

The following hold

i) the problem P has a solution u^*, provided that $\lambda = \lambda^*$;

ii) u^* is the weak limit in $H_0^1(\Omega)$ of stable solutions $u(\lambda)$, if $\lambda \nearrow \lambda^*$;

iii) for every $v \in H_0^1(\Omega)$,

$$f'(u^*)v^2 \in L^1(\Omega)$$

and

$$\lambda^* \int_\Omega f'(u^*)v^2 dx \leq \int_\Omega |\nabla v|^2 dx\,.$$

Proof. For every $v \in H_0^1(\Omega)$ and $\lambda \in [0, \lambda^*)$ we have, by Amann's Theorem,

$$\lambda^* \int_\Omega f'(u(\lambda))v^2 dx \leq \int_\Omega |\nabla v|^2 dx\,. \tag{2.5}$$

Choosing here $v = u(\lambda)$, we find

$$\lambda^* \int_\Omega f'(u(\lambda))u^2(\lambda) dx \leq \int_\Omega |\nabla u(\lambda)|^2 dx =$$

$$= \lambda \int_\Omega f(u(\lambda))u(\lambda) dx\,.$$

So, if for a as in our hypothesis we define

$$\Omega(\lambda) = \{x \in \Omega;\ u(\lambda)(x) > a\},$$

then

$$\int_\Omega f'(u(\lambda))u^2(\lambda) dx \leq \frac{\lambda}{\lambda^*} \int_\Omega f(u(\lambda))u(\lambda) dx \leq \frac{\lambda}{\lambda^*} \left[\int_{\Omega(\lambda)} f(u(\lambda))u(\lambda) dx + a \cdot |\Omega| \cdot f(a) \right].$$

By our hypotheses on f, we have

$$f'(u(\lambda))u^2(\lambda) \geq \mu f(u(\lambda))u(\lambda) \quad \text{in} \ \ \Omega(\lambda)\,.$$

Therefore

$$(\mu - 1) \int_{\Omega(\lambda)} f(u(\lambda))u(\lambda) dx \leq C, \tag{2.6}$$

where the constant C depends only on λ. This constant can be chosen sufficiently large so that

$$\int_{\Omega \setminus \Omega(\lambda)} f(u(\lambda))u(\lambda) dx \leq C\,. \tag{2.7}$$

By (2.6), (2.7) and $\mu > 1$ it follows that there exists $C > 0$ independent of λ such that

$$\int_\Omega f(u(\lambda))u(\lambda)dx \leq C\,. \tag{2.8}$$

From here and from

$$\int_\Omega |\nabla u(\lambda)|^2 dx = \lambda \int_\Omega f(u(\lambda))u(\lambda)dx,$$

it follows that $u(\lambda)$ is bounded in $H_0^1(\Omega)$, independently with respect to λ. Consequently, up to a subsequence, we may suppose that there exists $u^* \in H_0^1(\Omega)$ such that

$$u(\lambda) \rightharpoonup u^* \quad \text{weakly in} \ \ H_0^1(\Omega) \ \ \text{if} \ \ \lambda \to \lambda^* \tag{2.9}$$

$$u(\lambda) \to u^* \quad \text{a.e. in} \ \ \Omega \ \ \text{if} \ \ \lambda \to \lambda^*\,. \tag{2.10}$$

Hence

$$f(u(\lambda)) \to f(u^*) \quad \text{a.e. in} \ \ \Omega \ \ \text{if} \ \ \lambda \to \lambda^*\,.$$

By (2.8) we get

$$\int_\Omega f(u(\lambda))dx \leq C\,.$$

Since the mapping $\lambda \longmapsto f(u(\lambda))$ is increasing and the integral is bounded, we find, by the Monotone Convergence Theorem, $f(u^*) \in L^1(\Omega)$ and

$$f(u(\lambda)) \to f(u^*) \quad \text{in} \ \ L^1(\Omega) \quad \text{if} \ \ \lambda \to \lambda^*\,.$$

Let us now choose $v \in H_0^1(\Omega)$, $v \geq 0$. So,

$$\int_\Omega \nabla u(\lambda)\cdot\nabla v dx = \lambda \int_\Omega f(u(\lambda))v dx\,.$$

On the other hand, we have already remarked that

$$f(u(\lambda))v \to f(u^*)v \quad \text{a.e. in} \ \ \Omega, \ \ \text{if}\ \lambda \to \lambda^*\,.$$

and

$$\lambda \longmapsto f(u(\lambda))v \quad \text{is increasing}\,.$$

By (2.9), it follows that

$$\int_\Omega f(u(\lambda))v \leq C\,.$$

Now, again by the Monotone Convergence Theorem,

$$f(u(\lambda))v \to f(u^*)v \quad \text{in} \ \ L^1(\Omega) \ \ \text{if} \ \ \lambda \to \lambda^*$$

and

$$\int_\Omega \nabla u^* \cdot \nabla v dx = \lambda^* \int_\Omega f(u^*)v dx\,.$$

If $v \in H_0^1(\Omega)$ is arbitrary, we find the same conclusion if we consider $v = v^+ - v^-$.

So, for every $v \in H_0^1(\Omega)$,

$$f(u(\lambda))v \to f(u^*)v \quad \text{in} \quad L^1(\Omega)$$

$$\int_\Omega \nabla u^* \cdot \nabla v dx = \lambda^* \int_\Omega f(u^*) v dx$$

and $u^* \in H_0^1(\Omega)$. Consequently, u^* is a weak solution of the problem (2.2). Moreover, for every $v \in H_0^1(\Omega)$,

$$\lambda^* \int_\Omega f'(u(\lambda))v^2 dx \leq \int_\Omega |\nabla v|^2 dx\,.$$

Applying again the Monotone Convergence Theorem we find that

$$f'(u^*)v^2 \in L^1(\Omega)$$

and

$$\lambda^* \int_\Omega f'(u^*)v^2 dx \leq \int_\Omega |\nabla v|^2 dx\,.$$

□

The facts that

$$f(u^*) \in L^1(\Omega), \quad f(u^*)u^* \in L^1(\Omega)$$

and

$$f'(u^*)v^2 \quad \text{in} \quad L^1(\Omega), \qquad \text{for every} \;\; v \in H_0^1(\Omega),$$

imply a supplementary regularity of u^*. In many concrete situations one may show that there exists n_0 such that, if $N \leq n_0$, then $u^* \in L^\infty(\Omega)$ and $f(u^*) \in L^\infty(\Omega)$.

We shall deduce in what follows some known results, for the special case $f(t) = e^t$.

Theorem 12. *Let $f(t) = e^t$. Then*

$$f(u(\lambda)) \to f(u^*) \quad \textit{in} \quad L^p(\Omega), \quad \textit{if} \;\; \lambda \to \lambda^*,$$

for every $p \in [1, 5)$.

Consequently

$$u(\lambda) \to u^* \quad \textit{in} \quad W^{2,p}(\Omega), \quad \textit{if} \;\; \lambda \to \lambda^*,$$

for every $p \in [1, 5)$.

Moreover, if $N \leq 9$, then

$$u^* \in L^\infty(\Omega) \quad \textit{and} \quad f(u^*) \in L^\infty(\Omega)\,.$$

Proof. As we have already remarked, for every $v \in H_0^1(\Omega)$,

$$\int_\Omega \nabla u(\lambda) \cdot \nabla v dx = \lambda \int_\Omega e^{u(\lambda)} v dx \tag{2.11}$$

$$\lambda\int_\Omega e^{u(\lambda)}v^2dx \le \int_\Omega |\nabla v|^2 dx\,. \tag{2.12}$$

In (2.11) we choose $v = e^{(p-1)u(\lambda)} - 1 \in H_0^1(\Omega)$, for $p > 1$ arbitrary. We find

$$(p-1)\int_\Omega e^{(p-1)u(\lambda)}|\nabla u(\lambda)|^2dx = \lambda\int_\Omega e^{u(\lambda)}[e^{(p-1)u(\lambda)} - 1]dx\,. \tag{2.13}$$

In (2.12) we put

$$v = e^{\frac{p-1}{2}u(\lambda)} - 1 \in H_0^1(\Omega)\,.$$

Hence

$$\begin{cases} \lambda\displaystyle\int_\Omega e^{u(\lambda)}[e^{\frac{p-1}{2}u(\lambda)} - 1]^2dx \le \\ \le \frac{(p-1)^2}{4}\displaystyle\int_\Omega e^{(p-1)u(\lambda)}|\nabla u(\lambda)|^2dx\,. \end{cases} \tag{2.14}$$

Taking into account the relation (2.13), our relation (2.14) becomes

$$\begin{cases} \lambda\displaystyle\int_\Omega e^{pu(\lambda)}dx - 2\lambda\int_\Omega e^{\frac{p+1}{2}u(\lambda)}dx + \lambda\int_\Omega e^{u(\lambda)}dx \le \\ \le \frac{p-1}{4}[\lambda\displaystyle\int_\Omega e^{pu(\lambda)}dx - \lambda\int_\Omega e^{u(\lambda)}dx]\,. \end{cases} \tag{2.15}$$

By Hölder's Inequality, our relation (2.15) yields

$$\lambda\left(1 - \frac{p-1}{4}\right)\int_\Omega e^{pu(\lambda)}dx + \lambda\left(1 + \frac{p-1}{4}\right)\int_\Omega e^{u(\lambda)}dx \le 2\lambda\int_\Omega e^{\frac{p+1}{2}u(\lambda)}dx \le C\left(\int_\Omega e^{pu(\lambda)}dx\right)^2,$$

where C is a constant which does not depend on λ.

So, if $1 - \frac{p-1}{4} > 0$, that is $p < 5$, then the mapping $e^{u(\lambda)}$ is bounded in $L^p(\Omega)$.

We have already proved that if $\lambda \to \lambda^*$, then

$$e^{u(\lambda)} \to e^{u^*} \quad \text{a.e. in } \ \Omega\,.$$

Moreover

$$e^{u(\lambda)} \le e^{u^*} \quad \text{in } \ \Omega\,.$$

By the Dominated Convergence Theorem, we find that, for every $1 \le p < 5$,

$$e^{u^*} \in L^p(\Omega)$$

and

$$e^{u(\lambda)} \to e^{u^*} \quad \text{in } \ L^p(\Omega)$$

Taking into account the relation (2.2) which is fulfilled by $u(\lambda)$, and using a standard regularity theorem for elliptic equations, it follows that

$$u(\lambda) \to u^* \quad \text{in } \ W^{2,p}(\Omega) \ \text{ if } \ \lambda \to \lambda^*\,.$$

On the other hand, by Sobolev inclusions,

$$W^{2,p} \subset L^\infty(\Omega), \quad \text{if} \;\; N < 2p\,.$$

So, if $N \le 9$,

$$u(\lambda) \to u^* \quad \text{in} \;\; L^\infty(\Omega), \;\; \text{if} \;\; \lambda \to \lambda^*\,.$$

Hence

$$u^* \in L^\infty(\Omega) \quad \text{and} \quad e^{u^*} \in L^\infty(\Omega)\,.$$

$\square$

Remark 5. *The above result is optimal, since D. Joseph and T. Lundgren showed in [47] that if $N = 10$ and Ω is an open ball, then $u^* \notin L^\infty(\Omega)$.*

2.4 Bifurcation problems associated to convex positive asymptotic linear functions

Throughout this section we assume that f satisfies the same general assumptions as in the preceding ones and, moreover, f is asymptotic linear at infinity, in the sense that

$$\lim_{t\to\infty} \frac{f(t)}{t} = a \in (0, +\infty)\,. \tag{2.16}$$

Under these hypotheses, we shall study in detail the problem (2.2), more exactly, we shall try to ask the following questions:

i) what happens if $\lambda = \lambda^*$?

ii) what about the behaviour of the minimal solution $u(\lambda)$ provided λ approaches λ^*?

iii) the existence of other solutions except the minimal one and, in this case, their behaviour.

First of all we make the following notations:

a) if $\alpha \in L^\infty(\Omega)$, let $\lambda_j(\alpha)$ and $\varphi_j(\alpha)$ be the jth-eigenvalue (resp., eigenfunction) of the operator $-\Delta - \alpha$. Moreover, we can assume that $\varphi_1(\alpha) > 0$ and

$$\int_\Omega \varphi_j(\alpha)\varphi_k(\alpha)dx = \delta_{jk}\,.$$

If $\alpha = 0$ we denote λ_j (resp., φ_j).

b) a solution u of the problem (P) is said to be *stable* if $\lambda_1(\lambda f'(u)) > 0$ and *unstable*, in the opposite case.

Theorem 13. *Under the preceding hypotheses, we also assume that*

$$\lim_{t\to\infty} (f(t) - at) = l \ge 0\,.$$

Then

i) $\lambda^* = \frac{\lambda_1}{a}$.
ii) $\lim\limits_{\lambda \nearrow \lambda^*} u(\lambda) = \infty$, *uniformly on compact subsets of* Ω.
iii) for every $\lambda \in (0, \lambda^*)$, *the problem (2.2) has only the minimal solution.*
iv) the problem (2.2) has no solution if $\lambda = \lambda^*$.

In the proof of this theorem we shall make use of the following auxiliary results:

Lemma 1. *Let* $\alpha \in L^\infty(\Omega)$, $w \in H_0^1(\Omega) \setminus \{0\}$, $w \geq 0$, *such that* $\lambda_1(\alpha) \leq 0$ *and*

$$-\Delta w \geq \alpha w \,, \tag{2.17}$$

Then
i) $\lambda_1(\alpha) = 0$.
ii) $-\Delta w = \alpha w$.
iii) $w > 0$ *in* Ω.

Proof of the lemma. Multiplying (2.17) by $\varphi_1(\alpha)$ and integrating by parts, we find

$$\int_\Omega \alpha \varphi_1(\alpha) w + \lambda_1(\alpha) \int_\Omega \varphi_1(\alpha) w \geq \int_\Omega \alpha \varphi_1(\alpha) w \,.$$

Since $\lambda_1(\alpha) \leq 0$, it follows that $\lambda_1(\alpha) = 0$ and $-\Delta w = \alpha w$. But $w \geq 0$ and $w \not\equiv 0$. Therefore there exists $C > 0$ such that $w = C\varphi_1(\alpha)$, that is $w > 0$ in Ω. □

Lemma 2. *If there exist* $a, b > 0$ *such that* $f(t) = at + b$ *for all* $t \geq 0$, *then*
i) $\lambda^* = \frac{\lambda_1}{a}$.
ii) the problem (2.2) has no solution if $\lambda = \lambda^*$.

Proof of the lemma. For every $0 < \lambda < \frac{\lambda_1}{a}$, the linear problem

$$\begin{cases} -\Delta u - \lambda a u = \lambda b, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{2.18}$$

has a unique solution in $H_0^1(\Omega)$, which, moreover, is positive, by Stampacchia's Maximum Principle.

Since Ω is smooth and $-\Delta u = \lambda a u + \lambda b \in H_0^1(\Omega)$, we find $u \in H^3(\Omega)$. From now on, with a standard bootstrap regularization, it follows that $u \in H^\infty(\Omega)$, that is $u \in C^\infty(\overline{\Omega})$. So, we have proved the existence of a smooth solution to the problem (2.2), for every $0 < \lambda < \frac{\lambda_1}{a}$.

For concluding the proof, it is sufficient to show that the problem (2.2) has no solution if $\lambda^* = \frac{\lambda_1}{a}$. Indeed, if u would be a solution, by multiplication in (2.2) with φ_1 and integration by parts, it follows that $\displaystyle\int_\Omega \varphi_1 = 0$, which contradicts $\varphi_1 > 0$ in Ω. □

Lemma 3. *The following hold:*

i) $\lambda^* \geq \frac{\lambda_1}{a}$.

ii) if (2.2) has a solution for $\lambda = \lambda^*$*, then it is necessarily unstable.*

iii) the problem (2.2) has at most one solution for $\lambda = \lambda^*$*.*

iv) $u(\lambda)$ *is the unique solution* u *of the problem (2.2) such that* $\lambda_1(\lambda f'(u)) \geq 0$*.*

Proof. i) By the theorem of sub and super solutions, it is sufficient to show that, for every $0 < \lambda < \frac{\lambda_1}{a}$, the problem has a sub and a super solution. More precisely, we show that there exists $\underline{U}, \overline{U} \in C^2(\Omega) \cap C(\overline{\Omega})$ with $\underline{U} \leq \overline{U}$ and such that

$$\begin{cases} -\Delta \overline{U} \geq \lambda f(\overline{U}), & \text{in } \Omega \\ \overline{U} \geq 0, & \text{on } \partial\Omega \end{cases}$$

and $\underline{U}$ verifies a similar inequality, but with reversed signs.

Let $\overline{U}$ be the solution of the problem (2.18) for $b = f(0)$ and $\underline{U} \equiv 0$. From $f(t) \leq at + b$, for every $t > 0$ and $\overline{U} > 0$ in Ω it follows that $f(\overline{U}) \leq a\overline{U} + b$, which implies $-\Delta \overline{U} \geq \lambda f(\overline{U})$, in Ω.

The fact that $\underline{U}$ is subsolution is obvious.

ii) Assume that the problem (2.2) has a stable solution u^* for $\lambda = \lambda^*$. Consider the operator

$$G : \{u \in C^{2,\frac{1}{2}}(\overline{\Omega});\ u = 0 \ \text{ on } \ \partial\Omega\} \times \mathbb{R} \to C^{0,\frac{1}{2}}(\overline{\Omega}),$$

defined by

$$G(u, \lambda) = -\Delta u - \lambda f(u)\,.$$

Applying the Implicit Function Theorem to G it follows that the problem (2.2) has a solution for λ in a neighbourhood of λ^*, which contradicts the maximality of λ^*.

iii) Let u be a solution corresponding to $\lambda = \lambda^*$. Then u is a supersolution for the problem (2.2), for every $\lambda \in (0, \lambda^*)$, that is $u \geq u(\lambda)$, for every $\lambda \in (0, \lambda^*)$. Since the map $\lambda \longmapsto u(\lambda)$ is increasing, we obtain, by the monotone convergence theorem, that there exists $u^* \leq u$ such that

$$u(\lambda) \to u^* \quad \text{in} \quad L^1(\Omega)\,.$$

Since $-\Delta u(\lambda) = \lambda f(u(\lambda))$, for every $\lambda \in (0, \lambda^*)$, it follows that $-\Delta u^* = \lambda^* f(u^*)$. In order to show that u^* is solution of the problem (2.2) for $\lambda = \lambda^*$, it is enough to show that $u \in H_0^1(\Omega)$. Indeed, it follows then, by bootstrap, that if $N > 2$, then

$$-\Delta u^* \in L^{2^*}(\Omega)$$

and, consequently,

$$u^* \in W^{2,2^*}(\Omega)\,.$$

(Here we have denoted by 2^* the critical Sobolev exponent of 2, that is, $2^* = \frac{2N}{N-2}$).

If $N = 1, 2$, since $u^* \in H_0^1(\Omega)$, it follows that $-\Delta u^* \in L^4(\Omega)$ and, by Theorems 8.34 and 9.15 in [42], we find that

$$u^* \in C^{0,\frac{1}{2}}(\overline{\Omega}).$$

Applying now Theorem 4.3 in Gilbarg-Trudinger [42], it follows that u^* is a solution for the problem (2.2).

Let us show now that $u(\lambda)$ is bounded in $H_0^1(\Omega)$. Indeed, by multiplication in (2.2) with $u(\lambda)$ and integration by parts, we find

$$\int_\Omega |\nabla u(\lambda)|^2 = \lambda \int_\Omega f(u(\lambda))u(\lambda) \leq \lambda^* \int_\Omega u f(u)\,.$$

Hence there exists $v \in H_0^1(\Omega)$ such that, up to a subsequence,

$$u(\lambda) \rightharpoonup v \quad \text{weakly in } \; H_0^1(\Omega), \text{ if } \; \lambda \to \lambda^*$$

and

$$u(\lambda) \to v \quad \text{a.e. in } \; \Omega, \text{ if } \; \lambda \to \lambda^*\,.$$

But $u(\lambda) \to u^*$ a.e. in Ω. Consequently, $v = u^*$, that is $u^* \in H_0^1(\Omega)$ and

$$u(\lambda) \rightharpoonup u^* \quad \text{weakly in } \; H_0^1(\Omega), \text{ if } \; \lambda \to \lambda^*\,.$$

For concluding the proof, it remains to show that $u = u^*$. Let $w = u - u^* \geq 0$. Then

$$-\Delta w = \lambda^*(f(u) - f(u^*)) \geq \lambda^* f'(u^*)w\,. \tag{2.19}$$

We also have

$$\lambda_1(\lambda^* f'(u^*)) \leq 0\,.$$

By Lemma 1 it follows that, either $w = 0$, or $w > 0$ and, in both cases, $-\Delta w = \lambda^* f'(u^*)w$. If $w > 0$, then, by the last equality and by (2.19) it follows that f is linear in all intervals $[u^*(x), u(x)]$, for every $x \in \Omega$. This implies easily that f is linear in the interval $[0, \max_\Omega u]$, which contradicts Lemma 2.

iv) Assume that the problem (2.2) has a solution $u \neq u(\lambda)$ with $\lambda_1(\lambda f'(u)) \geq 0$. Then, by Hopf's strong maximum principle, (Theorem 3.5. in [42]), $u > u(\lambda)$. Put $w = u - u(\lambda) > 0$. It follows that

$$-\Delta w = \lambda(f(u) - f(u(\lambda))) \leq \lambda f'(u)w\,. \tag{2.20}$$

Multiplying (2.20) by $\varphi = \varphi_1(\lambda f'(u))$ and integrating by parts, we find

$$\lambda \int_\Omega f'(u)\varphi w + \lambda_1(\lambda f'(u)) \int_\Omega \varphi w \leq \lambda \int_\Omega f'(u)\varphi w\,.$$

So, $\lambda_1(\lambda f'(u)) = 0$ and in (2.20) the equality holds, which means that f is linear in the interval $[0, \max_\Omega u]$. Therefore

$$0 = \lambda_1(\lambda f'(u)) = \lambda_1(\lambda f'(u(\lambda))),$$

contradiction. □

The following result is a reformulation of Theorem 4.1.9. in Hörmander [45].

Lemma 4. *Let (u_n) be a sequence of superharmonic nonnegative functions defined on Ω. The following alternative holds:*

either

i) $\lim_{n\to\infty} u_n = \infty$*, uniformly on compact subsets of* Ω

or

ii) (u_n) *contains a subsequence which converges in* $L^1_{\rm loc}(\Omega)$ *to some* u^*.

Lemma 5. *The following conditions are equivalent:*

i) $\lambda^* = \frac{\lambda_1}{a}$.

ii) the problem (2.2) has no solution if $\lambda = \lambda^*$.

iii) $\lim_{\lambda\to\lambda^*} u(\lambda) = \infty$*, uniformly on every compact subset of* Ω.

Proof. i)$\Longrightarrow$ii) Let us first assume that there exists a solution u provided $\lambda = \lambda^*$. As we have already observed in Lemma 3, u is necessarily unstable. On the other hand,

$$\lambda_1(\lambda^* f'(u)) \geq \lambda_1(\lambda^* a) = 0$$

Hence $\lambda_1(\lambda^* f'(u)) = 0$, that is $f'(u) = a$, which contradicts Lemma 2.

ii)$\Longrightarrow$iii) Assume the contrary. We first prove that the sequence $u(\lambda)$, for $0 < \lambda < \lambda^*$, is bounded in $L^2(\Omega)$. Indeed, if not, passing eventually to a subsequence, we may assume that $u(\lambda) = k(\lambda)w(\lambda)$, with

$$\int_\Omega w^2(\lambda) = 1 \quad \text{and} \quad \lim_{\lambda\nearrow\lambda^*} k(\lambda) = \infty\,.$$

Thus, by Lemma 4, going again to a subsequence,

$$u(\lambda) \to u^* \quad \text{in} \quad L^1_{\rm loc}(\Omega), \quad \text{if } \lambda \to \lambda^*\,.$$

Therefore

$$\frac{\lambda}{k(\lambda)} f(u(\lambda)) \to 0 \quad \text{in} \quad L^1_{\rm loc}(\Omega),$$

that is,

$$-\Delta w(\lambda) \to 0 \quad \text{in} \quad L^1_{\rm loc}(\Omega)\,. \tag{2.21}$$

We prove in what follows that $(w(\lambda))$ is bounded in $H^1_0(\Omega)$. Indeed,

$$\int_\Omega |\nabla w(\lambda)|^2 = \int_\Omega -\Delta w(\lambda) w(\lambda) = \int_\Omega \frac{\lambda}{k(\lambda)} f(u(\lambda)) w(\lambda) \leq$$

$$\leq \lambda^* \int_\Omega (a w^2(\lambda) + \frac{f(0)}{k(\lambda)} w(\lambda)) \leq \lambda^* a + c \int_\Omega w(\lambda) \leq$$

$$\leq \lambda^* a + C\sqrt{|\Omega|}\,,$$

where $C > 0$ is a constant independent on λ.

Let $w \in H_0^1(\Omega)$ be such that, passing again to a subsequence,

$$w(\lambda) \rightharpoonup w \quad \text{weakly in} \quad H_0^1(\Omega), \quad \text{if} \;\; \lambda \to \lambda^* \tag{2.22}$$

$$w(\lambda) \to w \quad \text{in} \;\; L^2(\Omega) \;\; \text{if } \lambda \to \lambda^* \,.$$

By (2.21) and (2.22) it follows that

$$-\Delta w = 0, \quad w \in H_0^1(\Omega), \quad \int_\Omega w^2 = 1,$$

which yields a contradiction. Thus $(u(\lambda))$ is bounded in $L^2(\Omega)$. With the same arguments as above, $(u(\lambda))$ is bounded in $H_0^1(\Omega)$. Let $u \in H_0^1(\Omega)$ be such that, up to a subsequence,

$$u(\lambda) \rightharpoonup u \quad \text{weakly in} \;\; H_0^1(\Omega), \;\; \text{if} \;\; \lambda \to \lambda^*$$

$$u(\lambda) \to u \quad \text{in} \;\; L^2(\Omega), \;\; \text{if} \;\; \lambda \to \lambda^* \,.$$

It follows that $-\Delta u = \lambda^* f(u)$, that is u is a solution of the problem (2.2) for $\lambda = \lambda^*$, contradiction.

iii)$\Longrightarrow$ii) As observed, if (2.2) has a solution provided $\lambda = \lambda^*$, then it is necessarily equal to $\lim_{\lambda \to \lambda^*} u(\lambda)$, which is not possible in our case.

[iii) and ii)]$\Longrightarrow$i) Let $u(\lambda) = k(\lambda) w(\lambda)$, with $k(\lambda)$ and $w(\lambda)$ as above. With the same proof one can show that, $(w(\lambda))$ is bounded in $H_0^1(\Omega)$. Let $w \in H_0^1(\Omega)$ such that , up to a subsequence,

$$w(\lambda) \rightharpoonup w \quad \text{weakly in} \;\; H_0^1(\Omega), \quad \text{if} \;\; \lambda \to \lambda^*$$

$$w(\lambda) \to w \quad \text{in} \;\; L^2(\Omega), \quad \text{if} \;\; \lambda \to \lambda^* \,.$$

Then

$$-\Delta w(\lambda) \to -\Delta w, \quad \text{in} \;\; \mathcal{D}'(\Omega)$$

and

$$\frac{\lambda}{k(\lambda)} f(u(\lambda)) \to \lambda^* a w, \quad \text{in} \;\; L^2(\Omega) \,.$$

It follows that

$$-\Delta w = \lambda^* a w \quad \text{in} \;\; \Omega, \;\; w \in H_0^1(\Omega), \;\; w \geq 0 \;\; \text{and} \int_\Omega w^2 = 1 \,.$$

This means that $\lambda^* = \frac{\lambda_1}{a}$ (and $w = \varphi_1$). $\square$

Lemma 6. *The following conditions are equivalent:*

i) $\lambda^* > \frac{\lambda_1}{a}$.

ii) the problem (2.2) has exactly one solution, say u^*, corresponding to $\lambda = \lambda^*$.

iii) $u(\lambda)$ converges uniformly in $\overline{\Omega}$ to u^*, which is the unique solution of the problem (2.2) provided $\lambda = \lambda^*$.

Proof. We have already remarked that $\lambda^* \geq \frac{\lambda_1}{a}$. So, this lemma becomes a reformulation of the preceding one, but with the difference that the limit appearing in iii) is uniform in $\overline{\Omega}$. Since $(u(\lambda))$ converges a.e. in Ω to u^*, it is sufficient to show that $u(\lambda)$ has a limit in $C(\overline{\Omega})$ if $\lambda \to \lambda^*$. Even less, it is enough to show that $u(\lambda)$ is relatively compact in $C(\overline{\Omega})$. This follows easily by the Arzela-Ascoli theorem, if we show that $u(\lambda)$ is bounded in $C^{0,\frac{1}{2}}(\overline{\Omega})$. From $0 < u(\lambda) < u^*$ we get $0 < f(u(\lambda)) < f(u^*)$, which yields a uniform bound for $-\Delta u(\lambda)$ in $L^{2N}(\Omega)$. The desired bound for $u(\lambda)$ is now a consequence of the theorem 8.34 and of the remark at the page 212 in [42], as well as of the Closed Graph Theorem. $\square$

Proof of Theorem 13 By Lemma 5, the assertions i), ii) and iv) are equivalent. We shall deduce that $\lambda^* = \frac{\lambda_1}{a}$, by showing that the problem (2.2) has no solution if $\lambda = \frac{\lambda_1}{a}$. Indeed, if u would be such a solution then

$$-\Delta u = \lambda f(u) \geq \lambda_1 u \,. \tag{2.23}$$

Multiplying (2.23) by φ_1 and integrating by parts, it follows that $\lambda f(u) = \lambda_1 u$, which contradicts $f(0) > 0$.

iii) By Lemma 3 iv), it is enough to show that if $0 < \lambda < \frac{\lambda_1}{a}$, then every solution u verifies $\lambda_1(\lambda f'(u)) \geq 0$. On the other hand,

$$-\Delta - \lambda f'(u) \geq -\Delta - \lambda a,$$

which means that

$$\lambda_1(\lambda f'(u)) \geq \lambda_1(\lambda a) = \lambda_1 - \lambda a > 0 \,.$$

$\square$

Theorem 14. *Assume that*

$$\lim_{t\to\infty} (f(t) - at) = l < 0 \,.$$

Then

i) $\lambda^* \in (\frac{\lambda_1}{a}, \frac{\lambda_1}{\lambda_0})$, *where* $\lambda_0 = \min\{\frac{f(t)}{t} \;;\; t > 0\}$.

ii) for every $\lambda = \lambda^*$, *the problem (2.2) has exactly one solution, say* u^*.

iii) $\lim_{\lambda\to\lambda^*} u(\lambda) = u^*$, *uniformly in* Ω.

iv) if $\lambda \in (0, \frac{\lambda_1}{a}]$, *then* $u(\lambda)$ *is the unique solution of the problem (2.2).*

v) if $\lambda \in (\frac{\lambda_1}{a}, \lambda^*)$, *the the problem (2.2) has at least an unstable solution, say* $v(\lambda)$. *Moreover, for every such a solution* $v(\lambda)$,

vi) $\lim_{\lambda\to\frac{\lambda_1}{a}} v(\lambda) = \infty$, *uniformly on compact subsets of* Ω.

vii) $\lim_{\lambda\to\lambda^*} v(\lambda) = u^*$, *uniformly in* Ω.

Proof. i) In order to show that $\lambda^* \leq \frac{\lambda_1}{\lambda_0}$, we shall prove that the problem (2.2) has no solution if $\lambda = \frac{\lambda_1}{\lambda_0}$. Contrary, let u be such a solution. Multiplying (2.2) by φ_1 and integrating by parts, we find

$$\lambda_1 \int_\Omega \varphi_1 u = \lambda_1 \int_\Omega \varphi_1 f(u) \,. \tag{2.24}$$

Since $\lambda = \frac{\lambda_1}{\lambda_0}$, the relation (2.24) becomes

$$\lambda_1 \int_\Omega \varphi_1 u = \frac{\lambda_1}{\lambda_0} \int_\Omega \varphi_1 f(u) \geq \lambda_1 \int_\Omega \varphi_1 u,$$

which implies $f(u) = \lambda_0 u$. As above, this equality contradicts $f(0) > 0$.

The inequality $\lambda^* > \frac{\lambda_1}{a}$, as well as ii), iii) are equivalent, by Lemmas 3 and 6. We shall prove by contradiction that $\lambda^* > \frac{\lambda_1}{a}$. In this case,

$$\lambda^* = \frac{\lambda_1}{a}$$

and

$$\lim_{\lambda \to \lambda^*} u(\lambda) = \infty, \quad \text{uniformly on compact subsets of} \quad \Omega\,.$$

By (2.24),

$$0 = \int_\Omega \varphi_1 \cdot [\lambda_1 u(\lambda) - \lambda f(u(\lambda))] =$$

$$= \int_\Omega \varphi_1 \cdot [(\lambda_1 - a\lambda) u(\lambda) - \lambda (f(u(\lambda)) - a u(\lambda))] \geq$$

$$\geq -\lambda \int_\Omega \varphi_1 \cdot [f(u(\lambda)) - a u(\lambda))]\,.$$

Passing to the limit as $\lambda \nearrow \lambda^*$ and taking into account that $l < 0$, we find

$$0 \geq -l\lambda \int_\Omega \varphi_1 > 0,$$

a contradiction.

Since $\lambda^* \leq \frac{\lambda_1}{\lambda_0}$, and the problem (2.2) has a solution if $\lambda = \lambda^*$, it follows that $\lambda^* < \frac{\lambda_1}{\lambda_0}$.

In order to prove iv) we follow the same technique as in the proof of Theorem 13, iii).

v) Consider the functional

$$J : H_0^1(\Omega) \to \mathbb{R}, \quad J(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \int_\Omega F(u),$$

where

$$F(t) = \lambda \int_0^t f(s) ds\,.$$

For $\lambda \in (\frac{\lambda_1}{a}, \lambda^*)$ we shall obtain the unstable solution as a critical point of J, which is different from $u(\lambda)$.

We shall make use the following result, which can be found in Brezis-Nirenberg [21]:

Lemma 7. *The functional J has the following properties:*

i) J is of class C^1.

ii) for every $u, v \in H_0^1(\Omega)$,

$$\langle J'(u), v \rangle = \int_\Omega \nabla u \cdot \nabla v - \lambda \int_\Omega f(u) v$$

iii) $u_0 = u(\lambda)$ is a local minimum point of J.

The idea is to apply the Mountain-Pass Lemma. For this aim we shall modify the functional J, such that u_0 would become a strict minimum point. Therefore, for every $\varepsilon > 0$, let us define

$$J_\varepsilon : H_0^1(\Omega) \to \mathbb{R}, \quad J_\varepsilon(u) = J(u) + \frac{\varepsilon}{2} \int_\Omega |\nabla(u - u_0)|^2 .$$

By the preceding Lemma,

i) J is of class C^1.

ii) $\langle J_\varepsilon'(u), v\rangle = \displaystyle\int_\Omega \nabla u \cdot \nabla v - \lambda \int_\Omega f(u)v + \varepsilon \int_\Omega \nabla(u - u_0) \cdot \nabla v$.

iii) for every $\varepsilon > 0$, u_0 is a strict local minimum point of J_ε.

Lemma 8. *Let* $\varepsilon_0 = \frac{\lambda a - \lambda_1}{2\lambda_1}$. *Then there exists* $v_0 \in H_0^1(\Omega)$ *such that* $J_\varepsilon(v_0) < J_\varepsilon(u_0)$, *for every* $\varepsilon \in [0, \varepsilon_0]$.

Proof of the lemma. We first remark that $J_\varepsilon(u)$ is bounded by $J_0(u)$ and $J_{\varepsilon_0}(u)$. Therefore, for concluding the proof it is enough to show that

$$\lim_{t\to\infty} J_{\varepsilon_0}(t\varphi_1) = -\infty .$$

But

$$J_\varepsilon(t\varphi_1) = \frac{\lambda_1}{2} t^2 + \frac{\varepsilon_0}{2} \lambda_1 t^2 - \varepsilon_0 \lambda_1 t^2 \int_\Omega \varphi_1 u_0 + \frac{\varepsilon_0}{2} \int_\Omega |\nabla u_0|^2 - \int_\Omega F(t\varphi_1) . \tag{2.25}$$

Set $\alpha = \frac{3a\lambda + \lambda_1}{4\lambda} < a$. It follows that there exists a real number β such that $f(s) \geq \alpha s + \beta$, for every $s \in \mathbb{R}$. Consequently, for every $t \geq 0$,

$$F(t) \geq \frac{\alpha\lambda}{2} t^2 + \beta\lambda t .$$

¿From here and from (2.25) we find

$$\limsup_{t\to\infty} \frac{1}{t^2} J_{\varepsilon_0}(t\varphi_1) \leq \frac{\lambda_1 + \varepsilon_0\lambda_1 - \lambda\alpha}{2} < 0,$$

by our choice of α. □

Lemma 9. *The Palais-Smale condition is satisfied uniformly with respect to* ε. *More precisely, if*

$$(J_{\varepsilon_n}(u_n)) \quad \textit{is bounded in } \mathbb{R}, \textit{ for } \varepsilon_n \in [0, \varepsilon_0] \tag{2.26}$$

and

$$J'_{\varepsilon_n}(u_n) \to 0 \quad \textit{in} \quad H^{-1}(\Omega), \tag{2.27}$$

then (u_n) *is relatively compact in* $H_0^1(\Omega)$.

Proof of the lemma. It is enough to show that (u_n) has a subsequence converging in $H_0^1(\Omega)$. Indeed, in this case, up to another subsequence, there exist $u \in H_0^1(\Omega)$ and $\varepsilon \geq 0$ such that

$$u_n \rightharpoonup u \quad \text{weakly in} \quad H_0^1(\Omega)$$

$$u_n \to u \quad \text{in} \quad L^2(\Omega)$$

$$u_n \to u \quad \text{a.e. in} \quad \Omega$$

$$\varepsilon_n \to \varepsilon\,.$$

By (2.27) it follows that

$$-\Delta u_n - \lambda f(u_n) - \varepsilon_n \Delta(u_n - u_0) \to 0 \quad \text{in} \quad \mathcal{D}'(\Omega)\,. \tag{2.28}$$

By $|f(u_n) - f(u)| \leq a|u_n - u)|$ it follows that

$$f(u_n) \to f(u) \quad \text{in} \quad L^2(\Omega)\,.$$

Using now (2.28) we obtain

$$-(1+\varepsilon_n)\Delta u_n \to \lambda f(u) - \varepsilon \Delta u_0 \quad \text{in} \quad \mathcal{D}'(\Omega),$$

that is,

$$-\Delta u - \lambda f(u) - \varepsilon \Delta(u - u_0) = 0\,.$$

Multiplying this equality by u and integrating we find

$$(1+\varepsilon)\int_\Omega |\nabla u|^2 - \lambda \int_\Omega u f(u) - \varepsilon\lambda \int_\Omega u f(u_0) = 0\,. \tag{2.29}$$

By multiplication in (2.27) with (u_n) and integration we obtain

$$(1+\varepsilon_n)\int_\Omega |\nabla u_n|^2 - \lambda \int_\Omega u_n f(u_n) - \varepsilon_n\lambda \int_\Omega u_n f(u_0) \to 0, \tag{2.30}$$

by the boundedness of (u_n). The second term appearing in (2.30) tends to $-\lambda \int_\Omega u f(u)$, while the last tends to $-\varepsilon\lambda \int_\Omega u f(u_0)$, by the convergence in $L^2(\Omega)$ of the sequences (u_n) and $(f(u_n))$. So, by comparing the first terms of (2.29) and (2.30), it follows that

$$u_n \to u \quad \text{in} \quad H_0^1(\Omega)\,.$$

At this stage it is sufficient to prove that (u_n) contains a subsequence which is bounded in $L^2(\Omega)$. Indeed, the boundedness in $L^2(\Omega)$ of the sequence (u_n) implies the boundedness in $H_0^1(\Omega)$, as follows by (2.26). Arguing by contradiction, let us assume that

$$\|u_n\|_{L^2(\Omega)} \to \infty\,.$$

Put $u_n = k_n w_n$, with

$$k_n > 0,\ k_n \to \infty \ \text{ and } \int_\Omega w_n^2 = 1\,.$$

We can assume that $\varepsilon_n \to \varepsilon$. So,

$$0 = \lim_{n\to\infty} \frac{J_{\varepsilon_n(u_n)}}{k_n^2} = \tag{2.31}$$

$$= \lim_{n\to\infty} [\frac{1}{2}\int_\Omega |\nabla w_n|^2 - \frac{1}{k_n^2}\int_\Omega F(u_n) + \frac{\varepsilon_n}{2}\int_\Omega |\nabla (w_n - \frac{u_0}{k_n})|^2]\,.$$

On the other hand,

$$\int_\Omega |\nabla (w_n - \frac{u_0}{k_n})|^2 = \int_\Omega |\nabla w_n|^2 + \frac{1}{k_n^2}\int_\Omega |\nabla u_0|^2 - \frac{2\lambda}{k_n}\int_\Omega w_n f(u_0)\,.$$

Relation (2.31) becomes

$$\lim_{n\to\infty} [\frac{1+\varepsilon_n}{2}\int_\Omega |\nabla w_n|^2 - \frac{1}{k_n^2}\int_\Omega F(u_n)] = 0\,.$$

But

$$|F(u_n)| = |F(k_n w_n)| \le \frac{\lambda a}{2} k_n^2 w_n^2 + \lambda b |k_n w_n|,$$

because $|f(t)| \le a|t| + b$, where $b = f(0)$. Thus the sequence

$$\left(\frac{1}{k_n^2}\int_\Omega F(u_n)\right)$$

is bounded, which implies also the boundedness of (w_n) in $H_0^1(\Omega)$. Let $w \in H_0^1(\Omega)$ such that, up to a subsequence,

$$w_n \rightharpoonup w \quad \text{weakly in } \ H_0^1(\Omega)\,,$$

$$w_n \to w \quad \text{strongly in } \ L^2(\Omega)\,,$$

$$w_n \to w \quad \text{a.e. in } \ \Omega\,.$$

We also remark that $\displaystyle\int_\Omega w^2 = 1$.

We prove in what follows that

$$-(1+\varepsilon)\Delta w = \lambda a w^+\,. \tag{2.32}$$

Indeed, dividing in (2.27) by k_n we find

$$(1+\varepsilon_n)\int_\Omega \nabla w_n \cdot \nabla v - \lambda \int_\Omega \frac{f(u_n)}{k_n} v - \frac{\varepsilon_n \lambda}{k_n}\int_\Omega f(u_0) v \to 0\,, \tag{2.33}$$

for all $v \in H_0^1(\Omega)$. We remark that

$$(1+\varepsilon_n)\int_\Omega \nabla w_n \cdot \nabla v \to (1+\varepsilon)\int_\Omega \nabla w \cdot \nabla v\,.$$

Relation (2.32) follows from (2.33) if we show that the sequence $(\frac{1}{k_n} f(u_n))$ contains a subsequence which converges in $L^2(\Omega)$ to $a w^+$.

Since

$$\frac{1}{k_n}f(u_n) = \frac{1}{k_n}f(k_n w_n),$$

it is obvious that the needed limit aw^+ is in the set

$$\{x \in \Omega; \quad w_n(x) \to w(x) \neq 0\}.$$

If $w(x) = 0$ and $w_n(x) \to w(x)$, let $\varepsilon > 0$ and n_0 such that $|w_n(x)| < \varepsilon$, for every $n \geq n_0$. So,

$$\frac{f(k_n w_n)}{k_n} \leq \varepsilon a + \frac{b}{k_n}, \quad \text{for every} \quad n \geq n_0,$$

that is the asked limit is 0. Hence

$$\frac{f(u_n)}{k_n} \to aw^+, \quad \text{a.e. in} \quad \Omega.$$

Since $w_n \to w$ in $L^2(\Omega)$, it follows that (see Theorem IV.9 in Brezis [18]), up to a subsequence, (w_n) is dominated in $L^2(\Omega)$. From

$$\frac{1}{k_n}f(u_n) \leq a|w_n| + \frac{1}{k_n}b,$$

it follows that $(\frac{1}{k_n}f(u_n))$ is also dominated in $L^2(\Omega)$. It follows that the relation (2.32) is true.

By the Maximum Principle applied in (2.32), we find that $w \geq 0$ and

$$\begin{cases} -\Delta w = \dfrac{\lambda a}{1+\varepsilon} w \\ w \geq 0 \\ \displaystyle\int_\Omega w^2 = 1. \end{cases} \tag{2.34}$$

So, $\frac{\lambda a}{1+\varepsilon} = \lambda_1$ and $w = \varphi_1$, which contradicts $\varepsilon \in [0, \varepsilon_0]$ and the choice of ε_0. □

For $u_0 = u(\lambda)$ and v_0 found in Lemma 8, let

$$\mathcal{P} = \{p \in C([0,1], H_0^1(\Omega)); \; p(0) = u_0, \; p(1) = v_0\}$$

and

$$c_\varepsilon = \inf_{p \in \mathcal{P}} \max_{t \in [0,1]} J_\varepsilon(p(t)).$$

Lemma 10. *c_ε is uniformly bounded.*

Proof. The fact that J_ε increases with ε implies that $c_0 \leq c_\varepsilon \leq c_{\varepsilon_0}$, for every $0 \leq \varepsilon \leq \varepsilon_0$. □

Proof of Theorem 14 v) continued For every $\varepsilon \in (0, \varepsilon_0]$, let $v_\varepsilon \in H_0^1(\Omega)$ such that

$$-\Delta v_\varepsilon = \frac{\lambda}{1+\varepsilon}f(v_\varepsilon) + \frac{\lambda\varepsilon}{1+\varepsilon}f(u_0) \tag{2.35}$$

and

$$J_\varepsilon(v_\varepsilon) = c_\varepsilon\,. \tag{2.36}$$

By Lemmas 9, 10 and from (2.36) it follows that there exists $v \in H_0^1(\Omega)$ such that

$$v_\varepsilon \to v \quad \text{in} \quad H_0^1(\Omega), \quad \text{if} \quad \varepsilon \searrow 0\,.$$

Taking into account the relation (2.35) we obtain

$$-\Delta v = \lambda f(v)\,.$$

It remains to prove that $v \neq u_0$ and $v \in C^2(\Omega) \cap C(\overline{\Omega})$.

We first remark that v_ε is a solution of the equation (2.35), which is different from u_0, so it is unstable, that is

$$\lambda_1(\frac{\lambda}{1+\varepsilon} f'(v_\varepsilon)) \leq 0\,.$$

Indeed, (2.35) is an equation of the form

$$-\Delta u = g(u) + h(x),$$

where g as convex and positive, while h is positive. Therefore, by the results established in Brezis-Nirenberg [21], if this equation has a solution, then it has also a minimal solution u, with $\lambda_1(g'(u)) \geq 0$. By the proof of Lemma 3 iv) it follows that if v is another solution, then $\lambda_1(g'(v)) < 0$. In our case, u_0 plays the role of u, while v_ε plays the role of v. It remains to prove now that the limit of a sequence of unstable solutions is unstable, too.

Lemma 11. *Let $u_n \to u$ in $H_0^1(\Omega)$ and $\mu_n \to \mu$ such that $\lambda_1(\mu_n f'(u_n)) \leq 0$.*
Then $\lambda_1(\mu f'(u)) \leq 0$.

Proof of the lemma. The fact that $\lambda_1(\alpha) \leq 0$ is equivalent to the existence of some $\varphi \in H_0^1(\Omega)$ such that

$$\int_\Omega |\nabla\varphi|^2 \leq \int_\Omega \alpha\varphi^2 \quad \text{and} \quad \int_\Omega \varphi^2 = 1\,.$$

This assertion follows from the Courant-Hilbert Principle.

Let $\varphi_n \in H_0^1(\Omega)$ be such that

$$\int_\Omega |\nabla\varphi_n|^2 \leq \int_\Omega \mu_n f'(u_n)\varphi_n^2 \tag{2.37}$$

and

$$\int_\Omega \varphi_n^2 = 1\,. \tag{2.38}$$

Since $f' \leq a$, we obtain by the relation (2.37) that (φ_n) is bounded in $H_0^1(\Omega)$. Let $\varphi \in H_0^1(\Omega)$ such that, up to a subsequence,

$$\varphi_n \rightharpoonup \varphi \quad \text{weakly in} \quad H_0^1(\Omega)$$

$$\varphi_n \to \varphi \quad \text{strongly in} \quad L^2(\Omega)\,.$$

So, again up to a subsequence, the right hand side of (2.37) converges to $\mu \int_\Omega f'(u)\varphi^2$. This assertion can be proved by extracting from (φ_n) a subsequence dominated in $L^2(\Omega)$, as in the proof of Theorem IV.9 in Brezis [18]. By the relations

$$\int_\Omega \varphi^2 = 1 \quad \text{and} \quad \int_\Omega |\nabla\varphi|^2 \le \liminf_{n\to\infty} \int_\Omega |\nabla\varphi_n|^2$$

our conclusion follows. □

The fact that $v \in C^2(\Omega) \cap C(\overline{\Omega})$ follows by a standard bootstrap argument:

$$v \in H_0^1(\Omega) \Longrightarrow f(v) \in L^{2^*}(\Omega) \Longrightarrow v \in W^{2,2^*}(\Omega) \Longrightarrow \ldots$$

We point out that the main tools are:

a) if $v \in L^p(\Omega)$, then $f(v) \in L^p(\Omega)$.

b) a standard regularity result for elliptic equations (Theorem 9.15 in Gilbarg-Trudinger [42]).

c) the Sobolev embeddings.

vi) Assume the contrary. Thus, there exists $\mu_n \to \frac{\lambda_1}{a}$ and a sequence (v_n) of unstable solutions of the problem (2.2) corresponding to $\lambda = \mu_n$, as well as $v \in L^1_{\mathrm{loc}}(\Omega)$ such that $v_n \to v$ in $L^1_{\mathrm{loc}}(\Omega)$.

We first prove that (v_n) is unbounded in $H_0^1(\Omega)$. Indeed, if not, Let $w \in H_0^1(\Omega)$ such that, up to a subsequence,

$$v_n \rightharpoonup w \quad \text{weakly in} \quad H_0^1(\Omega)\,,$$
$$v_n \to w \quad \text{in} \quad L^2(\Omega)\,.$$

Therefore

$$-\Delta v_n \to -\Delta w \quad \text{in} \quad \mathcal{D}'(\Omega)$$

and

$$f(v_n) \to f(w) \quad \text{in} \quad L^2(\Omega),$$

which means that

$$-\Delta w = \frac{\lambda_1}{a} f(w)\,.$$

It follows that $w \in C^2(\Omega) \cap C(\overline{\Omega})$, that is w is solution of the problem (2.2).

We prove now that

$$\lambda_1\Big(\frac{\lambda_1}{a} f'(w)\Big) \le 0\,. \tag{2.39}$$

Indeed, since v_n is an unstable solution, it is sufficient to show that $v_n \to w$ in $H_0^1(\Omega)$ and to apply then Lemma 11. But

$$\mu_n f(v_n) \to \frac{\lambda_1}{a} f(w) \quad \text{in} \quad H^{-1}(\Omega)$$

(because the convergence holds in $L^2(\Omega)$).

The operator $-\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$ is bicontinuous, so (2.39) holds. Hence $w \neq u(\frac{\lambda_1}{a})$, which contradicts iv).

The fact that (v_n) is unbounded in $H_0^1(\Omega)$ implies its unboundedness in $L^2(\Omega)$. Indeed, we have observed that the boundedness in $L^2(\Omega)$ implies the boundedness in $H_0^1(\Omega)$. Let $v_n = k_n w_n$ with $k_n > 0$, $\displaystyle\int_\Omega w_n^2 = 1$ and, up to a subsequence, $k_n \to \infty$. Hence

$$-\Delta w_n = \frac{\mu_n}{k_n} f(u_n) \to 0 \quad \text{in} \quad L^1_{\text{loc}}(\Omega),$$

so also in $\mathcal{D}'(\Omega)$. The sequence (w_n) may be assumed to be bounded, with an argument already done. Let w be its weak limit in $H_0^1(\Omega)$. It follows that

$$-\Delta w = 0 \quad \text{and} \quad \int_\Omega w^2 = 1,$$

which is the desired contradiction.

vii) As above, it is enough to prove the $L^2(\Omega)$-boundedness of $v(\lambda)$ for λ in a neighbourhood of λ^* and to apply the uniqueness property of u^*. Assuming the contrary, let $\mu_n \to \lambda^*$ and v_n be a solution of the problem (2.2) for $\lambda = \mu_n$, with $\|v_n\|_{L^2(\Omega)} \to \infty$. Writing, as above, $v_n = k_n w_n$, it follows that

$$-\Delta w_n = \frac{\mu_n}{k_n} f(u_n)\,.$$

The right hand side of this relation is bounded in $L^2(\Omega)$, so (w_n) is bounded in $H_0^1(\Omega)$. Let $w \in H_0^1(\Omega)$ such that, up to a subsequence,

$$w_n \rightharpoonup w \quad \text{weakly in} \quad H_0^1(\Omega)$$

$$w_n \to w \quad \text{strongly in} \quad L^2(\Omega)\,.$$

With an argument already done it follows that

$$-\Delta w = \lambda^* a w, \quad w \geq 0, \quad \int_\Omega w^2 = 1\,.$$

This forces the equality $\lambda^* = \frac{\lambda_1}{a}$, which yields a contradiction. $\square$

Chapter 3

Nonsmooth Mountain-Pass Theory

3.1 Basic properties of locally Lipschitz functionals

Throughout this chapter, X will denote a real Banach space. Let X^* be its dual and, for every $x \in X$ and $x^* \in X^*$, let $\langle x^*, x\rangle$ be the duality pairing between X^* and X.

Definition 3. *A functional $f : X \to \mathbb{R}$ is said to be locally Lipschitz provided that, for every $x \in X$, there exists a neighbourhood V of x and a positive constant $k = k(V)$ depending on V such that*

$$|f(y) - f(z)| \leq k||y - z||,$$

for each $y, z \in V$.

The set of all locally Lipschitz mappings defined on X with real values will be denoted by $\mathrm{Lip}_{loc}(X, \mathbb{R})$.

Definition 4. *Let $f : X \to \mathbb{R}$ be a locally Lipschitz functional and $x, v \in X$. We call the generalized directional derivative of f in x with respect to the direction v the number*

$$f^0(x, v) = \limsup_{\substack{y \to x \\ \lambda \searrow 0}} \frac{f(y + \lambda v) - f(y)}{\lambda}\,.$$

It is obvious that if f is a locally Lipschitz functional, then $f^0(x, v)$ is a finite number and

$$|f^0(x, v)| \leq k||v||\,. \tag{3.1}$$

Moreover, if $x \in X$ is fixed, then the mapping $v \longmapsto f^0(x, v)$ is positive homogeneous and subadditive, so convex continuous. By the Hahn-Banach theorem, there exists a linear map $x^* : X \to \mathbb{R}$ such that for every $v \in X$,

$$x^*(v) \leq f^0(x, v)\,.$$

The continuity of x^* is an immediate consequence of the above inequality and of (3.1).

Definition 5. *Let $f : X \to \mathbb{R}$ be a locally Lipschitz functional and $x \in X$. The generalized gradient (Clarke subdifferential) of f at the point x is the nonempty subset $\partial f(x)$ of X^* which is defined by*

$$\partial f(x) = \{x^* \in X^*;\ f^0(x,v) \geq \langle x^*, v\rangle, \quad \textit{for all} \ \ v \in X\}\,.$$

We also point out that if f is convex then $\partial f(x)$ coincides with the subdifferential of f in x in the sense of the convex analysis, that is

$$\partial f(x) = \{x^* \in X^*;\ f(y) - f(x) \geq \langle x^*, y - x\rangle, \quad \text{for all} \ \ y \in X\}\,.$$

We list in what follows the main properties of the Clarke gradient of a locally Lipschitz functional. We refer to [26], [27], [24] for further details and proofs.

a) For every $x \in X$, $\partial f(x)$ is a convex and $\sigma(X^*, X)$-compact set.

b) For every $x, v \in X$ the following holds

$$f^0(x,v) = \max\{\langle x^*, v\rangle;\ x^* \in \partial f(x)\}\,.$$

c) The multivalued mapping $x \longmapsto \partial f(x)$ is upper semicontinuous, in the sense that for every $x_0 \in X,\ \varepsilon > 0$ and $v \in X$, there exists $\delta > 0$ such that, for any $x^* \in \partial f(x)$ satisfying $||x - x_0|| < \delta$, there is some $x_0^* \in \partial f(x_0)$ satisfying $|\langle x^* - x_0^*, v\rangle| < \varepsilon$.

d) The functional $f^0(\cdot, \cdot)$ is upper semicontinuous.

e) If x is an extremum point of f, then $0 \in \partial f(x)$.

f) The mapping

$$\lambda(x) = \min_{x^* \in \partial f(x)} ||x^*||$$

exists and is lower semicontinuous.

g) $\quad \partial(-f)(x) = -\partial f(x)$.

h) Lebourg's Mean Value Theorem: if x and y are two distinct point in X then there exists a point z situated in the open segment joining x and y such that

$$f(y) - f(x) \in \langle \partial f(z), y - x\rangle$$

i) If f has a Gâteaux derivative f' which is continuous in a neighbourhood of x, then $\partial f(x) = \{f'(x)\}$. If X is finite dimensional, then $\partial f(x)$ reduces at one point if and only if f is Fréchet-differentiable at x.

Definition 6. *A point $x \in X$ is said to be a critical point of the locally Lipschitz functional $f : X \to \mathbb{R}$ if $0 \in \partial f(x)$, that is $f^0(x,v) \geq 0$, for every $v \in X$. A number c is a critical value of f provided that there exists a critical point $x \in X$ such that $f(x) = c$.*

Remark that a minimum point is a critical point. Indeed, if x is a local minimum point, then for every $v \in X$,

$$0 \leq \limsup_{\lambda \searrow 0} \frac{f(x+\lambda v) - f(x)}{\lambda} \leq f^0(x, v)\,.$$

We now introduce a compactness condition for locally Lipschitz functionals. This condition was used for the first time, in the case of C^1-functionals, by R. Palais and S. Smale (in the global variant) and by H. Brezis, J.M. Coron and L. Nirenberg (in its local variant).

Definition 7. *If $f : X \to \mathbb{R}$ is a locally Lipschitz functional and c is a real number, we say that f satisfies the Palais-Smale condition at the level c (in short, $(\mathrm{PS})_c$) if any sequence (x_n) in X satisfying $f(x_n) \longrightarrow c$ and $\lambda(x_n) \longrightarrow 0$, contains a convergent subsequence. The mapping f satisfies the Palais-Smale condition (in short, (PS)) if every sequence (x_n) which satisfies $(f(x_n))$ is bounded and $\lambda(x_n) \longrightarrow 0$, has a convergent subsequence.*

3.2 Ekeland's Variational Principle

Theorem 15. *(Ekeland) Let (M, d) be a complete metric space and let $\psi : M \to (-\infty, +\infty]$, $\psi \not\equiv +\infty$, be a lower semicontinuous function which is bounded from below. Then the following hold: for every $\varepsilon > 0$ and for any $z_0 \in M$ there exists $z \in M$ such that*

(i) $\psi(z) \leq \psi(z_0) - \varepsilon\, d(z, z_0)$;

(ii) $\psi(x) \geq \psi(z) - \varepsilon\, d(x, z)$, for any $x \in M$.

Proof. We may assume without loss of generality that $\varepsilon = 1$. Define the following binary relation on M:

$$y \leq x \qquad \text{if and only if} \qquad \psi(y) - \psi(x) + d(x, y) \leq 0\,.$$

We verify obviously that "$\leq$" is an order relation. For arbitrary $x \in M$, set

$$S(x) = \{y \in M; y \leq x\}\,.$$

Let (ε_n) be a sequence of positive numbers such that $\varepsilon_n \to 0$ and fix $z_0 \in M$. For any $n \geq 0$, let $z_{n+1} \in S(z_n)$ be such that

$$\psi(z_{n+1}) \leq \inf_{S(z_n)} \psi + \varepsilon_{n+1}\,.$$

The existence of z_{n+1} follows by the definition of the set $S(x)$. We will prove that the sequence (z_n) converges to some element z which satisfies (i) and (ii).

Let us first remark that $S(y) \subset S(x)$, provided that $y \leq x$. Hence, $S(z_{n+1}) \subset S(z_n)$. It follows that, for any $n \geq 0$,

$$\psi(z_{n+1}) - \psi(z_n) + d(z_n, z_{n+1}) \leq 0\,,$$

which implies $\psi(z_{n+1}) \leq \psi(z_n)$. Since ψ is bounded from below, it follows that the sequence $\{\psi(z_n)\}$ converges.

We prove in what follows that (z_n) is a Cauchy sequence. Indeed, for any n and p we have

$$\psi(z_{n+p}) - \psi(z_n) + d(z_{n+p}, z_n) \leq 0\,. \tag{3.2}$$

Therefore

$$d(z_{n+p}, z_n) \leq \psi(z_n) - \psi(z_{n+p}) \to 0\,, \quad \text{as } n \to \infty\,,$$

which shows that (z_n) is a Cauchy sequence, so it converges to some $z \in M$. Now, taking $n = 0$ in (3.2), we find

$$\psi(z_p) - \psi(z_0) + d(z_p, z_0) \leq 0\,.$$

So, as $p \to \infty$, we find (i).

In order to prove (ii), let us choose an arbitrary $x \in M$. There are two cases:

- $x \in S(z_n)$, for any $n \geq 0$. It follows that $\psi(z_{n+1}) \leq \psi(x) + \varepsilon_{n+1}$ which implies that $\psi(z) \leq \psi(x)$.
- there exists an integer $N \geq 1$ such that $x \notin S(z_n)$, for any $n \geq N$ or, equivalently,

$$\psi(x) - \psi(z_n) + d(x, z_n) > 0\,, \qquad \text{for every } n \geq N\,.$$

Passing at the limit in this inequality as $n \to \infty$ we find (ii). □

Corollary 2. *Assume the same hypotheses on M and ψ. Then, for any $\varepsilon > 0$, there exists $z \in M$ such that*

$$\psi(z) < \inf_M \psi + \varepsilon$$

and

$$\psi(x) \geq \psi(z) - \varepsilon\, d(x, z)\,, \qquad \textit{for any } x \in M\,.$$

The conclusion follows directly from Theorem 15.

The following will be of particular interest in our next arguments.

Corollary 3. *Let E be a Banach space and let $\psi : E \to \mathbb{R}$ be a C^1 function which is bounded from below. Then, for any $\varepsilon > 0$ there exists $z \in E$ such that*

$$\psi(z) \leq \inf_E \psi + \varepsilon \qquad \textit{and} \qquad \|\psi'(z)\|_{E^\star} \leq \varepsilon\,.$$

Proof. The first part of the conclusion follows directly from Theorem 15. For the second part we have

$$\|\psi'(z)\|_{E^\star} = \sup_{\|u\|=1} \langle \psi'(z), u \rangle\,.$$

But

$$\langle \psi'(z), u \rangle = \lim_{\delta \to 0} \frac{\psi(z + \delta u) - \psi(z)}{\delta \|u\|}\,.$$

So, by Theorem 15,

$$\langle \psi'(z), u \rangle \geq -\varepsilon\,.$$

Replacing now u by $-u$ we find

$$\langle \psi'(z), u \rangle \leq \varepsilon \,,$$

which concludes our proof. □

We give in what follows a variant of Ekeland's Theorem, whose proof use in an essential manner the fact that the dimension of the space is finite.

Theorem 16. *Let $\psi : \mathbb{R}^N \to (-\infty, +\infty]$ be a lower semicontinuous function, $\psi \not\equiv +\infty$, bounded from below. Let $x_\varepsilon \in \mathbb{R}^N$ be such that*

$$\inf \psi \leq \psi(x_\varepsilon) \leq \inf \psi + \varepsilon \,. \tag{3.3}$$

Then, for every $\lambda > 0$, there exists $z_\varepsilon \in \mathbb{R}^N$ such that

(i) $\psi(z_\varepsilon) \leq \psi(x_\varepsilon)$;

(ii) $\|z_\varepsilon - x_\varepsilon\| \leq \lambda$;

(iii) $\psi(z_\varepsilon) \leq \psi(x) + \frac{\varepsilon}{\lambda} \|z_\varepsilon - x\|$, for every $x \in \mathbb{R}^N$.

Proof. Given x_ε satisfying (3.3), let us consider the function $\varphi : \mathbb{R}^N \to (-\infty, +\infty]$ defined by

$$\varphi(x) = \psi(x) + \frac{\varepsilon}{\lambda} \|x - x_\varepsilon\| \,.$$

By our hypotheses on ψ it follows that φ is lower semicontinuous and bounded from below. Moreover, $\varphi(x) \to +\infty$ as $\|x\| \to +\infty$. Therefore there exists $z_\varepsilon \in \mathbb{R}^N$ which minimizes φ on $\mathbb{R}^N$, that is, for every $x \in \mathbb{R}^N$,

$$\psi(z_\varepsilon) + \frac{\varepsilon}{\lambda} \|z_\varepsilon - x_\varepsilon\| \leq \psi(x) + \frac{\varepsilon}{\lambda} \|x - x_\varepsilon\| \,. \tag{3.4}$$

By letting $x = x_\varepsilon$ we find

$$\psi(z_\varepsilon) + \frac{\varepsilon}{\lambda} \|z_\varepsilon - x_\varepsilon\| \leq \psi(x_\varepsilon) \,,$$

and (i) follows. Now, since $\psi(x_\varepsilon) \leq \inf \psi + \varepsilon$, we clearly deduce from the above that $\|z_\varepsilon - x_\varepsilon\| \leq \lambda$.

We infer from (3.4) that, for every $x \in \mathbb{R}^N$,

$$\psi(z_\varepsilon) \leq \psi(x) + \frac{\varepsilon}{\lambda} \left(\|x - x_\varepsilon\| - \|z_\varepsilon - x_\varepsilon\|\right) \leq \psi(x) + \frac{\varepsilon}{\lambda} \|x - z_\varepsilon\|$$

which is exactly the desired inequality (iii). □

The above result shows that, the closer to x_ε we desire z_ε to be, the larger the perturbation of ψ that must be accepted. In practise, a good compromise is to take $\lambda = \sqrt{\varepsilon}$.

It is striking to remark that the Ekeland Variational Principle characterizes the completeness of a metric space in a certain sense. More precisely we have

Theorem 17. *Let (M, d) be a metric space. Then M is complete if and only if the following holds: for every application $\psi : M \to (-\infty, +\infty]$, $\psi \not\equiv +\infty$, which is bounded from below and for every $\varepsilon > 0$, there exists $z_\varepsilon \in M$ such that*

(i) $\psi(z_\varepsilon) \leq \inf_M \psi + \varepsilon$

and

(ii) $\psi(z) > \psi(z_\varepsilon) - \varepsilon\, d(z, z_\varepsilon)$, for any $z \in M \setminus \{z_\varepsilon\}$.

Proof. The "only if" part follows directly from Corollary 2.

For the converse, let us assume that M is an arbitrary metric space satisfying the hypotheses. Let $(v_n) \subset M$ be an arbitrary Cauchy sequence and consider the function $f : M \to \mathbb{R}$ defined by

$$f(u) = \lim_{n\to\infty} d(u, v_n)\,.$$

the function f is continuous and, since (v_n) is a Cauchy sequence, then $\inf f = 0$. In order to justify the completeness of M it is enough to find $v \in M$ such that $f(v) = 0$. For this aim, choose an arbitrary $\varepsilon \in (0, 1)$. Now, from our hypotheses (i) and (ii), there exists $v \in M$ with $f(v) \le \varepsilon$ and

$$f(w) + \varepsilon\, d(w, v) > f(v)\,, \qquad \text{for any } \ w \in M \setminus \{v\}\,.$$

From the definition of f and the fact that (v_n) is a Cauchy sequence we can take $w = v_k$ for k large enough such that $f(w)$ is arbitrarily small and $d(w, v) \le \varepsilon + \eta$, for any $\eta > 0$, because $f(v) \le \varepsilon$. Using (ii) we obtain that, in fact, $f(v) \le \varepsilon^k$. Repeating the argument we may conclude that $f(v) \le \varepsilon^n$, for all $n \ge 1$ and so $f(v) = 0$, as required. □

3.3 Mountain Pass and Saddle Point type theorems

Let $f : X \to \mathbb{R}$ be a locally Lipschitz functional. Consider K a compact metric space and K^* a closed nonempty subset of K. If $p^* : K^* \to X$ is a continuous mapping, set

$$\mathcal{P} = \{p \in C(K, X);\ p = p^* \ \text{ on } \ K^*\}\,.$$

By a celebrated theorem of Dugundji [34], the set $\mathcal{P}$ is nonempty.

Define

$$c = \inf_{p\in\mathcal{P}} \max_{t\in K} f(p(t))\,. \tag{3.5}$$

Obviously, $c \ge \max_{t\in K^*} f(p^*(t))$.

Theorem 18. *Assume that*

$$c > \max_{t\in K^*} f(p^*(t))\,. \tag{3.6}$$

Then there exists a sequence (x_n) in X such that

i) $\lim_{n\to\infty} f(x_n) = c$;

ii) $\lim_{n\to\infty} \lambda(x_n) = 0$.

For the proof of this theorem we need the following auxiliary result:

Lemma 12. *Let M be a compact metric space and let $\varphi : M \to 2^{X^*}$ be an upper semicontinuous mapping such that, for every $t \in M$, the set $\varphi(t)$ is convex and $\sigma(X^*, X)$-compact. For $t \in M$, denote*

$$\gamma(t) = \inf\{\ ||x^*||;\ \ x^* \in \varphi(t)\}$$

and

$$\gamma = \inf_{t\in M} \gamma(t)\,.$$

Then, for every fixed $\varepsilon > 0$, there exists a continuous mapping $v : M \to X$ such that for every $t \in M$ and $x^* \in \varphi(t)$,

$$||v(t)|| \le 1 \quad \textit{and} \quad \langle x^*, v(t)\rangle \ge \gamma - \varepsilon\,.$$

Proof of Lemma. Assume, without loss of generality, that $\gamma > 0$ and $0 < \varepsilon < \gamma$. Denoting by B_r the open ball in X^* centered at the origin and with radius r, then, for every $t \in M$ we have

$$B_{\gamma-\frac{\varepsilon}{2}} \cap \varphi(t) = \emptyset$$

Since $\varphi(t)$ and $B_{\gamma-\frac{\epsilon}{2}}$ are convex, disjoint and $\sigma(X^*, X)$-compact sets, it follows from the separation theorem in locally convex spaces (Theorem 3.4 in [77]), applied to the space $(X^*, \sigma(X^*, X))$, and from the fact that the dual of this space is X, that: for every $t \in M$, there exists $v_t \in X$ such that

$$||v_t|| = 1 \quad \text{and} \quad \langle \xi, v_t\rangle \le \langle x^*, v_t\rangle\,,$$

for any $\xi \in B_{\gamma-\frac{\epsilon}{2}}$ and for every $x^* \in \varphi(t)$.

Hence, for each $x^* \in \varphi(t)$,

$$\langle x^*, v_t\rangle \ge \sup_{\xi\in B_{\gamma-\frac{\varepsilon}{2}}} \langle \xi, v_t\rangle = \gamma - \frac{\varepsilon}{2}\,.$$

Since φ is upper semicontinuous, there exists an open neighbourhood $V(t)$ of t such that for every $t' \in V(t)$ and $x^* \in \varphi(t')$,

$$\langle x^*, v_t\rangle > \gamma - \varepsilon\,.$$

Therefore, since M is compact and $M = \bigcup_{t\in M} V(t)$, there exists an open covering $\{V_1, ..., V_n\}$ of M. Let $v_1, ..., v_n$ be on the unit sphere of X such that

$$\langle x^*, v_i\rangle > \gamma - \varepsilon,$$

for every $1 \le i \le n$, $t \in V_i$ and $x^* \in \varphi(t)$.

If $\rho_i(t) = \operatorname{dist}(t, \partial V_i)$, define

$$\zeta_i(t) = \frac{\rho_i(t)}{\sum_{j=1}^n \rho_j(t)} \quad \text{and} \quad v(t) = \sum_{i=1}^n \zeta_i(t) v_i$$

It follows easily that v satisfies our conditions. $\square$

Proof of Theorem 18 We apply Ekeland's Principle to

$$\psi(p) = \max_{t\in K} f(p(t)),$$

defined on $\mathcal{P}$, which is a complete metric space if it is endowed with the usual metric. The mapping ψ is continuous and bounded from below because, for every $p \in \mathcal{P}$,

$$\psi(p) \ge \max_{t\in K^*} f(p^*(t))$$

Since

$$c = \inf_{p \in \mathcal{P}} \psi(p),$$

it follows that for every $\varepsilon > 0$, there is some $p \in \mathcal{P}$ such that

$$\psi(q) - \psi(p) + \varepsilon d(p, q) \geq 0, \quad \text{for all} \;\; q \in \mathcal{P}\,; \tag{3.7}$$

$$c \leq \psi(p) \leq c + \varepsilon\,.$$

Set

$$B(p) = \{t \in K;\; f(p(t)) = \psi(p)\}\,.$$

For concluding the proof it is sufficient to show that there exists $t' \in B(p)$ such that

$$\lambda(p(t')) \leq 2\varepsilon\,.$$

Indeed the conclusion of the theorem follows then easily by choosing $\varepsilon = \frac{1}{n}$ and $x_n = p(t')$.

Applying Lemma 12 for $M = B(p)$ and $\varphi(t) = \partial f(p(t))$, we obtain a continuous map $v : B(p) \to X$ such that, for every $t \in B(p)$ and $x^* \in \partial f(p(t))$, we have

$$||v(t)|| \leq 1 \quad \text{and} \quad \langle x^*, v(t)\rangle \geq \gamma - \varepsilon\,,$$

where

$$\gamma = \inf_{t \in B(p)} \lambda(p(t))\,.$$

It follows that for every $t \in B(p)$,

$$f^0(p(t), -v(t)) = \max\{\langle x^*, -v(t)\rangle;\; x^* \in \partial f(p(t))\} =$$

$$= -\min\{\langle x^*, v(t);\; x^* \in \partial f(p(t))\} \leq -\gamma + \varepsilon\,.$$

By (3.6) we have $B(p) \cap K^* = \emptyset$. So, there exists a continuous extension $w : K \to X$ of v such that $w = 0$ pe K^* and, for every $t \in K$,

$$||w(t)|| \leq 1\,.$$

Choose in the place of q in (3.7) small perturbations of the path p:

$$q_h(t) = p(t) - hw(t),$$

where $h > 0$ is small enough.

It follows from (3.7) that, for every $h > 0$,

$$-\varepsilon \leq -\varepsilon||w||_\infty \leq \frac{\psi(q_h) - \psi(p)}{h}\,. \tag{3.8}$$

In what follows, $\varepsilon > 0$ will be fixed, while $h \to 0$. Let $t_h \in K$ be such that $f(q_h(t_h)) = \psi(q_h)$. We may also assume that the sequence (t_{h_n}) converges to some t_0, which, obviously, is in $B(p)$. Observe that

$$\frac{\psi(q_h) - \psi(p)}{h} = \frac{\psi(p - hw) - \psi(p)}{h} \leq \frac{f(p(t_h) - hw(t_h)) - f(p(t_h))}{h}\,.$$

It follows from this relation and from (3.8) that

$$-\varepsilon \leq \frac{f(p(t_h)-hw(t_h))-f(p(t_h))}{h} \leq$$

$$\leq \frac{f(p(t_h)-hw(t_0))-f(p(t_h))}{h} + \frac{f(p(t_h)-hw(t_h))-f(p(t_h)-hw(t_0))}{h}.$$

Using the fact that f is a locally Lipschitz map and $t_{h_n} \to t_0$, we find that

$$\lim_{n\to\infty} \frac{f(p(t_{h_n})-h_n w(t_{h_n}))-f(p(t_{h_n})-h_n w(t_0))}{h_n} = 0\,.$$

Therefore

$$-\varepsilon \leq \limsup_{n\to\infty} \frac{f(p(t_0)+z_n-h_n w(t_0))-f(p(t_0)+z_n)}{h_n},$$

where $z_n = p(t_{h_n}) - p(t_0)$. Consequently,

$$-\varepsilon \leq f^0(p(t_0), -w(t_0)) = f^0(p(t_0), -v(t_0)) \leq -\gamma + \varepsilon\,.$$

It follows that

$$\gamma = \inf\{||x^*||;\ x^* \in \partial f(p(t)), t \in B(p)\} \leq 2\varepsilon\,.$$

Taking into account the lower semicontinuity of λ, we find the existence of some $t' \in B(p)$ such that

$$\lambda(p(t')) = \inf\{||x^*||;\ x^* \in \partial f(p(t'))\} \leq 2\varepsilon\,. \qquad \blacksquare$$

Corollary 4. *If f satisfies the condition* $(\mathrm{PS})_c$ *and the hypotheses of Theorem 18, then c is a critical value of f corresponding to a critical point which is not in $p^*(K^*)$.*

The proof of this result follows easily by applying Theorem 18 and the fact that λ is lower semicontinuous. $\square$

The following result generalizes the classical Ambrosetti-Rabinowitz Theorem.

Corollary 5. *Let $f : X \to \mathbb{R}$ be a locally Lipschitz functional such that $f(0) = 0$ and there exists $v \in X \setminus \{0\}$ so that $f(v) \leq 0$. Set*

$$\mathcal{P} = \{p \in C([0,1], X);\ p(0) = 0 \ \ \text{and} \ \ p(1) = v\}$$

$$c = \inf_{p\in\mathcal{P}} \max_{t\in[0,1]} f(p(t))\,.$$

If $c > 0$ and f satisfies the condition $(\mathrm{PS})_c$*, then c is a critical value of f.*

For the proof of this result it is sufficient to apply Corollary 4 for $K = [0,1]$, $K^* = \{0,1\}$, $p^*(0) = 0$ and $p^*(1) = v$. $\square$

Corollary 6. *Let $f : X \to \mathbb{R}$ be a locally Lipschitz mapping. Assume that there exists a subset S of X such that, for every $p \in \mathcal{P}$,*

$$p(K) \cap S \neq \emptyset\,.$$

If

$$\inf_{x \in S} f(x) > \max_{t \in K^*} f(p^*(t)),$$

then the conclusion of Theorem 18 holds.

Proof. It suffices to observe that

$$\inf_{p \in \mathcal{P}} \max_{t \in K} f(p(t)) \geq \inf_{x \in S} f(x) > \max_{t \in K^*} f(p^*(t))\,.$$

■

Using now Theorem 18 we may prove the following result, which is originally due to Brezis-Coron-Nirenberg (see Theorem 2 in [19]):

Corollary 7. *Let $f : X \to \mathbb{R}$ be a Gâteaux differentiable functional such that $f' : (X, ||\cdot||) \to (X^*, \sigma(X^*, X))$ is continuous. If f satisfies (3.6), then there exists a sequence (x_n) in X such that*

i) $\lim_{n \to \infty} f(x_n) = c\,;$

ii) $\lim_{n \to \infty} \|f'(x_n)\| = 0\,.$

Moreover, if f satisfies the condition $(\mathrm{PS})_c$*, then there exists $x \in X$ such that $f(x) = c$ and $f'(x) = 0$.*

Proof. Observe first that f' is locally bounded. Indeed, if (x_n) is a sequence converging to x_0, then

$$\sup_n |\langle f'(x_n), v\rangle| < \infty,$$

for every $v \in X$. Thus, by the Banach-Steinhaus Theorem,

$$\limsup_{n \to \infty} ||f'(x_n)|| < \infty$$

For $\lambda > 0$ small enough and $h \in X$ sufficiently small we have

$$|f(x_0 + h + \lambda v) - f(x_0 + h)| = |\lambda \langle f'(x_0 + h + \lambda\theta v), v\rangle| \leq C||\lambda v||\,, \tag{3.9}$$

where $\theta \in (0, 1)$. Therefore $f \in \mathrm{Lip}_{loc}(X, \mathbb{R})$ and $f^0(x_0, v) = \langle f'(x_0), v\rangle$, by the continuity assumption on f'. In 3.9 the existence of C follows from the local boundedness property of f'.

Since f^0 is linear in v, we get

$$\partial f(x) = \{f'(x)\}$$

and, for concluding the proof, it remains to apply just Theorem 18 and Corollary 4. □

A very useful result in applications is the following variant of the "Saddle Point" Theorem of P.H. Rabinowitz.

Corollary 8. *Assume that X admits a decomposition of the form $X = X_1 \oplus X_2$, where X_2 is a finite dimensional subspace of X. For some fixed $z \in X_2$, suppose that there exists $R > ||z||$ such that*

$$\inf_{x \in X_1} f(x+z) > \max_{x \in K^*} f(x)\,,$$

where

$$K^* = \{x \in X_2;\ ||x|| = R\}\,.$$

Set

$$K = \{x \in X_2;\ ||x|| \leq R\}$$

$$\mathcal{P} = \{p \in C(K, X);\ p(x) = x \quad if \quad ||x|| = R\}\,.$$

If c is chosen as in (3.5) and f satisfies the condition $(\mathrm{PS})_c$*, then c is a critical value of f.*

Proof. Applying Corollary 6 for $S = z + X_1$, we observe that it is sufficient to prove that, for every $p \in \mathcal{P}$,

$$p(K) \cap (z + X_1) \neq \emptyset\,.$$

If $P : X \to X_2$ is the canonical projection, then the above condition is equivalent to the fact that, for every $p \in \mathcal{P}$, there exists $x \in K$ such that

$$P(p(x) - z) = P(p(x)) - z = 0\,.$$

For proving this claim, we shall make use of an argument based on the topological degree theory. Indeed, for every fixed $p \in \mathcal{P}$ we have

$$P \circ p = \mathrm{Id} \quad \text{on } K^* = \partial K\,.$$

Hence

$$d(P \circ p - z, \mathrm{Int}\ K, 0) = d(P \circ p, \mathrm{Int}\ K, z) =$$

$$= d(\mathrm{Id}, \mathrm{Int}\ K, z) = 1\,.$$

Now, by the existence property of the Brouwer degree, we may find $x \in \mathrm{Int}\ K$ such that

$$(P \circ p)(x) - z = 0,$$

which concludes our proof. □

It is natural to ask us what happens if the condition (3.6) fails to be valid, more precisely, if

$$c = \max_{t \in K^*} f(p^*(t))\,.$$

The following example shows that in this case the conclusion of Theorem 18 does not hold.

Example 1. *Let $X = \mathbb{R}^2$, $K = [0,1] \times \{0\}$, $K^* = \{(0,0),(1,0)\}$ and let p^* be the identic map of K^*. As locally Lipschitz functional we choose*

$$f : X \to \mathbb{R}, \quad f(x,y) = x + |y|\,.$$

In this case,

$$c = \max_{t \in K^*} f(p^*(t)) = 1\,.$$

An elementary computation shows that

$$\partial f(x,y) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \text{ if } \; y > 0\,;$$

$$\partial f(x,y) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \text{ if } \; y < 0\,;$$

$$\partial f(x,0) = \left\{ \begin{pmatrix} 1 \\ a \end{pmatrix};\ \text{if } \; -1 \le a \le 1 \right\}.$$

It follows that f satisfies the condition Palais-Smale. However f has no critical point.

The following result gives a sufficient condition for that Theorem 18 holds provided that the condition (3.6) fails.

Theorem 19. *Assume that for every $p \in \mathcal{P}$ there exists $t \in K \setminus K^*$ such that $f(p(t)) \ge c$.*

Then there exists a sequence (x_n) in X such that

i) $\lim_{n\to\infty} f(x_n) = c\,;$

ii) $\lim_{n\to\infty} \lambda(x_n) = 0\,.$

Moreover, if f satisfies the condition $(\mathrm{PS})_c$*, then c is a critical value of f. Furthermore, if (p_n) is an arbitrary sequence in $\mathcal{P}$ satisfying*

$$\lim_{n\to\infty} \max_{t \in K} f(p_n(t)) = c,$$

then there exists a sequence (t_n) in K such that

$$\lim_{n\to\infty} f(p_n(t_n)) = c \quad \textit{and} \quad \lim_{n\to\infty} \lambda(p_n(t_n)) = 0\,.$$

Proof. For every $\varepsilon > 0$ we apply Ekeland's Principle to the perturbed functional

$$\psi_\varepsilon : \mathcal{P} \to \mathbb{R}, \quad \psi_\varepsilon(p) = \max_{t \in K}(f(p(t)) + \varepsilon d(t))\,,$$

where

$$d(t) = \min\{\mathrm{dist}(t, K^*), 1\}\,.$$

If

$$c_\varepsilon = \inf_{p \in \mathcal{P}} \psi_\varepsilon(p),$$

then

$$c \leq c_\varepsilon \leq c + \varepsilon\,.$$

Thus, by Ekeland's Principle, there exists a path $p \in \mathcal{P}$ such that, for every $q \in \mathcal{P}$,

$$\psi_\varepsilon(q) - \psi_\varepsilon(p) + \varepsilon d(p,q) \geq 0 \tag{3.10}$$

$$c \leq c_\varepsilon \leq \psi_\varepsilon(p) \leq c_\varepsilon + \varepsilon \leq c + 2\varepsilon\,.$$

Denoting

$$B_\varepsilon(p) = \{t \in K;\ f(p(t)) + \varepsilon d(t) = \psi_\varepsilon(p)\},$$

it remains to show that there is some $t' \in B_\varepsilon(p)$ such that $\lambda(p(t')) \leq 2\varepsilon$. Indeed, the conclusion of the first part of the theorem follows easily, by choosing $\varepsilon = \frac{1}{n}$ and $x_n = p(t')$.

Now, by Lemma 12 applied for $M = B_\varepsilon(p)$ and $\varphi(t) = \partial f(p(t))$, we find the existence of a continuous mapping $v : B_\varepsilon(p) \to X$ such that, for every $t \in B_\varepsilon(p)$ and $x^* \in \partial f(p(t))$,

$$||v(t)|| \leq 1 \quad \text{and} \quad \langle x^*, v(t)\rangle \geq \gamma_\varepsilon - \varepsilon,$$

where

$$\gamma_\varepsilon = \inf_{t \in B_\varepsilon(p)} \lambda(p(t))\,.$$

On the other hand, it follows by our hypothesis that

$$\psi_\varepsilon(p) > \max_{t \in K^*} f(p(t))\,.$$

Hence

$$B_\varepsilon(p) \cap K^* = \emptyset\,.$$

So, there exists a continuous extension w of v, defined on K and such that

$$w = 0 \quad \text{on} \quad K^* \qquad \text{and} \quad ||w(t)|| \leq 1, \quad \text{for any} \quad t \in K\,.$$

Choose as q in relation (3.10) small variations of the path p:

$$q_h(t) = p(t) - hw(t),$$

for $h > 0$ sufficiently small.

In what follows $\varepsilon > 0$ will be fixed, while $h \to 0$.

Let $t_h \in B_\varepsilon(p)$ be such that

$$f(q(t_h)) + \varepsilon d(t_h) = \psi_\varepsilon(q_h)\,.$$

There exists a sequence (h_n) converging to 0 and such that the corresponding sequence (t_{h_n}) converges to some t_0, which, obviously, lies in $B_\varepsilon(p)$. It follows that

$$-\varepsilon \leq -\varepsilon||w||_\infty \leq \frac{\psi_\varepsilon(q_h) - \psi_\varepsilon(p)}{h} = \frac{f(q_h(t_h)) + \varepsilon d(t_h) - \psi_\varepsilon(p)}{h} \leq$$

$$\leq \frac{f(q_h(t_h)) - f(p(t_h))}{h} = \frac{f(p(t_h) - hw(t_h)) - f(p(t_h))}{h}.$$

With the same arguments as in the proof of Theorem 18 we obtain the existence of some $t' \in B_\varepsilon(p)$ such that

$$\lambda(p(t')) \leq 2\varepsilon .$$

Furthermore, if f satisfies $(\mathrm{PS})_c$ then c is a critical value of f, since λ is lower semicontinuous.

For the second part of the proof, applying again Ekeland's Variational Principle, we deduce the existence of a sequence of paths (q_n) in $\mathcal{P}$ such that, for every $q \in \mathcal{P}$,

$$\psi_{\varepsilon_n^2}(q) - \psi_{\varepsilon_n^2}(q_n) + \varepsilon_n d(q, q_n) \geq 0 ;$$

$$\psi_{\varepsilon_n^2}(q_n) \leq \psi_{\varepsilon_n^2}(p_n) - \varepsilon_n d(p_n, q_n) ,$$

where (ε_n) is a sequence of positive numbers converging to 0 and (p_n) are paths in $\mathcal{P}$ such that

$$\psi_{\varepsilon_n^2}(p_n) \leq c + 2\varepsilon_n^2 .$$

Applying the same argument for q_n, instead of p, we find $t_n \in K$ such that

$$c - \varepsilon_n^2 \leq f(q_n(t_n)) \leq c + 2\varepsilon_n^2 ;$$

$$\lambda(q_n(t_n)) \leq 2\varepsilon_n .$$

We shall prove that this is the desired sequence (t_n). Indeed, by the Palais-Smale condition $(\mathrm{PS})_c$, there exists a subsequence of $(q_n(t_n))$ which converges to a critical point. The corresponding subsequence of $(p_n(t_n))$ converges to the same limit. A standard argument, based on the continuity of f and the lower semicontinuity of λ shows that for all the sequence we have

$$\lim_{n\to\infty} f(p_n(t_n)) = c$$

and

$$\lim_{n\to\infty} \lambda(p_n(t_n)) = 0 .$$

$\square$

Corollary 9. *Let $f : X \to \mathbb{R}$ be a locally Lipschitz functional which satisfies the Palais-Smale condition. If f has two different minimum points, then f possesses a third critical point.*

Proof. Let x_0 and x_1 be two different minimum points of f.

Case 1. $f(x_0) = f(x_1) = a$. Choose $0 < R < \frac{1}{2}\|x_1 - x_0\|$ such that $f(x) \geq a$, for all $x \in B(x_0, R) \cup B(x_1, R)$.

Set $A = \overline{B}(x_0, \frac{R}{2}) \cup \overline{B}(x_1, \frac{R}{2})$.

Case 2. $f(x_0) > f(x_1)$. Choose $0 < R < \|x_1 - x_0\|$ such that $f(x) \geq f(x_0)$, for every $x \in B(x_0, R)$. Put $A = \overline{B}(x_0, \frac{R}{2}) \cup \{x_1\}$.

In both cases, fix $p^* \in C([0,1],X)$ such that $p^*(0) = x_0$ and $p^*(1) = x_1$. If $K^* = (p^*)^{-1}(A)$ then, by Theorem 19, we obtain the existence of a critical point of f, which is different from x_0 and x_1, as we can easily deduce by examining the proof of Theorem 19. $\square$

With the same proof as of Corollary 7 one can show

Corollary 10. *Let X be a Banach space and let $f : X \to \mathbb{R}$ be a Gâteaux-differentiable functional such that the operator $f' : (X, \|\cdot\|) \to (X^*, \sigma(X^*, X))$ is continuous. Assume that for every $p \in \mathcal{P}$ there exists $t \in K \setminus K^*$ such that $f(p(t)) \geq c$.*

Then there exists a sequence (x_n) in X so that

i) $\lim_{n\to\infty} f(x_n) = c$;

ii) $\lim_{n\to\infty} \|f'(x_n)\| = 0$.

If, furthermore, f satisfies $(\mathrm{PS})_c$ *, then there exists $x \in X$ such that $f(x) = c$ and $f'(x) = 0$.*

The following result is a strengthened variant of Theorems 18 and 19.

Theorem 20. *Let $f : X \to \mathbb{R}$ be a locally Lipschitz functional and let F be a closed subset of X, with no common point with $p^*(K^*)$. Assume that*

$$f(x) \geq c, \quad \text{for every } \ x \in F \tag{3.11}$$

and

$$p(K) \cap F \neq \emptyset \ , \quad \text{for all} . p \in \mathcal{P} \tag{3.12}$$

Then there exists a sequence (x_n) in X such that

i) $\lim_{n\to\infty} \mathrm{dist}\ (x_n, F) = 0$;

ii) $\lim_{n\to\infty} f(x_n) = c$;

iii) $\lim_{n\to\infty} \lambda(x_n) = 0$.

Proof. Fix $\varepsilon > 0$ such that

$$\varepsilon < \min\{1;\ \mathrm{dist}\ (p^*(K^*), F)\}\,.$$

Choose $p \in \mathcal{P}$ so that

$$\max_{t\in K} f(p(t)) \leq c + \frac{\varepsilon^2}{4}\,.$$

The set

$$K_0 = \{t \in K;\ \mathrm{dist}\ (p(t), F) \geq \varepsilon\}$$

is bounded and contains K^*. Define

$$\mathcal{P}_0 = \{q \in C(K,X);\ q = p \ \text{ on } \ K_0\}\,.$$

Set

$$\eta : X \to \mathbb{R}, \quad \eta(x) = \max\{0;\ \varepsilon^2 - \varepsilon\ \mathrm{dist}\ (x, F)\}\,.$$

Define $\psi : \mathcal{P}_0 \to \mathbb{R}$ by

$$\psi(q) = \max_{t \in K} \, (f + \eta)(q(t)) \, .$$

The functional ψ is continuous and bounded from below. By Ekeland's Principle, there exists $p_0 \in \mathcal{P}_0$ such that, for every $q \in \mathcal{P}_0$,

$$\psi(p_0) \leq \psi(q) \, ,$$

$$d(p_0, q) \leq \frac{\varepsilon}{2} \, , \tag{3.13}$$

$$\psi(p_0) \leq \psi(q) + \frac{\varepsilon}{2} \, d(q, p_0) \, . \tag{3.14}$$

The set

$$B(p_0) = \{t \in K; \quad (f + \eta)(p_0(t)) = \psi(p_0)\}$$

is closed. For concluding the proof, it is sufficient to show that there exists $t \in B(p_0)$ such that

$$\text{dist}\,(p_0(t), F) \leq \frac{3\varepsilon}{2} \, , \tag{3.15}$$

$$c \leq f(p_0(t)) \leq c + \frac{5\varepsilon^2}{4} \, , \tag{3.16}$$

$$\lambda(p_0(t)) \leq \frac{5\varepsilon}{2}. \tag{3.17}$$

Indeed, it is enough to choose then $\varepsilon = \frac{1}{n}$ and $x_n = p_0(t)$.

Proof of (3.15): It follows by the definition of $\mathcal{P}_0$ and (3.12) that, for every $q \in \mathcal{P}_0$, we have

$$q(K \setminus K_0) \cap F \neq \emptyset \, .$$

Therefore, for any $q \in \mathcal{P}_0$,

$$\psi(q) \geq c + \varepsilon^2 \, .$$

On the other hand,

$$\psi(p) \leq c + \frac{\varepsilon^2}{4} + \varepsilon^2 = c + \frac{5\varepsilon^2}{4}.$$

Hence

$$c + \varepsilon^2 \leq \psi(p_0) \leq \psi(p) \leq c + \frac{5\varepsilon^2}{4} \, . \tag{3.18}$$

So, for each $t \in B(p_0)$,

$$c + \varepsilon^2 \leq \psi(p_0) = (f + \eta)(p_0(t)).$$

Moreover, if $t \in K_0$, then

$$(f + \eta)(p_0(t)) = (f + \eta)(p(t)) = f(p(t)) \leq c + \frac{\varepsilon^2}{4}.$$

This implies that

$$B(p_0) \subset K \setminus K_0.$$

By the definition of K_0 it follows that, for every $t \in B(p_0)$ we have

$$\text{dist } (p(t), F) \leq \varepsilon.$$

Now, the relation (3.13) yields

$$\text{dist } (p_0(t), F) \leq \frac{\varepsilon}{2}.$$

Proof of (3.16): For every $t \in B(p_0)$ we have

$$\psi(p_0) = (f + \eta)(p_0(t)).$$

Using (3.18) and taking into account that

$$0 \leq \eta \leq \varepsilon^2,$$

it follows that

$$c \leq f(p_0(t)) \leq c + \frac{5\varepsilon^2}{4}.$$

Proof of (3.17): Applying Lemma 12 for $\varphi(t) = \partial f(p_0(t))$, we find a continuous mapping $v : B(p_0) \to X$ such that, for every $t \in B(p_0)$,

$$\|v(t)\| \leq 1.$$

Moreover, for every $t \in B(p_0)$ and $x^* \in \partial f(p_0(t))$,

$$\langle x^*, v(t) \rangle \geq \gamma - \varepsilon,$$

where

$$\gamma = \inf_{t \in B(p_0)} \lambda(p_0(t)).$$

Hence for every $t \in B(p_0)$,

$$f^0(p_0(t), -v(t)) = \max\{\langle x^*, -v(t) \rangle; x^* \in \partial f(p_0(t))\} =$$

$$= -\min\{\langle x^*, v(t) \rangle; x^* \in \partial f(p_0(t))\} \leq -\gamma + \varepsilon.$$

Since $B(p_0) \cap K_0 = \emptyset$, there exists a continuous extension w of v to the set K such that $w = 0$ on K_0 and $\|w(t)\| \leq 1$, for all $t \in K$.

Now, by (3.14) it follows that for every $\lambda > 0$,

$$-\frac{\varepsilon}{2} \leq -\frac{\varepsilon}{2} \|w\|_\infty \leq \frac{\psi(p_0 - \lambda w) - \psi(p_0)}{\lambda}. \tag{3.19}$$

For every n, there exists $t_n \in K$ such that

$$\psi(p_0 - \frac{1}{n} w) = (f + \eta)(p_0(t_n) - \frac{1}{n} w(t_n)).$$

Passing eventually to a subsequence, we may suppose that (t_n) converges to t_0, which, clearly, lies in $B(p_0)$. On the other hand, for every $t \in K$ and $\lambda > 0$ we have

$$f(p_0(t) - \lambda w(t)) \leq f(p_0(t)) + \lambda\varepsilon.$$

Hence

$$n[\psi(p_0 - \lambda w) - \psi(p_0)] \leq n[f(p_0(t_n) - \frac{1}{n}w(t_n)) + \frac{\varepsilon}{n} - f(p_0(t_n))].$$

Therefore, by (3.19) it follows that

$$-\frac{3\varepsilon}{2} \leq n[\psi(p_0(t_n) - \frac{1}{n}w(t_n)) - f(p_0(t_n))] \leq$$

$$\leq n[\psi(p_0(t_n) - \frac{1}{n}w(t_0)) - f(p_0(t_n))] +$$

$$+ n[f(p_0(t_n) - \frac{1}{n}w(t_n)) - f(p_0(t_n) - \frac{1}{n}w(t_0)).$$

Using the fact that f is locally Lipschitz and $t_n \to t_0$ we find

$$\limsup_{n\to\infty} n\,[f(p_0(t_n) - \frac{1}{n}w(t_n)) - f(p_0(t_n) - \frac{1}{n}w(t_0))] = 0.$$

Therefore

$$-\frac{3\varepsilon}{2} \leq \limsup_{n\to\infty} n\,[f(p_0(t_0) + z_n - \frac{1}{n}w(t_0)) - f(p_0(t_0) + z_n)],$$

where $z_n = p_0(t_n) - p_0(t_0)$. Hence

$$-\frac{3\varepsilon}{2} \leq f^0(p_0(t_0), -w(t_0)) \leq -\gamma + \varepsilon.$$

So

$$\gamma = \inf\{\|x^*\|; x^* \in \partial f(p_0(t)), t \in B(p_0)\} \leq \frac{5\varepsilon}{2}.$$

Now, by the lower semicontinuity of λ, we find $t \in B(p_0)$ such that

$$\lambda(p_0(t)) = \inf_{x^* \in \partial f(p_0(t))} \|x^*\| \leq \frac{5\varepsilon}{2},$$

which ends our proof. □

Corollary 11. *Under hypotheses of Theorem 20, if f satisfies,* (PS)$_c$*, then c is a critical value of f.*

Remark 6. *If*

$$\inf_{x\in X_1} f(x+z) = \max_{x\in K^*} f(x),$$

then the conclusion of Corollary 8 remains valid, with an argument based on Theorem 10.

Corollary 12. *(Ghoussoub-Preiss Theorem). Let $f : X \to \mathbb{R}$ be a locally Lipschitz Gâteaux-differentiable functional such that $f' : (X, \|\cdot\|) \to (X^*, \sigma(X^*, X))$ is continuous. Let a and b be in X and define*

$$c = \inf_{p \in \mathcal{P}} \max_{t \in [0,1]} f(p(t)),$$

where $\mathcal{P}$ is the set of continuous paths $p : [0,1] \to X$ such that $p(0) = a$ and $p(1) = b$. Let F be a closed subset of X which does not contain a and b and $f(x) \geq c$, for all $x \in F$. Suppose, in addition that, for every $p \in \mathcal{P}$,

$$p([0,1]) \cap \mathcal{P} \neq \emptyset.$$

Then there exists a sequence (x_n) în X so that
i) $\lim_{n\to\infty} \operatorname{dist}(x_n, F) = 0\,;$
ii) $\lim_{n\to\infty} f(x_n) = c\,;$
iii) $\lim_{n\to\infty} \|f'(x_n)\| = 0\,.$
Moreover, if f satisfies $(\mathrm{PS})_c$, then there exists $x \in F$ such that $f(x) = c$ and $f'(x) = 0$.

Proof. With the same arguments as in the proof of Corollary 7 we deduce that the functional f is locally Lipschitz and

$$\partial f(x) = \{f'(x)\}.$$

Applying Theorem 20 for $K = [0,1]$, $K^* = \{0,1\}$, $p^*(0) = a$ $p^*(1) = b$, our conclusion follows.

The last part of the theorem follows from Corollary 11. □

3.4 Applications of the Mountain-Pass Theorem

The simplest model

Let $1 < p < \frac{N+2}{N-2}$, if $N \geq 3$, and $1 < p < +\infty$, provided that $N = 1, 2$. Consider the problem

$$\begin{cases} -\Delta u = u^p\,, & \text{in } \Omega \\ u > 0\,, & \text{in } \Omega \\ u = 0\,, & \text{on } \partial\Omega\,. \end{cases} \tag{3.20}$$

Our aim is to prove in what follows the following

Theorem 21. *There exists a solution of the problem (3.20), which is not necessarily unique. Furthermore, this solution is unstable.*

Remark 7. *If $p = \frac{N+2}{N-2}$ then the energy functional associated to the problem (3.20) does not have the Palais-Smale property. The case $p \geq \frac{N+2}{N-2}$ is difficult; for instant, there is no solution even in the simplest case where $\Omega = B(0,1)$. If $p = 1$ then the existence of a solution depends on the geometry of the domain: if 1 is not an eigenvalue of $(-\Delta)$ in $H_0^1(\Omega)$ then there is no solution to our problem (3.20).*

If $0 < p < 1$ then there exists a unique solution (since the mapping $u \longmapsto f(u)/u = u^{p-1}$ is decreasing) and, moreover, this solution is stable. The arguments may be done in this case by using the method of sub and super solutions.

Proof. We first argue the instability of the solution. So, in order to justify that $\lambda_1(-\Delta - pu^{p-1}) < 0$, let φ be an eigenfunction corresponding to λ_1. We have

$$-\Delta\varphi - pu^{p-1}\varphi = \lambda_1\varphi\,, \qquad \text{in } \Omega\,.$$

Integrating by parts this equality we find

$$(1-p)\int_\Omega u^p\varphi = \lambda_1\int_\Omega \varphi u\,,$$

which implies $\lambda_1 < 0$, since $u > 0$ in Ω and $p > 1$.

We will prove the existence of a solution by using two different methods:

1. A variational proof. Let

$$m = \inf\{\int_\Omega \mid \nabla v \mid^2;\ v \in H_0^1(\Omega) \text{ and } \|v\|_{L^{p+1}} = 1\}\,.$$

First step: m is achieved. Let $(u_n) \subset H_0^1(\Omega)$ be a minimizing sequence. Since $p < \frac{N+2}{N-2}$ then $H_0^1(\Omega)$ is compactly embedded in $L^{p+1}(\Omega$. It follows that

$$\int_\Omega \mid \nabla u_n \mid^2 \to m\,,$$

$$\|u_n\|_{L^{p+1}} = 1\,.$$

So, up to a subsequence,

$$u_n \rightharpoonup u\,, \qquad \text{weakly in } H_0^1(\Omega)$$

and

$$u_n \to u\,, \qquad \text{strongly in } L^{p+1}(\Omega)\,.$$

By the lower semicontinuity of the functional $\|\cdot\|_{L^2}$ we find that

$$\int_\Omega \mid \nabla u \mid^2 \leq \liminf_{n\to\infty}\int_\Omega \mid \nabla u_n \mid^2 = m$$

which implies $\int_\Omega \mid \nabla u \mid^2 = m$. Since $\|u\|_{L^{p+1}} = 1$, it follows that m is achieved by u.

We remark that we have even $u_n \to u$, strongly in $H_0^1(\Omega)$. This follows by the weak convergence of (u_n) in $H_0^1(\Omega)$ and by $\|u_n\|_{H_0^1} \to \|u\|_{H_0^1}$.

2. $u \geq 0$, a.e. in Ω. We may assume that $u \geq 0$, a.e. in Ω. Indeed, if not, we may replace u by $\mid u \mid$. This is possible since $\mid u \mid \in H_0^1(\Omega)$ and so, by Stampacchia's theorem,

$$\nabla \mid u \mid = (\operatorname{sign} u)\,\nabla u\,, \qquad \text{if } u \neq 0\,.$$

Moreover, on the level set $[u=0]$ we have $\nabla u = 0$, so

$$\mid \nabla \mid u \mid\mid = \mid \nabla u \mid , \qquad \text{a.e. in } \Omega .$$

3. u verifies $-\Delta u = u^p$ in weak sense. We have to prove that, for every $w \in H_0^1(\Omega)$,

$$\int_\Omega \nabla u \nabla w = m \int_\Omega u^p w .$$

Put $v = u + \varepsilon w$ in the definition of m. It follows that

$$\int_\Omega \mid \nabla v \mid^2 = \int_\Omega \mid \nabla u \mid^2 + 2\varepsilon \int_\Omega \nabla u \nabla w + \varepsilon^2 \int_\Omega \mid \nabla w \mid^2$$

and

$$\begin{aligned}\int_\Omega \mid u + \varepsilon w \mid^{p+1} = & \int_\Omega \mid u \mid^{p+1} + \varepsilon(p+1) \int_\Omega u^p w + o(\varepsilon) = \\ & 1 + \varepsilon(p+1) \int_\Omega u^p w + o(\varepsilon) .\end{aligned}$$

Therefore

$$\|v\|^2_{L^{p+1}} = \left(1 + \varepsilon(p+1) \int_\Omega u^p w + o(\varepsilon)\right)^{2/(p+1)} = 1 + 2\varepsilon \int_\Omega u^p w + o(\varepsilon) .$$

Hence

$$m = \int_\Omega \mid u \mid^2 \leq \frac{m + 2\varepsilon \displaystyle\int_\Omega \nabla u \nabla w + o(\varepsilon)}{1 + 2\varepsilon \displaystyle\int_\Omega u^p w + o(\varepsilon)} = m + 2\varepsilon \left(\int_\Omega \nabla u \nabla w - m \int_\Omega u^p w \right) + o(\varepsilon) ,$$

which implies

$$\int_\Omega \nabla u \nabla w = m \int_\Omega u^p w , \qquad \text{for every } w \in H_0^1(\Omega) .$$

Consequently, the function $u_1 = m^\alpha u$ ($\alpha = \frac{1}{p-1}$) is a weak solution of our problem (3.20), that is $u = u_1 m^{-\alpha}$ is weak solution to (3.20).

4. Regularity of u. We know until now that $u \in H_0^1(\Omega) \subset L^{2^\star}(\Omega)$. In a general framework, assuming that $u \in L^q$, it follows that $u^p \in L^{q/p}$, that is, by Schauder regularity and Sobolev embeddings, $u \in W^{2,q/p} \subset L^s$, where $\frac{1}{s} = \frac{p}{q} - \frac{2}{N}$. So, assuming that $q_1 > (p-1)\,\frac{N}{2}$, we have $u \in L^{q_2}$, where $\frac{1}{q_2} = \frac{p}{q_1} - \frac{2}{N}$. In particular, $q_2 > q_1$. Let (q_n) be the increasing sequence we may construct in this manner and set $q_\infty = \lim_{n\to\infty} q_n$. Assuming, by contradiction, that $q_n < \frac{Np}{2}$ we obtain, passing at the limit as $n \to \infty$, that $q_\infty = \frac{N(p-1)}{2} < q_1$, contradiction. This shows that there exists $r > \frac{N}{2}$ such that $u \in L^r(\Omega)$ which implies $u \in W^{2,r}(\Omega) \subset L^\infty(\Omega)$. Therefore $u \in W^{2,r}(\Omega) \subset C^k(\overline{\Omega})$, where k denotes the integer part of $2 - \frac{N}{r}$. Now, by Hölder continuity, $u \in C^2(\overline{\Omega})$.

2. Second proof (Mountain-Pass Lemma). Set

$$F(u) = \frac{1}{2} \int_\Omega \mid \nabla u \mid^2 - \frac{1}{p+1} \int_\Omega (u^+)^{p+1} , \qquad u \in H_0^1(\Omega) .$$

Standard arguments show that F is a C^1 functional and u is a critical point of F if and only if u is a solution to the problem (3.20). We observe that $F'(u) = -\Delta u - (u^+)^p \in H^{-1}(\Omega)$. So, if u is a critical point of F then $-\Delta u = (u^+)^p \geq 0$ in Ω and hence, by the Maximum Principle, $u \geq 0$ in Ω.

We verify the hypotheses of the Mountain-Pass Lemma. Obviously, $F(0) = 0$. On the other hand,

$$\int_\Omega (u^+)^{p+1} \leq \int_\Omega | u |^{p+1} = \|u\|^{p+1}_{L^{p+1}} \leq C\|u\|^{p+1}_{H^1_0}.$$

Therefore

$$F(u) \geq \frac{1}{2}\|u\|^2_{H^1_0} - \frac{C}{p+1}\|u\|^{p+1}_{H^1_0} \geq \rho > 0,$$

provided that $\|u\|_{H^1_0} = R$, small enough.

Let us now prove the existence of some v_0 such that $\|v_0\| > R$ and $F(v_0) \leq 0$. For this aim, choose an arbitrary $w_0 \geq 0$, $w_0 \not\equiv 0$. We have

$$F(tw_0) = \frac{t^2}{2}\int_\Omega | \nabla w_0 |^2 - \frac{t^{p+1}}{p+1}\int_\Omega (w_0^+)^{p+1} \leq 0,$$

for $t > 0$ large enough. □

A bifurcation problem

Let us consider a C^1 convex function $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) > 0$ and $f'(0) > 0$. We also assume that there exists $1 < p < \frac{N+2}{N-2}$ such that

$$| f(u) | \leq C(1 + | u |^p)$$

and there exist $\mu > 2$ and $A > 0$ such that

$$\mu \int_0^u f(t)dt \leq uf(u), \qquad \text{for every } u \geq A.$$

A standard example of function satisfying these conditions is $f(u) = (1 + u)^p$.

Consider the problem

$$\begin{cases} -\Delta u = \lambda f(u), & \text{in } \Omega \\ u > 0, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.21}$$

We already know that there exists $\lambda^\star > 0$ such that, for every $\lambda < \lambda^\star$, there exists a minimal and stable solution $\underline{u}$ to the problem (3.21).

Theorem 22. *Under the above hypotheses on f, for every $\lambda \in (0, \lambda^\star)$, there exists a second solution $u \geq \underline{u}$ and, furthermore, u is unstable.*

Proof. We find a solution u of the form $u = \underline{u} + v$ with $v \geq 0$. It follows that v satisfies

$$\begin{cases} -\Delta v = \lambda(f(\underline{u}+v) - f(\underline{u})\,, & \text{in } \Omega \\ v > 0\,, & \text{in } \Omega \\ v = 0\,, & \text{on } \partial\Omega\,. \end{cases} \tag{3.22}$$

Hence v fulfills an equation of the form

$$-\Delta v + a(x)v = g(x,v)\,, \qquad \text{in } \Omega\,,$$

where $a(x) = -\lambda f'(\underline{u})$ and

$$g(x,v) = \lambda\,(f(\underline{u}(x)+v) - f(\underline{u}(x))) - \lambda f'(\underline{u}(x))v\,.$$

We verify easily the following properties:
(i) $g(x,0) = g_v(x,0) = 0$;
(ii) $\mid g(x,v) \mid \leq C(1+ \mid v \mid^p)$;
(iii) $\mu \int_0^v g(x,t)dt \leq vg(x,v)$, for every $v \geq A$ large enough;
(iv) the operator $-\Delta - \lambda f'(\underline{u})$ is coercive, since $\lambda_1(-\Delta - \lambda f'(\underline{u})) > 0$, for every $\lambda < \lambda_1$.

So, by the Mountain-Pass Lemma, the problem (3.21) has a solution which is, *a fortiori*, unstable. □

3.5 Critical points and coerciveness of locally Lipschitz functionals with the strong Palais-Smale property

The Palais-Smale property for C^1 functionals appears as the most natural compactness condition. In order to obtain corresponding results for non-differentiable functionals the Palais-Smale condition introduced in Definition 7 is not always an efficient tool, because of the nonlinearity of the Clarke subdifferential. For this aim, we shall define a stronger Palais-Smale type condition, which will be very useful in applications. In many cases, our compactness condition will be a local one, similar to $(\mathrm{PS})_c$ in Definition 7. The most efficient tool in our reasonings will be, as in the preceding paragraph, the Ekeland variational principle. As we shall remark the new Palais-Smale condition is in closed link with coerciveness properties of locally Lipschitz functionals.

As above, X will denote a real Banach space.

Definition 8. *The locally Lipschitz functional $f : X \to \mathbb{R}$ is said to satisfy the strong Palais-Smale condition at the point c (notation: (s-PS$)_c$) provided that, for every sequence (x_n) in X satisfying*

$$\lim_{n\to\infty} f(x_n) = c \tag{3.23}$$

and

$$f^0(x_n, v) \geq -\frac{1}{n}\,\|v\|, \quad \textit{for every } \; v \in X, \tag{3.24}$$

contains a convergent subsequence.

If this property holds for any real number c we shall say that f satisfies the strong Palais-Smale condition $(\mathrm{s}-\mathrm{PS})$*).*

Remark 8. *It follows from the continuity of f and the upper semicontinuity of $f^0(\cdot,\cdot)$ that if f satisfies the condition* $(\text{s-PS})_c$ *and there exist sequences (x_n) and (ε_n) such that the conditions (3.23) and (3.24) are fulfilled, then c is a critical value of f. Indeed, up to a subsequence, $x_n \to x$. It follows that $f(x) = c$ and, for every $v \in X$,*

$$f^0(x, v) \geq \limsup_{n\to\infty} f^0(x_n, v) \geq 0,$$

that is $0 \in \partial f(x)$.

Definition 9. *A mapping $f : X \to \mathbb{R}$ is said to be coercive provided that*

$$\lim_{\|x\|\to\infty} f(x) = +\infty.$$

For each $a \in \mathbb{R}$ and $f : X \to \mathbb{R}$, we shall denote from now on

$$[f = a] = \{x \in X; f(x) = a\};$$

$$[f \leq a] = \{x \in X; f(x) \leq a\};$$

$$[f \geq a] = \{x \in X; f(x) \geq a\}.$$

Proposition 1. *Let $f : X \to \mathbb{R}$ be a locally Lipschitz bounded from below functional. If $a = \inf_X f$ and f satisfies the condition* $(\text{s-PS})_a$ *, then there exists $\alpha > 0$ such that the set $[f \leq a + \alpha]$ is bounded.*

Proof. We assume, by contradiction, that for every $\alpha > 0$, the set $[f \leq a + \alpha]$ is unbounded. So, there exists a sequence (z_n) in X such that, for every $n \geq 1$,

$$a \leq f(z_n) \leq a + \frac{1}{n^2},$$

$$\|z_n\| \geq n.$$

Using Ekeland's Principle, for every $n \geq 1$ there is some $x_n \in X$ such that, for any $x \in X$,

$$a \leq f(x_n) \leq f(z_n),$$

$$f(x) - f(x_n) + \frac{1}{n}\|x - x_n\| \geq 0,$$

$$\|x_n - z_n\| \leq \frac{1}{n}.$$

Therefore

$$\|x_n\| \geq n - \frac{1}{n} \longrightarrow \infty\,,$$

$$f(x_n) \longrightarrow a\,,$$

and, for each $v \in X$,

$$f^0(x_n, v) \geq -\frac{1}{n}\,\|v\|\,.$$

Now, by (s-PS)$_a$, it follows that the unbounded sequence (x_n) contains a convergent subsequence, contradiction. □

The following is an immediate consequence of the above result

Corollary 13. *If f is a locally Lipschitz bounded from below functional satisfying the strong Palais-Smale condition, then f is coercive.*

This result was proved by S.J. Li [54] for C^1 functionals. He used in his proof the Deformation Lemma. Corollary 13 also follows from

Proposition 2. *Let $f : X \to \mathbb{R}$ be a locally Lipschitz functional satisfying*

$$a = \liminf_{\|x\|\to\infty} f(x) < +\infty.$$

Then there exists a sequence (x_n) in X such that

$$\|x_n\| \to \infty, \quad f(x_n) \longrightarrow a$$

and, for every $v \in X$,

$$f^0(x_n, v) \geq -\frac{1}{n}\,\|v\|.$$

Proof. For every $r > 0$ we define

$$m(r) = \inf_{\|x\|\geq r} f(x).$$

Obviously, the mapping m is non-decreasing and $\lim_{r\to\infty} m(r) = a$. For any integer $n \geq 1$ there exists $r_n > 0$ such that, for every $r \geq r_n$,

$$m(r) \geq a - \frac{1}{n^2}.$$

Remark that we can choose r_n so that $r_n \geq \frac{n}{2} + \frac{1}{n}$. Choose $z_n \in X$ such that $\|z_n\| \geq 2r_n$ and

$$f(z_n) \leq m(r_{2n}) + \frac{1}{n^2} \leq a + \frac{1}{n^2}. \tag{3.25}$$

Applying Ekeland's Principle to the functional f restricted to the set $\{x \in X;\ \|x\| \geq r_n\}$ and for $\varepsilon = \frac{1}{n}$, $z = z_n$, we get $x_n \in X$ such that $\|x_n\| \geq r_n$ and, for every $x \in X$ with $\|x\| \geq r_n$,

$$f(x) \geq f(x_n) - \frac{1}{n}\,\|x - x_n\|\,, \tag{3.26}$$

$$a-\frac{1}{n^2}\le m(r_n)\le f(x_n)\le f(z_n)-\frac{1}{n}\,\|z_n-x_n\|\,. \tag{3.27}$$

It follows from (3.25) and (3.27) that $\|x_n-z_n\|\le\frac{2}{n}$, which implies

$$\|x_n\|\ge 2r_n-\frac{2}{n}\longrightarrow+\infty.$$

On the other hand, $f(x_n)\longrightarrow a$. For every $v\in X$ and $\lambda>0$, putting $x=x_n+\lambda v$ in (3.26), we find

$$f^0(x_n,v)\ge\limsup_{\lambda\searrow 0}\frac{f(x_n+\lambda v)-f(x_n)}{\lambda}\ge-\frac{1}{n}\,\|v\|\,,$$

which concludes our proof. □

Proposition 3. *Let $f:X\to\mathbb{R}$ be a locally Lipschitz bounded from below functional. Assume there exists $c\in\mathbb{R}$ such that f satisfies* (s-PS)$_c$ *and, for every $a<c$, the set $[f\le a]$ is bounded.*

Then there exists $\alpha>0$ such that the set $[f\le c+\alpha]$ is bounded.

Proof. Arguing by contradiction, we assume that the set $[f\le c+\alpha]$ is unbounded, for every $\alpha>0$. It follows by our hypothesis that, for every $n\ge 1$, there exists $r_n\ge n$ such that

$$[f\le c-\frac{1}{n^2}]\subset B(0,r_n).$$

Set

$$c_n=\inf_{X\setminus B(0,r_n)}f\ge c-\frac{1}{n^2}.$$

Since the set $[f\le c+\frac{1}{n^2}]$ is unbounded, we obtain the existence of a sequence (z_n) in X such that

$$\|z_n\|\ge r_n+1+\frac{1}{n}$$

and

$$f(z_n)\le c+\frac{1}{n^2}.$$

So, $z_n\in X\setminus B(0,r_n)$ and

$$f(z_n)\le c_n+\frac{2}{n^2}.$$

Applying Ekeland's Principle to the functional f restricted to $X\setminus B(0,r_n)$, we find $x_n\in X\setminus B(0,r_n)$ such that, for every $x\in X$ with $\|x\|\ge r_n$, we have

$$c_n\le f(x_n)\le f(z_n)\,,$$

$$f(x)\ge f(x_n)-\frac{2}{n}\,\|x-x_n\|\,,$$

$$\|x_n-z_n\|\le\frac{1}{n}\,.$$

Hence

$$\|x_n\|\ge\|z_n\|-\|x_n-z_n\|\ge r_n+1\longrightarrow+\infty\,,$$

$$f(x_n) \longrightarrow c$$

and, for every $v \in X$,

$$f^0(x_n, v) \geq -\frac{2}{n} \|v\| .$$

Now, by (s-PS)$_c$, we obtain that the unbounded sequence (x_n) contains a convergent subsequence, contradiction. $\square$

Proposition 4. *Let $f : X \to \mathbb{R}$ be a locally Lipschitz bounded from below functional. Assume that f is not coercive. If*

$$c = \sup\{a \in \mathbb{R};\ [f \leq a] \ \text{ is bounded}\},$$

then f does not satisfy the condition (s-PS)$_c$ *.*

Proof. Denote

$$A = \{a \in \mathbb{R};\ [f \leq a] \ \text{ is bounded}\}.$$

It follows from the lower boundedness of f that the set A is nonempty. Since f is not coercive, it follows that

$$c = \sup A < +\infty.$$

Assume, by contradiction, that f satisfies (s-PS)$_c$. Then, by Proposition 3, there exists $\alpha > 0$ such that the set $[f \leq a + \alpha]$ is bounded, which contradicts the maximality of c. $\square$

Remark 9. *The real number c defined in Proposition 4 may also be characterized by*

$$c = \inf\{b \in \mathbb{R};\ [f \leq b] \ \text{is unbounded}\} .$$

Proposition 5. *Let $f : X \to \mathbb{R}$ be a locally Lipschitz functional satisfying (s-PS). Assume there exists $a \in \mathbb{R}$ such that the set $[f \leq a]$ is bounded.*

Then the functional f is coercive.

Proof. Without loss of generality, we may assume that $a = 0$. It follows now from our hypothesis that there exists an integer n_0 such that $f(x) > 0$,for any $x \in X$ with $\|x\| \geq n_0$. We assume, by contradiction, that

$$0 \leq c = \liminf_{\|x\| \to \infty} f(x) < +\infty.$$

Applying Proposition 2, we find a sequence (x_n) in X such that $\|x_n\| \to \infty$, $f(x_n) \longrightarrow c$ and, for every $v \in X$,

$$f^0(x_n, v) \geq -\frac{1}{n} \|v\|.$$

Using now the condition (s-PS) , we obtain that the unbounded sequence (x_n) contains a convergent subsequence, contradiction. So, the functional f is coercive. $\square$

Corollary 14. *Let $f : X \to \mathbb{R}$ be a locally Lipschitz bounded from below functional which satisfies* (s-PS)
.

Then every minimizing sequence of f contains a convergent subsequence.

Proof. Let (x_n) be a minimizing sequence of f. Passing eventually at a subsequence we have

$$f(x_n) \le \inf_X f + \frac{1}{n^2}.$$

By Ekeland's Principle, there exists $z_n \in X$ such that, for every $x \in X$,

$$f(x) \ge f(z_n) - \frac{1}{n}\|x - z_n\|,$$

$$f(z_n) \le f(x_n) - \frac{1}{n}\|x_n - z_n\|.$$

With an argument similar to that used in the proof of Proposition 2 we find

$$\|x_n - z_n\| \le \frac{2}{n}, \tag{3.28}$$

$$f(z_n) \le \inf_X f + \frac{1}{n^2},$$

and, for every $v \in X$,

$$f^0(z_n, v) \ge -\frac{1}{n}\ \|v\|.$$

Using now the condition (s-PS) ,we find that the sequence (z_n) is relatively compact. By (3.28) it follows that the corresponding subsequence of (x_n) is convergent, too. □

Define the map

$$M : [0, +\infty) \to \mathbb{R}, \qquad M(r) = \inf_{\|x\|=r} f(x).$$

We shall prove in what follows some elementary properties of this functional.

Proposition 6. *Let $f : X \to \mathbb{R}$ be a locally Lipschitz bounded from below functional which satisfies the condition (s-PS). Assume there exists $R > 0$ such that all the critical points of f are in the closed ball of radius R.*

Then the functional M is increasing and continuous at the right on the set $(R, +\infty)$.

For the proof of this result an auxiliary one. First we introduce a weaker variant of the condition (s-PS) for functionals defined on a circular crown.

Definition 10. *Let $0 < a < b$ and let f be a locally Lipschitz map defined on*

$$A = \{x \in X;\ a \le \|x\| \le b\}.$$

We say that f satisfies the Palais-Smale type condition $(\mathrm{PS})_A$ *provided that every sequence* (x_n) *satisfying*

$$a+\delta \le \|x_n\| \le b-\delta, \quad \textit{for some} \;\; \delta>0\,,$$

$$\sup_n |f(x_n)| < +\infty\,,$$

$$f^0(x_n,v) \ge -\frac{1}{n}\,\|v\|, \quad \textit{for some} \;\; v\in X,$$

contains a convergent subsequence.

Lemma 13. *Let A be as in Definition 10 and let f be a locally Lipschitz bounded from below functional defined on A. If f satisfies* $(\mathrm{PS})_A$ *and f does not have critical points which are interior point of A then, for every $a<r_1<r<r_2<b$,*

$$M(r) > \min\{M(r_1), M(r_2)\}\,. \tag{3.29}$$

Proof of Lemma. Without loss of generality, let us assume that f takes only positive values. Arguing by contradiction, let $r_1<r<r_2$ be such that the inequality (3.29) is not fulfilled. There exists a sequence (x_n) such that $\|x_n\|=r$ and

$$f(x_n) < M(r) + \frac{1}{n^2}.$$

Applying now Ekeland's Principle to f restricted to the set

$$B = \{x\in X;\; r_1 \le \|x\| \le r_2\},$$

we find $z_n\in B$ such that, for every $x\in B$,

$$f(x) \ge f(z_n) - \frac{1}{n}\|x-z_n\|\,,$$

$$f(z_n) \le f(x_n) - \frac{1}{n}\|x_n-z_n\|\,.$$

Moreover, $r_1 < \|z_n\| < r_2$, for n large enough. Indeed, if it would exist $n\ge 1$ such that $\|z_n\|=r_1$, then

$$M(r_1) \le f(x_n) - \frac{1}{n}\|x_n-z_n\| \le M(r) + \frac{1}{n^2} - \frac{1}{n}\|x_n-z_n\| \le$$
$$\le M(r) + \frac{1}{n^2} - \frac{1}{n}(r-r_1) \le M(r_1) + \frac{1}{n^2} - \frac{1}{n}(r-r_1).$$

It follows that $r-r_1 \le \frac{1}{n}$, which is not possible if n is sufficiently large. Therefore

$$\sup_n |f(z_n)| = \sup_n f(z_n) \le M(r)$$

and, for every $v\in X$,

$$f^0(z_n,v) \ge -\frac{1}{n}\,\|v\|\,.$$

Using now $(\mathrm{PS})_A$, the sequence (z_n) contains a subsequence which converges to a critical point of f belonging to B. This contradicts one of the hypotheses imposed to f. □

Proof of Proposition 6 If M is not increasing, there exists $r_1 < r_2$ such that $M(r_2) \leq M(r_1)$. On the other hand, by Corollary 13 we have

$$\lim_{r\to\infty} M(r) = +\infty.$$

Choosing now $r > r_2$ so that $M(r) \geq M(r_1)$, we find that $r_1 < r_2 < r$ and

$$M(r_2) \leq M(r_1) = \min\{M(r_1), M(r)\},$$

which contradicts Lemma 13. So, M is an increasing map.

The continuity at the right of M follows from its upper semicontinuity. □

There exists a local variant of Proposition 6 for locally Lipschitz functionals defined on the set $\{x \in X;\ \|x\| \leq R_0\}$, for some $R_0 > 0$.

Assume f satisfies the condition (s-PS) in the following sense: every sequence (x_n) with the properties

$$\|x_n\| \leq R < R_0\,,$$

$$\sup_n |f(x_n)| < +\infty$$

and

$$f^0(x_n, v) \geq -\frac{1}{n}\,\|v\|, \quad \text{for every} \ \ v \in X \ \ \text{and} \ \ n \geq 1,$$

is relatively compact.

Proposition 7. *Let f be a locally Lipschitz functional defined on $\|x\| \leq R_0$ and satisfying the condition* (s-PS) *. Assume $f(0) = 0$, $f(x) > 0$ provided $0 < \|x\| < R_0$ and f does not have critical point in the set $\{x \in X;\ 0 < \|x\| < R_0\}$.*

Then there exists $0 < r_0 \leq R_0$ such that M is increasing on $[0, r_0)$ and decreasing on $[r_0, R_0)$.

Proof. Let (R_n) an increasing sequence of positive numbers which converges to R_0. It follows by the upper semicontinuity of M restricted to $[0, R_n]$ that there exists $r_n \in (0, R_n]$ such that M achieves its maximum in r_n. Let $r_0 \in (0, R_0]$ be the limit of the increasing sequence (r_n). Our conclusion follows now easily by applying Lemma 13. □

Chapter 4

Critical point theorems of Lusternik-Schnirelmann type

4.1 Basic results on the notions of genus and Lusternik-Schnirelmann category

One of the most interesting problems related to the extremum problems is how to find estimates of the eigenvalues and eigenfunctions of a given operator. In this field the Lusternik-Schnirelmann theory plays a very important role. The starting point of this theory is the eigenvalue problem

$$Ax = \lambda x, \quad \lambda \in \mathbb{R}, \ \ x \in \mathbb{R}^n,$$

where $A \in M_n(\mathbb{R})$ is a symmetric matrix. This problem may be written, equivalently,

$$F'(x) = \lambda x, \quad \lambda \in \mathbb{R}, \ \ x \in \mathbb{R}^n,$$

where

$$F(x) = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij} x_i x_j,$$

provided that $A = (a_{ij})$ and $x = (x_1, ..., x_n)$.

The eigenvalues of the operator A are, by Courant's Principle,

$$\lambda_k = \max_{M \in \mathcal{V}_k} \min_{x \in M} \frac{\langle Ax, x \rangle}{\|x\|^2} = 2 \max_{A \in \mathcal{V}_k} \min_{x \in M} F(x)\,,$$

for $1 \leq k \leq n$, where $\mathcal{V}_k$ denotes the set of all vector subspaces of $\mathbb{R}^n$ with the dimension k.

The first result in the Lusternik-Schnirelmann theory was proved in 1930 and is the following:

Lusternik-Schnirelmann Theorem. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^1 even functional. Then f' has at least $2n$ distinct eigenfunctions on the sphere S^{n-1}.*

The ulterior achievements in mathematics showed the signifiance of this theorem. We only point out that the variational arguments play at this moment a very strong instrument in the study of potential operators. Hence it is not a coincidence the detail that the solutions of such a problem are found by analysing the extrema of a suitable functional.

L. Lusternik and L. Schnirelmann developed their theory using the notion of *Lusternik-Schnirelmann category* of a set. A simpler notion is that of *genus*, which is due to Coffman [28], but equivalent to that introduced by Krasnoselski.

Let X be a real Banach space and denote by $\mathcal{F}$ the family of all closed and symmetric with respect to the origin subsets of $X \setminus \{0\}$.

Definition 11. *A nonempty subset A of $\mathcal{F}$ has the genus k provided k is the least integer with the property that there exists a continuous odd mapping $h : A \to \mathbb{R}^k \setminus \{0\}$.*

We shall denote from now on by $\gamma(A)$ the genus of the set $A \in \mathcal{F}$.

By definition, $\gamma(\emptyset) = 0$ and $\gamma(A) = +\infty$, if $\gamma(A) \neq k$, for every integer k.

Lemma 14. *Let $D \subset \mathbb{R}^n$ be a bounded open and symmetric set which contains the origin. Let $f : \overline{D} \to \mathbb{R}^n$ be a continuous function which does not vanish on the boundary of D.*

Then $f(D)$ contains a neighbourhood of the origin.

Proof. By Borsuk's Theorem, $d[f; D, 0]$ is an odd number, so different from 0. Now, by the existence theorem for the topological degree, it follows that $0 \in f(D)$. The property of continuity of the topological degree implies the existence of some $\varepsilon > 0$ such that $a \in f(D)$, for all $a \in \mathbb{R}^n$ with $\|a\| < \varepsilon$. □

Lemma 15. *Let D be as in Lemma 14 and $g : \partial D \to \mathbb{R}^n$ a continuous odd function, such that the set $g(\partial D)$ is contained in a proper subspace of $\mathbb{R}^n$.*

Then there exists $z \in \partial D$ such that $g(z) = 0$.

Proof. We may suppose that $g(\partial D) \subset \mathbb{R}^{n-1}$. If g does not vanish on ∂D, then, by Tietze's Theorem, there exists an extension h of g at the set $\overline{D}$. By Lemma 14, the set $h(\overline{D})$ contains a neighbourhood of the origin in $\mathbb{R}^n$, which is not possible, because $h(\overline{D}) \subset \mathbb{R}^{n-1}$. Thus, there is some $z \in \partial D$ such that $g(z) = 0$. □

Lemma 16. *Let $A \in \mathcal{F}$ a set which is homeomorphic with S^{n-1} by an odd homeomorphism.*

Then $\gamma(A) = n$.

Proof. Obviously, $\gamma(A) \leq n$.

If $\gamma(A) = k < n$, then there exists $h : A \to \mathbb{R}^k \setminus \{0\}$ continuous and odd.

Let $f : A \to S^{n-1}$ the homeomorphism given in the hypothesis. Then $h \circ f^{-1} : S^{n-1} \to \mathbb{R}^k \setminus \{0\}$ is continuous and odd, which contradicts Lemma 15. Therefore $\gamma(A) = n$. □

The main properties of the notion of genus of a closed and symmetric set a listed in what follows:

Lemma 17. *Let $A, B \in \mathcal{F}$.*

i) If there exists $f : A \to B$ continuous and odd then $\gamma(A) \leq \gamma(B)$.

ii) If $A \subset B$, then $\gamma(A) \leq \gamma(B)$.

iii) If the sets A and B are homeomorphic, then $\gamma(A) = \gamma(B)$.

iv) $\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$.

v) If $\gamma(B) < +\infty$, then

$$\gamma(A) - \gamma(B) \leq \gamma(\overline{A \setminus B}).$$

vi) If A is compact, then $\gamma(A) < +\infty$.

vii) If A is compact, then there exists $\varepsilon > 0$ such that

$$\gamma(V_\varepsilon(A)) = \gamma(A),$$

where

$$V_\varepsilon(A) = \{x \in X;\ \mathrm{dist}(x, A) \leq \varepsilon\}.$$

Proof. i) If $\gamma(B) = n$, let $h : B \to \mathbb{R}^n \setminus \{0\}$ continuous and odd. Then the mapping $h \circ f : A \to \mathbb{R}^n \setminus \{0\}$ is also continuous and odd, that is $\gamma(A) \leq n$.

If $\gamma(B) = +\infty$, the result is trivial.

ii) We choose $f = \mathrm{Id}$ in the preceding proof.

iii) It follows from i), by interventing the sets A and B.

iv) Let $\gamma(A) = m$, $\gamma(B) = n$ and $f : A \to \mathbb{R}^m \setminus \{0\}$, $g : B \to \mathbb{R}^n \setminus \{0\}$ be continuous and odd. By Tietze's Theorem let $F : X \to \mathbb{R}^m$ and $G : X \to \mathbb{R}^n$ be continuous extensions of f and g. Moreover, let us assume that F and G are odd. If not, we replace the function F with

$$x \longmapsto \frac{F(x) - F(-x)}{2}.$$

Let

$$h = (F, G) : A \cup B \to \mathbb{R}^{m+n} \setminus \{0\}.$$

Clearly, h is continuous and odd, that is $\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$.

v) follows from ii), iv) and the fact that $A \subset (\overline{A \setminus B}) \cup B$.

vi) If $x \neq 0$ and $r < \|x\|$, then $B_r(x) \cap B_r(-x) = \emptyset$. So,

$$\gamma(B_r(x) \cup B_r(-x)) = 1.$$

By compactness arguments, we can cover the set A with a finite number of open balls, that is $\gamma(A) < +\infty$.

vii) Let $\gamma(A) = n$ and $f : A \to \mathbb{R}^n \setminus \{0\}$ be continuous and odd. With the same arguments as in iv), let $F : X \to \mathbb{R}^n$ be a continuous and odd extension of f.

Since f does not vanish on the compact set A, there is some $\varepsilon > 0$ such that F does not vanish in $V_\varepsilon(A)$. Thus $\gamma(V_\varepsilon(A)) \leq n = \gamma(A)$.

The reversed inequality follows from ii). $\square$

We give in what follows the notion of *Lusternik-Schnirelmann category* of a set. For further details and proof we refer to Mawhin-Willem [57] and Palais [65].

A topological space X is said to be *contractible* provided that the identic map is homotopic with a constant map, that is, there exist $u \in X$ and a continuous function $F : [0,1] \times X \to X$ such that, for every $x \in X$,

$$F(0,x) = x \quad \text{and} \quad F(1,x) = u\,.$$

A subset M of X is said to be *contractible in* X is there exist $u \in X$ and a continuous function $F : [0,1] \times M \to X$ such that, for every $x \in M$,

$$F(0,x) = x \quad \text{and} \quad F(1,x) = u\,.$$

If A is a subset of X, define the *category of* A *in* X, denoted by $\mathrm{Cat}_X(A)$, as follows:

$\mathrm{Cat}_X(A) = 0$, if $A = \emptyset$;

$\mathrm{Cat}_X(A) = n$, if n is the smallest integer such that A may be covered with n closed sets which are contractible in X;

$\mathrm{Cat}_X(A) = \infty$, if contrary.

Lemma 18. *Let A and B be subsets of X.*

i) If $A \subset B$, then $\mathrm{Cat}_X(A) \le \mathrm{Cat}_X(B)$.

ii) $\mathrm{Cat}_X(A \cup B) \le \mathrm{Cat}_X(A) + \mathrm{Cat}_X(B)\,.$

iii) Let $h : [0,1] \times A \to X$ be a continuous mapping such that $h(0,x) = x$, for every $x \in A$.

If A is closed and $B = h(1,A)$, then $\mathrm{Cat}_X(A) \le \mathrm{Cat}_X(B)$.

4.2 A finite dimensional version of the Lusternik-Schnirelmann theorem

The first version of the celebrated Lusternik-Schnirelmann Theorem, published in [55], was generalized in several ways. We shall prove in what follows a finite dimensional variant, by using the notion of *genus* of a set. Other variants of the Lusternik-Schnirelmann Theorem may be found in Krasnoselski [49], Palais [65], Rabinowitz [68], Struwe [80].

Let $f, g \in C^1(\mathbb{R}^n, \mathbb{R})$ and let $a > 0$ be a fixed real number.

Definition 12. *We say that the functional f has a critical point with respect to g and a if there exist $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ such that*

$$\begin{cases} f'(x) = \lambda g'(x) \\ \quad g(x) = a\,. \end{cases} \tag{4.1}$$

In this case, x is said to be a critical point of f (with respect to the mapping g and the number a), while $f(x)$ is called a critical value of f.

We say that the real number c is a critical value of f if the problem (2.1) admits a solution $x \in \mathbb{R}^n$ such that $f(x) = c$.

Lemma 19. *Let $g : \mathbb{R}^n \to \mathbb{R}$ be an even map which is Fréchet differentiable and such that*

1) $\operatorname{Ker} g = \{0\}$;

2) $\langle g'(x), x\rangle > 0$, *for every* $x \neq 0$;

3) $\lim\limits_{\|x\|\to\infty} g(x) = \infty$.

Then the sets $[g = a]$ *and* S^{n-1} *are homeomorphic.*

Proof. Let

$$h : [g = a] \to S^{n-1}, \qquad h(x) = \frac{x}{\|x\|} .$$

Evidently, h is well defined and continuous. We shall prove in what follows that h is one-to-one and onto.

Let $y \in S^{n-1}$. Consider the mapping

$$f : [0, \infty) \to \mathbb{R}, \quad f(t) = g(ty) .$$

Then f is differentiable and, for every $t > 0$,

$$f'(t) = \langle g'(ty), y\rangle > 0 .$$

Since $f(0) > 0$ and $\lim\limits_{t\to\infty} f(t) = \infty$, it follows that there exists a unique $t_0 > 0$ such that $f(t_0) = a$, that is $g(t_0 y) = a$. Thus, $t_0 y \in [g = a]$ and $h(t_0 y) = y$. Therefore h is surjective.

Let now $x, y \in [g = a]$ be such that $h(x) = h(y)$, that is

$$\frac{x}{\|x\|} = \frac{y}{\|y\|} .$$

If $x \neq y$, then there is some $t_0 > 0$, $t \neq 1$ such that $y = t_0 x$.

Consider the mapping

$$\psi : [0, \infty) \to \mathbb{R}, \quad \psi(t) = g(tx) .$$

It follows that $\psi(1) = \psi(t_0)$. But, for every $t > 0$,

$$\psi'(t) = \langle g'(tx), x\rangle > 0 ,$$

which implies that the equality $\psi(1) = \psi(t)$ is not possible provided $t \neq 1$. Thus, $x = y$, that is h is one-to-one.

The condition 3) from our hypotheses implies the continuity of h^{-1}. Moreover, h^{-1} is odd, because g is even. Thus, h is the desired homeomorphism. □

For every $1 \leq k \leq n$ define the set

$$\mathcal{V}_k = \{A;\ \ A \subset [g = a],\ A \text{ compact, symmetric and } \gamma(A) \geq k\} .$$

Let F be a vector subspace of $\mathbb{R}^n$ and of dimension k. If $S_k = F \cap S^{n-1}$, it follows by Lemma 19 that the set $A = h^{-1}(S_k)$ lies in $\mathcal{V}_k$, that is $\mathcal{V}_k \neq \emptyset$, for every $1 \leq k \leq n$. Moreover,

$$\mathcal{V}_n \subset \mathcal{V}_{n-1} \subset \mathcal{V}_1 .$$

Theorem 23. *Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be two even functionals of class C^1 and $a > 0$ fixed. Assume that g satisfies the following assumptions:*

1) $\text{Ker } g = \{0\}$;

2) $\langle g'(x), x\rangle > 0$, *for every* $x \in \mathbb{R}^n \setminus \{0\}$;

3) $\lim_{\|x\|\to\infty} g(x) = \infty$.

Under these assumptions, f admits at least $2n$ critical points with respect to the application g and the number a.

Proof. Observe first that the critical points appear in pairs, because of the evenness of the mappings f and g.

Step 1. *The characterization of the critical values of* f.

Let, for every $1 \leq k \leq n$,

$$c_k = \sup_{A\in\mathcal{V}_k} \min_{x\in A} f(x) .$$

We propose to show that c_k are critical values of f. This is not enough for concluding the proof, since it is possible that the numbers c_k are not distinct.

If c is a real number, let

$$A_c = \{x \in [g = a];\ f(x) \geq c\} .$$

We shall prove that, for every $1 \leq k \leq n$,

$$c_k = \sup\{r \in \mathbb{R};\ \gamma(A_r) \geq k\} .$$

Set

$$x_k = \sup\{r \in \mathbb{R};\ \gamma(A_r) \geq k\} .$$

¿From $\gamma(A_r) \geq k$ it follows that $\inf\{f(x);\ x \in A_r\} \leq c_k$, that is $x_k \leq c_k$.

If $x_k < c_k$, then there exists $A \in \mathcal{V}_k$ such that

$$c_k > \inf_{x\in A} f(x) = \alpha > x_k .$$

Thus, $A \subset A_\alpha$ and $k \leq \gamma(A) \leq \gamma(A_\alpha)$, which contradicts the definition of x_k. Consequently, $c_k = x_k$.

Using now the fact that, for every $\varepsilon > 0$, it follows that

$$\gamma(A_{c_k-\varepsilon}) \geq k .$$

Let K_c be the set of critical values of f corresponding to the critical value c. By the Deformation Lemma (Theorem A.4 in [68]), if V is a neighbourhood of K_c, there exist $\varepsilon > 0$ and $\eta \in C([0, 1]\times\mathbb{R}^n, \mathbb{R}^n)$ such that, for every fixed $t \in [0, 1]$, the mapping

$$x \longmapsto \eta(t, x)$$

is odd and

$$\eta(1, A_{c-\varepsilon} \setminus V) \subset A_{c+\varepsilon}\,.$$

Now, putting for every $x \in \mathbb{R}^n$,

$$s(x) = \eta(1, x),$$

we get

$$s(A_{c-\varepsilon} \setminus V) \subset A_{c+\varepsilon}\,. \tag{4.2}$$

In particular, if $K_c = \emptyset$, then

$$s(A_{c-\varepsilon}) \subset A_{c+\varepsilon}\,.$$

Step 2. *For every* $1 \leq k \leq n$, *the number* c_k *is a critical value of* f.

Indeed, if not, using the preceding result, there exists $\varepsilon > 0$ such that

$$s(A_{c_k-\varepsilon}) \subset A_{c_k+\varepsilon}\,.$$

¿From $\gamma(A_{c_k-\varepsilon}) \geq k$. By Lemma 17 ii), it follows that

$$\gamma(s(A_{c_k-\varepsilon})) \geq k\,.$$

The definition of c_k yields

$$c_k \geq \inf_{x \in s(A_{c_k-\varepsilon})} f(x)\,. \tag{4.3}$$

By (4.2) and (4.3) it follows that $c_k \geq c_k + \varepsilon$, contradiction.

Step 3. *A multiplicity argument.*

We study in what follows the case of multiple critical values. Let us assume that

$$c_{k+1} = ... = c_{k+p} = c, \quad p > 1\,.$$

In this case we shall prove that $\gamma(K_c) \geq p$.

If, by contradiction, $\gamma(K_c) \leq p - 1$, then, by Lemma 17 vii), there is some $\varepsilon > 0$ such that

$$\gamma(V_\varepsilon(K_c)) \leq p - 1\,.$$

Let $V = \text{Int } V_\varepsilon(K_c)$. By (4.2) it follows that

$$s(A_{c-\varepsilon} \setminus V) \subset A_{c+\varepsilon}\,.$$

Observe that

$$B = \overline{A_{c-\varepsilon} - V_\varepsilon(K_c)} = A_{c-\varepsilon} \setminus \text{Int } V_\varepsilon(K_c)\,.$$

But $\gamma(A_{c-\varepsilon}) \geq k + p$. Using now Lemma 17 v), we have

$$\gamma(B) \geq \gamma(A_{c-\varepsilon}) - \gamma(V_\varepsilon(K_c)) \geq k + 1\,.$$

By Lemma 17 i), it follows that

$$\gamma(s(B)) \geq k+1\,.$$

The definition of $c = c_{k+1}$ shows that

$$\inf_{x\in s(B)} f(x) \leq c\,. \tag{4.4}$$

The inclusion $s(B) \subset A_{c+\varepsilon}$ implies

$$\inf_{x\in s(B)} f(x) \geq c+\varepsilon,$$

which contradicts (4.4). $\square$

4.3 Critical points of locally Lipschitz Z-periodic functionals

Let X be a Banach space and let Z be a discrete subgroup of it. Therefore

$$\inf_{z\in Z\setminus\{0\}} \|z\| > 0\,.$$

Definition 13. *A function $f : X \to \mathbb{R}$ is said to be Z-periodic provided that $f(x+z) = f(x)$, for every $x \in X$ and $z \in Z$.*

If the locally Lipschitz functional $f : X \to \mathbb{R}$ is Z-periodic, then, for every $v \in X$, the mapping $x \longmapsto f^0(x, v)$ is Z-periodic and ∂f is Z-periodic, that is, for every $x \in X$ and $z \in Z$,

$$\partial f(x+z) = \partial f(x)\,.$$

Thus the functional λ inherits the property of Z-periodicity.

If $\pi : X \to X/Z$ is the canonical surjection and x is a critical point of f, then the set $\pi^{-1}(\pi(x))$ contains only critical points. Such a set is said to be a *critical orbit* of f. We also remark that X/Z becomes a complete metric space if it is endowed with the metric

$$d(\pi(x), \pi(y)) = \inf_{z\in Z} \|x-y-z\|\,.$$

A locally Lipschitz functional which is Z-periodic $f : X \to \mathbb{R}$ satisfies the Palais-Smale $(\mathrm{PS})_Z$ condition provided that, for every sequence (x_n) in X such that $(f(x_n))$ is bounded and $\lambda(x_n) \to 0$, there exists a convergent subsequence of $(\pi(x_n))$. Equivalently, this means that, up to a subsequence, there exists $z_n \in Z$ such that the sequence $(x_n - z_n)$ is convergent. If c is a real number, then f satisfies the local condition of type Palais-Smale $(\mathrm{PS})_{Z,c}$ if, for every sequence (x_n) in X such that $f(x_n) \to c$ and $\lambda(x_n) \to 0$, there exists a convergent subsequence of $(\pi(x_n))$.

Theorem 24. *Let $f : X \to \mathbb{R}$ be a locally Lipschitz functional which is Z-periodic and satisfies the assumption (3.6).*

If f satisfies the condition $(\mathrm{PS})_{Z,c}$, then c is a critical value of f, corresponding to a critical point which is not in $\pi^{-1}(\pi(p^*(K^*)))$.

Proof. With the same arguments as in the proof of Theorem 18 we find a sequence (x_n) in X such that

$$\lim_{n\to\infty} f(x_n) = c \quad \text{and} \quad \lim_{n\to\infty} \lambda(x_n) = 0\,.$$

The Palais-Smale condition $(\mathrm{PS})_{Z,c}$ implies the existence of some x such that , up to a subsequence, $\pi(x_n) \longrightarrow \pi(x)$. Passing now to the equivalence class mod Z, we may assume that $x_n \longrightarrow x$ in X. Moreover, x is a critical point of f, because

$$f(x) = \lim_{n\to\infty} f(x_n) = c$$

and

$$\lambda(x) \le \liminf_{n\to\infty} \lambda(x_n) = 0\,.$$

□

Lemma 20. *If n is the dimension of the vector space spanned by the discrete subgroup Z of X, then, for every $1 \le i \le n+1$, the set*

$$\mathcal{A}_i = \{A \subset X;\ A \textit{ is compact and } \ \mathrm{Cat}_{\pi(X)}\pi(A) \ge i\}$$

is nonempty. Moreover

$$\mathcal{A}_1 \supset \mathcal{A}_2 \supset \ldots \supset \mathcal{A}_{n+1}\,.$$

The proof of this result may be found in Mawhin-Willem [57].

Lemma 21. *For every $1 \le i \le n+1$, the set $\mathcal{A}_i$ becomes a complete metric space if it is endowed with the Hausdorff metric*

$$\delta(A,B) = \max\{\sup_{a\in A} \mathrm{dist}(a,B), \sup_{b\in B} \mathrm{dist}(b,A)\}\,.$$

The proof of this result may be found in Kuratovski [50].

Lemma 22. *If $f : X \to \mathbb{R}$ is continuous, then, for every $1 \le i \le n+1$, the mapping $\eta : \mathcal{A}_i \to \mathbb{R}$ defined by*

$$\eta(A) = \max_{x\in A} f(x)$$

is lower semicontinuous.

Proof. For any fixed i, let (A_n) be a sequence in $\mathcal{A}_i$ and $A \in \mathcal{A}_i$ such that $\delta(A_n, A) \longrightarrow 0$. For every $x \in A$ there exists a sequence (x_n) in X such that $x_n \in A_n$ and $x_n \to x$. Thus,

$$f(x) = \lim_{n\to\infty} f(x_n) \le \liminf_{n\to\infty} \eta(A_n)\,.$$

Since $x \in A$ is arbitrary, it follows that

$$\eta(A) \le \liminf_{n\to\infty} \eta(A_n)\,.$$

$\square$

In what follows, $f : X \to \mathbb{R}$ will be a locally Lipschitz functional which is Z-periodic and satisfies the condition $(\mathrm{PS})_Z$. Moreover, we shall assume that f is bounded from below. Let $\mathrm{Cr}(f,c)$ be the set of critical points of f having the real number c as corresponding critical value. Thus,

$$\mathrm{Cr}(f,c) = \{x \in X;\ f(x) = c \ \text{ and } \ \lambda(x) = 0\}\,.$$

If n is the dimension of the vector space spanned by the discrete group Z, then, for every $1 \le i \le n+1$, let

$$c_i = \inf_{A \in \mathcal{A}_i} \eta(A)\,.$$

It follows by Lemma 20 and the boundedness from below of f that

$$-\infty < c_1 \le c_2 \le \ldots \le c_{n+1} < +\infty\,.$$

Theorem 25. *Under the above hypotheses, the functional f has at least $n+1$ distinct critical orbits.*

Proof. It is enough to show that if $1 \le i \le j \le n+1$ and $c_i = c_j = c$, then the set $\mathrm{Cr}(f,c)$ contains at least $j-i+1$ distinct critical orbits. Arguing by contradiction, let us assume that there exists $i \le j$ such that the set $\mathrm{Cr}(f,c)$ has $k \le j-i$ distinct critical orbits, generated by $x_1, \ldots, x_k$. We first choose an open neighbourhood of $\mathrm{Cr}(f,c)$ defined by

$$V_r = \bigcup_{l=1}^{k} \bigcup_{z \in Z} B(x_l + z, r)\,.$$

Moreover we may assume that $r > 0$ is chosen so that π restricted to the set $\overline{B}\,(x_l, 2r)$ is one-to-one. This contradiction shows that, for every $1 \le l \le k$,

$$\mathrm{Cat}_{\pi(X)}\ \pi(\overline{B}(x_l, 2r)) = 1\,.$$

In the above arguments, $V_r = \emptyset$ if $k = 0$.

<u>Step 1.</u> We shall prove that there exists $0 < \varepsilon < \min\{\frac{1}{4}, r\}$ such that, for every $x \in [c - \varepsilon \le f \le c + \varepsilon] \setminus V_r$, we have

$$\lambda(x) > \sqrt{\varepsilon}\,. \tag{4.5}$$

Indeed, if not, there exists a sequence (x_m) in $X \setminus V_r$ such that, for every $m \ge 1$,

$$c - \frac{1}{m} \le f(x_m) \le c + \frac{1}{m}$$

and

$$\lambda(x_m) \le \frac{1}{\sqrt{m}}\,.$$

Since f satisfies the condition $(\mathrm{PS})_Z$, passing eventually to a subsequence, $\pi(x_m) \longrightarrow \pi(x)$, for some $x \in V \setminus V_r$. By the Z-periodicity property of f and λ, we may assume that $x_m \longrightarrow x$. The continuity of f and the lower semicontinuity of λ imply $f(x) = 0$ and $\lambda(x) = 0$, contradiction, because $x \in V \setminus V_r$.

Step 2. For ε found above and taking into account the definition of c_j, there exists $A \in \mathcal{A}_j$ such that

$$\max_{x \in A} f(x) < c + \varepsilon^2 .$$

Putting $B = A \setminus V_{2r}$ and applying Lemma 18 we find

$$j \leq \mathrm{Cat}_{\pi(X)} \pi(A) \leq \mathrm{Cat}_{\pi(X)} (\pi(B) \cup (\overline{V}_{2r})) \leq$$

$$\leq \mathrm{Cat}_{\pi(X)} \pi(B) + \mathrm{Cat}_{\pi(X)} \pi(\overline{V}_{2r}) \leq \mathrm{Cat}_{\pi(X)} \pi(B) + k \leq$$

$$\leq \mathrm{Cat}_{\pi(X)} \pi(B) + j - i .$$

Thus

$$\mathrm{Cat}_{\pi(X)} \pi(B) \geq i,$$

that is, $B \in \mathcal{A}_i$.

Step 3. For ε and B as above, we apply Ekeland's Principle to the functional η defined in Lemma 22. Thus, there exists $C \in \mathcal{A}_i$ such that, for every $D \in \mathcal{A}_i$, $D \neq C$,

$$\eta(C) \leq \eta(B) \leq \eta(A) \leq c + \varepsilon^2 ,$$

$$\delta(B, C) \leq \varepsilon ,$$

$$\eta(D) > \eta(C) - \varepsilon \delta(C, D) . \tag{4.6}$$

Since $B \cap V_{2r} = \emptyset$ and $\delta(B, C) \leq \varepsilon < r$, we have $C \cap V_r = \emptyset$. In particular, the set $F = [f \geq c - \varepsilon]$ is contained in $[c - \varepsilon \leq f \leq c + \varepsilon]$ and $F \cap V_r = \emptyset$. Applying now Lemma 12 for $\varphi = \partial f$ defined on F, we find a continuous map $v : F \to X$ such that, for every $x \in F$ and $x^* \in \partial f(x)$,

$$\|v(x)\| \leq 1$$

and

$$\langle x^*, v(x) \rangle \geq \inf_{x \in F} \lambda(x) - \varepsilon \geq \inf_{x \in C} \lambda(x) - \varepsilon \geq \sqrt{\varepsilon} - \varepsilon,$$

the last inequality being justified by the relation (4.5). Thus, for every $x \in F$ and $x^* \in \partial f(x)$,

$$f^0(x, -v(x)) = \max_{x^* \in \partial f(x)} \langle x^*, -v(x) \rangle = - \min_{x^* \in \partial f(x)} \langle x^*, v(x) \rangle \leq$$

$$\leq \varepsilon - \sqrt{\varepsilon} < -\varepsilon,$$

by the choice of ε.

By the upper semicontinuity of f^0 and the compactness of F there exists $\delta > 0$ such that, for every $x \in F,\ y \in X,\ \|y - x\| \leq \delta$ we have

$$f^0(y, -v(x)) < -\varepsilon\,. \tag{4.7}$$

Since $C \cap \mathrm{Cr}(f, c) = \emptyset$ and C is compact and $\mathrm{Cr}(f, c)$ is closed, there exists a continuous extension $w : X \to X$ of v such that the restriction of w to $\mathrm{Cr}(f, c)$ is the identic map and, for every $x \in X$, $\|w(x)\| \leq 1$.

Let $\alpha : X \to [0, 1]$ be a continuous function which is Z-periodic and such that $\alpha = 1$ on $[f \geq c]$ and $\alpha = 0$ on $[f \leq c - \varepsilon]$. Let $h : [0, 1] \times X \to X$ be the continuous map defined by

$$h(t, x) = x - t\delta\alpha(x)w(x)$$

If $D = h(1, C)$, it follows by Lemma 18 that

$$\mathrm{Cat}_{\pi(X)}\pi(D) \geq \mathrm{Cat}_{\pi(X)}\pi(C) \geq i,$$

which shows that $D \in \mathcal{A}_i$, because D is compact.

<u>Step 4.</u> By Lebourg's Mean Value Theorem we find that, for every $x \in X$, there is some $\theta \in (0, 1)$ such that

$$f(h(1, x)) - f(h(0, x)) \in \langle \partial f(h(\theta, x)), -\delta\alpha(x)w(x)\rangle\,.$$

Thus, there exists $x^* \in \partial f(h(\theta, x))$ such that

$$f(h(1, x)) - f(h(0, x)) = \alpha(x)\langle x^*, -\delta w(x)\rangle\,.$$

It follows now by (4.7) that if $x \in F$, then

$$\begin{aligned} f(h(1, x)) - f(h(0, x)) &= \delta\alpha(x)\langle x^*, -w(x)\rangle \leq \\ &\leq \delta\alpha(x)f^0(x - \theta\delta\alpha(x)w(x), -v(x)\rangle \leq -\varepsilon\delta\alpha(x)\,. \end{aligned} \tag{4.8}$$

Thus, for every $x \in C$,

$$f(h(1, x)) \leq f(x)\,.$$

Let $x_0 \in C$ be such that $f(h(1, x_0)) = \eta(D)$. Therefore

$$c \leq f(h(1, x_0)) \leq f(x_0)\,.$$

By the definitions of α and F it follows that $\alpha(x_0) = 1$ and $x_0 \in F$. Thus, by (4.8), we have

$$f(h(1, x_0)) - f(x_0) \leq -\varepsilon\delta\,.$$

Hence

$$\eta(D) + \varepsilon\delta \leq f(x_0) \leq \eta(C)\,. \tag{4.9}$$

Now, by the definition of D,

$$\delta(C, D) \leq \delta\,.$$

Thus

$$\eta(D) + \varepsilon\delta(C, D) \leq \eta(C),$$

that is, by (4.6), we find $C = D$, which contradicts the relation (4.9). $\square$

Chapter 5

Applications to the study of multivalued elliptic problems

5.1 Multivalued variants of some results of Brezis-Nirenberg and Mawhin-Willem

Let Ω be an open bounded set with the boundary sufficiently smooth in $\mathbb{R}^N$. Let g be a measurable function defined on $\Omega \times \mathbb{R}$ and such that

$$|g(x,t)| \leq C(1+|t|^p), \quad a.e. \;\; (x,t) \in \Omega \times \mathbb{R}\,, \tag{5.1}$$

where C is a positive constant and $1 \leq p < \frac{N+2}{N-2}$ (if $N \geq 3$) and $1 \leq p < \infty$ (if $N = 1, 2$).

Define the functional $\psi : L^{p+1}(\Omega) \to \mathbb{R}$ by

$$\psi(u) = \int_\Omega \int_0^{u(x)} g(x,t)dtdx\,.$$

We first prove that ψ is a locally Lipschitz map. Indeed the growth condition (5.1) and Hölder's Inequality yield

$$|\psi(u) - \psi(v)| \leq C' \; (|\Omega|^{\frac{p}{p+1}} + \max_{w \in U} \|w\|^{\frac{p}{p+1}}_{L^{p+1}(\Omega)}) \cdot \|u - v\|_{L^{p+1}(\Omega)},$$

where U is an open ball containing u and v.

Put

$$\underline{g}(x,t) = \lim_{\varepsilon \searrow 0} essinf\{g(x,s);\; |t-s| < \varepsilon\}\,,$$

$$\overline{g}(x,t) = \lim_{\varepsilon \searrow 0} esssup\{g(x,s);\; |t-s| < \varepsilon\}\,.$$

Lemma 23. *The mappings $\underline{g}$ and $\overline{g}$ are measurable.*

Proof. Observe that

$$\overline{g}(x,t) = \lim_{\varepsilon \searrow 0} esssup\{g(x,s);\; s \in [t-\varepsilon, t+\varepsilon]\} =$$

$$= \lim_{n\to\infty} esssup\{g(x,s);\ s \in [t-\tfrac{1}{n}, t+\tfrac{1}{n}]\}\,.$$

Replacing, locally, the map g by $g+M$, for M large enough, we may assume that $g \geq 0$. It follows that

$$\overline{g}(x,t) = \lim_{n\to\infty} \|g(x,\cdot)\|_{L^\infty([t-\frac{1}{n}, t+\frac{1}{n}])} =$$
$$= \lim_{n\to\infty} \lim_{m\to\infty} \|g(x,\cdot)\|_{L^m([t-\frac{1}{n}, t+\frac{1}{n}])} =$$
$$= \lim_{n\to\infty} \lim_{m\to\infty} g_{m,n}(x,t),$$

where

$$g_{m,n}(x,t) = \left(\int_{t-\frac{1}{n}}^{t+\frac{1}{n}} (g(x,s))^m ds\right)^{\frac{1}{m}}$$

Thus, it is sufficient to prove that if $h \in L^\infty_{loc}(\Omega\times\mathbb{R})$ and $a>0$, then the mapping

$$k : \Omega\times\mathbb{R} \to \mathbb{R}, \qquad k(x,t) = \int_{t-a}^{t+a} h(x,s)ds$$

is measurable and to apply this result for $h(x,s) = (g(x,s))^m$ and $a = \frac{1}{n}$.

Observe that, assuming we have already proved that

$$\Omega\times\mathbb{R}\times\mathbb{R} \ni (x,s,t) \stackrel{l}{\longmapsto} h(x,s+t)$$

is measurable, then the conclusion follows by

$$k(x,t) = \int_{-a}^{a} l(x,s,t)ds$$

and Fubini's Theorem applied to the locally integrable function l (the local integrability follows from its local boundedness). So, it is sufficient to justify the measurability of the mapping l. In order to do this, it is enough to prove that reciprocal images of Borel sets (resp. negligible sets) through the function

$$\Omega\times\mathbb{R}\times\mathbb{R} \ni (x,t,s) \stackrel{\omega}{\longmapsto} (x,s+t) \in \Omega\times\mathbb{R}$$

are Borel sets (resp., negligible). The first condition is, obviously, fulfilled. For the second, let A be a measurable set of null measure in $\Omega\times\mathbb{R}$. Consider the map

$$\mathbb{R}^N\times\mathbb{R}\times\mathbb{R} \ni (x,t,s) \stackrel{\eta}{\longmapsto} (x,s+t,t) \in \mathbb{R}^N\times\mathbb{R}\times\mathbb{R}\,.$$

We observe that η is invertible and of class C^1. Moreover

$$\eta(\omega^{-1}(A)) \subset A\times\mathbb{R},$$

which is negligible. So, $\omega^{-1}(A)$ which is negligible, too. $\square$

Let $G : \Omega\times\mathbb{R} \to \mathbb{R}$ be defined by

$$G(x,t) = \int_0^t g(x,s)ds\,.$$

Lemma 24. *Let g be a locally bounded measurable function, defined on $\Omega \times \mathbb{R}$ and $\underline{g}, \overline{g}$ as above.*

Then the Clarke subdifferential of G with respect to t is given by

$$\partial_t G(x,t) = [\underline{g}(x,t), \overline{g}(x,t)], \quad a.e. \ (x,t) \in \Omega \times \mathbb{R}\,.$$

The condition "a.e." can be removed if, for every x, the mapping $t \longmapsto g(x,t)$ is measurable and locally bounded.

Proof. We show that we have equality on the set

$$\{(x,t) \in \Omega \times \mathbb{R};\ g(x,\cdot) \ \text{ is locally bowhered and measurable}\}\,.$$

In order to prove our result, it is enough to consider mappings g which do not depend on x. For this aim the equality that we have to prove is equivalent to

$$G^0(t;1) = \overline{g}(t) \quad \text{and} \quad G^0(t;-1) = \underline{g}(t)\,. \tag{5.2}$$

Examining the definitions of $G^0, \overline{g}$ and $\underline{g}$, it follows that

$$\underline{g}(t) = -(\overline{-g})(t) \quad \text{and} \quad G^0(t,-1) = -(-G)^0(t,1)\,.$$

So, the second equality appearing in (5.2) is equivalent to the first one.

The inequality $G^0(t,1) \leq \overline{g}(t)$ is proved in Chang [24]. For the reversed inequality, we assume by contradiction that there exists $\varepsilon > 0$ such that

$$G^0(t,1) = \overline{g}(t) - \varepsilon\,.$$

Let $\delta > 0$ be such that

$$\frac{G(\tau+\lambda) - G(\tau)}{\lambda} < \overline{g}(t) - \frac{\varepsilon}{2}\,,$$

if $0 < |\tau - t| < \delta$ and $0 < \lambda < \delta$. Then

$$\frac{1}{\lambda}\int_\tau^{\tau+\lambda} g(s)ds < \overline{g}(t) - \frac{\varepsilon}{2}, \quad \text{if } \ |\tau - t| < \delta,\ \lambda > 0\,. \tag{5.3}$$

We now justify the existence of some $\lambda_n \searrow 0$ such that

$$\frac{1}{\lambda_n}\int_\tau^{\tau+\lambda_n} g(s)ds \longrightarrow g(\tau), \quad a.e. \ \ \tau \in (t-\delta, t+\delta)\,. \tag{5.4}$$

Assume, for the moment, that (5.4) has already been proved. Then by (5.3) and (5.4) it follows that for every $\tau \in (t-\delta, t+\delta)$,

$$g(\tau) \leq \overline{g}(t) - \frac{\varepsilon}{2}\,.$$

Thus we get the contradiction

$$\overline{g}(t) \leq \ esssup\,\{g(s);\ s \in [t-\delta, t+\delta]\} \leq \overline{g}(t) - \frac{\varepsilon}{2}\,.$$

For concluding the proof it is sufficient to prove (5.4). Observe that we can "cut off" the mapping g, in order to have $g \in L^\infty \cap L^1$. Then (5.4) is nothing else that the classical result

$$T_\lambda \longrightarrow \mathrm{Id}_{L^1(\mathbb{R})}, \quad \text{if } \lambda \searrow 0\,. \tag{5.5}$$

in $\mathcal{L}(L^1(\mathbb{R}))$, where

$$T_\lambda\, u(t) = \frac{1}{\lambda}\int_t^{t+\lambda} u(s)ds\,,$$

for $\lambda > 0,\ t \in \mathbb{R},\ u \in L^1(\mathbb{R})$.

Indeed, we observe easily that T_λ is linear and continuous in in $L^1(\mathbb{R})$ and that

$$\lim_{\lambda\searrow 0} T_\lambda\, u = u \quad \text{ in } \mathcal{D}(\mathbb{R}),$$

for $u \in \mathcal{D}(\mathbb{R})$. The relation (5.5) follows now by a density argument. □

Returning to our problem, it follows by Theorem 2.1 in Chang [24] that

$$\partial\psi_{|H_0^1(\Omega)}\,(u) \subset \partial\psi(u)\,. \tag{5.6}$$

For obtaining further information concerning $\partial\psi$, we need the following refinement of Theorem 2.1 in [24].

Theorem 26. *If $u \in L^{p+1}(\Omega)$ then*

$$\partial\psi(u)(x) \subset [\underline{g}(x,u(x)), \overline{g}(x,u(x))], \quad a.e.\ \ x \in \Omega,$$

in the sense that if $w \in \partial\psi(u)$, then

$$\underline{g}(x,u(x)) \le w(x) \le \overline{g}(x,u(x)), \quad a.e.\ \ x \in \Omega\,. \tag{5.7}$$

Proof. Let h be a Borel function such that $h = g$ a.e. in $\Omega \times \mathbb{R}$. Then

$$A = \Omega \setminus \{x \in \Omega;\ h(x,t) = g(x,t) \quad a.e.\ \ t \in \mathbb{R}\}$$

is a negligible set. Thus

$$B = \{x \in \Omega;\ \text{ there exists } \ t \in \mathbb{R} \ \text{ such that } \ \underline{g}(x,t) \ne \underline{h}(x,t)\}$$

is a negligible set. A similar reasoning may be done for $\overline{g}$ and $\overline{h}$.

It follows that we can assume g is a Borel function.

Lemma 25. *Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a locally bounded Borel function. Then $\overline{g}$ is a Borel function.*

Proof of Lemma. Since the restriction is local, we may assume that g is nonnegative and bounded by 1. Since

$$g = \lim_{n\to\infty} \lim_{m\to\infty} g_{m,n}\,,$$

where

$$g_{m,n}(x,t) = (\int_{t-\frac{1}{n}}^{t+\frac{1}{n}} |g^m(x,s)|ds)^{\frac{1}{m}},$$

it is enough to show that $g_{m,n}$ is a Borel function.

Set

$$\mathcal{M} = \{g : \Omega \times \mathbb{R} \to \mathbb{R};\ |g| \le 1 \text{ and } \ g \ \text{ is borelian}\}\,,$$

$$\mathcal{N} = \{g \in \mathcal{M};\ g_{m,n} \ \text{ is borelian}\}\,.$$

Evidently, $\mathcal{N} \subset \mathcal{M}$. By a classical result from Measure Theory (see Berberian [16], p.178) and the Lebesgue Dominated Convergence Theorem, we find $\mathcal{M} \subset \mathcal{N}$. Consequently, $\mathcal{M} = \mathcal{N}$. □

Proof of Theorem 26 continued. Let $v \in L^\infty(\Omega)$. There exist the sequences $\lambda_i \searrow 0$ and $h_i \to 0$ in $L^{p+1}(\Omega)$ such that

$$\psi^0(u,v) = \lim_{i\to\infty} \frac{1}{\lambda_i} \int_\Omega \int_{u(x)+h_i(x)}^{u(x)+h_i(x)+\lambda_i v(x)} g(x,s)dsdx\,.$$

We may assume that $h_i \to 0$ a.e. So

$$\psi^0(u,v) = \lim_{i\to\infty} \frac{1}{\lambda_i} \int_{[v>0]} \int_{u(x)+h_i(x)}^{u(x)+h_i(x)+\lambda_i v(x)} g(x,s)dsdx \le$$

$$\le \int_{[v>0]} (\limsup_{i\to\infty} \frac{1}{\lambda_i} \int_{u(x)+h_i(x)}^{u(x)+h_i(x)+\lambda_i v(x)} g(x,s)ds)dx \le$$

$$\le \int_{[v>0]} \overline{g}(x,u(x))v(x)dx\,.$$

So, for every $v \in L^\infty(\Omega)$,

$$\psi^0(u,v) \le \int_{[v>0]} \overline{g}(x,u(x))v(x)dx \tag{5.8}$$

Let us now assume that (5.7) is not true. So, there exist $\varepsilon > 0$, a set E with $|E| > 0$ and $w \in \partial\psi(u)$ such that, for every $x \in E$,

$$w(x) \ge \overline{g}(x,u(x)) + \varepsilon\,. \tag{5.9}$$

Putting $v = \chi_E$ in (5.8) it follows that

$$\langle w,v\rangle = \int_E w \le \psi^0(u,v) \le \int_E \overline{g}(x,u(x))dx,$$

which contradicts (5.9). □

We assume in what follows that

$$g(x,0)=0 \quad \text{and} \quad \lim_{\varepsilon\searrow 0} esssup \left\{ |\frac{g(x,t)}{t}|;\ (x,t)\in\Omega\times[-\varepsilon,\varepsilon] \right\} < \lambda_1\,, \tag{5.10}$$

where λ_1 is the first eigenvalue of the operator $(-\Delta)$ in $H_0^1(\Omega)$. Furthermore, we shall assume that the following "tehnical" condition is fulfilled:

there exist $\mu>2$ and $r\geq 0$ such that

$$\mu G(x,t) \leq \begin{cases} t\underline{g}(x,t), & a.e.\ x\in\Omega, t\geq r \\ t\overline{g}(x,t), & a.e.\ x\in\Omega, t\leq -r \end{cases} \quad \text{and } g(x,t)\geq 1 \ a.e.\ x\in\Omega, t\geq r\,. \tag{5.11}$$

It follows from (5.1) and (5.10) that there exist some constants $0<C_1<\lambda_1$ and $C_2>0$ such that

$$|g(x,t)|\leq C_1|t|+C_2|t|^p, \quad a.e.\ (x,t)\in\Omega\times\mathbb{R}\,. \tag{5.12}$$

Theorem 27. *Let $\alpha=\frac{\lambda_1}{\lambda_1-C_1}>1$ and $a\in L^\infty(\Omega)$ be such that the operator $-\Delta+\alpha a$ is coercive. Assume, further, that the conditions (5.1), (5.10) and (5.11) are fulfilled.*

Then the multivalued elliptic problem

$$-\Delta u(x)+a(x)u(x)\in[\underline{g}(x,u(x)),\overline{g}(x,u(x))], \quad a.e.\ x\in\Omega \tag{5.13}$$

has a solution in $H_0^1(\Omega)\cap W^{2,q}(\Omega)\setminus\{0\}$, where q is the conjugated exponent of $p+1$.

Remark. The technical condition imposed to a is automatically fulfilled if $a\geq 0$ or, more general, if $\|a^-\|_{L^\infty(\Omega)}<\lambda_1\alpha^{-1}$.

Proof. Consider in the space $H_0^1(\Omega)$ the locally Lipschitz functional

$$\varphi(u)=\frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)}+\frac{1}{2}\int_\Omega a(x)u^2(x)dx-\psi(u)\,.$$

In order to prove the theorem, it is enough to show that φ has a critical point $u_0\in H_0^1(\Omega)$ corresponding to a positive critical value. Indeed, it is obvious that

$$\partial\varphi(u)=-\Delta u+a(x)u-\partial\psi_{|H_0^1(\Omega)}(u), \quad \text{in} \quad H^{-1}(\Omega)\,.$$

If u_0 is a critical point of φ, it follows that there exists $w\in\partial\psi_{|H_0^1(\Omega)}(u_0)$ such that

$$-\Delta u_0+a(x)u_0=w, \quad \text{in} \quad H^{-1}(\Omega)$$

But $w\in L^q(\Omega)$. By a standard regularity result for elliptic equations, we find that $u_0\in W^{2,q}(\Omega)$ and that u_0 is a solution of (5.13).

In order to prove that φ has such a critical point we shall apply Corollary 5. More precisely, we shall prove that φ satisfies the Palais-Smale condition, as well as the following "geometric" hypotheses:

$$\varphi(0)=0 \quad \text{and there exists } v\in H_0^1(\Omega)\setminus\{0\} \text{ such that } \varphi(v)\leq 0\,; \tag{5.14}$$

$$\text{there exist } \ c>0 \text{ and } 0<R<\|v\| \text{ such that } \varphi\geq c \text{ pe } \partial B(0,R)\,. \tag{5.15}$$

<u>Verification of (5.14).</u> Evidently, $\varphi(0)=0$. For the other assertion appearing in (5.15) we need

Lemma 26. *There exist positive constants C_1 and C_2 such that*

$$g(x,t) \geq C_1 t^{\mu-1} - C_2, \quad a.e. \quad (x,t) \in \Omega \times [r,\infty)\,. \tag{5.16}$$

Proof of Lemma. We shall show that

$$\underline{g} \leq g \leq \overline{g}, \qquad \text{a.e. in} \quad \Omega \times [r,\infty)\ . \tag{5.17}$$

Assume for the moment that the relation (5.17) was proved. Then, by (5.11),

$$\mu \underline{G}(x,t) \leq t\underline{g}(x,t), \quad a.e. \quad (x,t) \in \Omega \times \mathbb{R}\,, \tag{5.18}$$

where

$$\underline{G}(x,t) = \int_0^t \underline{g}(x,s)ds\,.$$

Consequently it is sufficient to prove (5.16) for $\underline{g}$ instead of g.

Since $\underline{g}(x,\cdot) \in L^\infty_{loc}(\mathbb{R})$, it follows that $\underline{G}(x,\cdot) \in W^{1,\infty}_{loc}(\mathbb{R})$. if we choose $C > 0$ sufficiently large so that

$$\mu \underline{G}(x,t) \leq C + t\underline{g}(x,t), \quad a.e. \quad (x,t) \in \Omega \times [0,\infty)\,. \tag{5.19}$$

We observe that (5.19) shows that

$$(0,+\infty) \ni t \longmapsto \frac{\underline{G}(x,t)}{t^\mu} - \frac{C}{\mu t^\mu}$$

is increasing. So, there exist R large enough and positive constants K_1, K_2 such that

$$\underline{G}(x,t) \geq K_1 t^\mu - K^2, \quad a.e. \quad (x,t) \in \Omega \times [R,+\infty)\,.$$

Relation (5.16) follows now from the above inequality and from (5.18).

For proving the second inequality appearing in (5.17), we observe that

$$\overline{g} = \lim_{n\to\infty} g_n \quad \text{in} \quad L^\infty_{loc}(\Omega)\,, \tag{5.20}$$

where

$$g_n(x,t) = esssup\ \{g(x,s); \quad |t-s| \leq \frac{1}{n}\}\,.$$

For fixed $x \in \Omega$, it is sufficient to show that for every interval $I = [a,b] \subset \mathbb{R}$ we have

$$\int_I \overline{g}(x,t)dt \geq \int_I g(x,t)dt$$

and to use then a standard argument from Measure Theory.

Taking into account (5.20), it is enough to show that

$$\liminf_{n\to\infty} \int_I g_n(x,t)dt \geq \int_I g(x,t)dt\,.$$

We have

$$\int_I g_n(x,t)dt = \int_I esssup\ \{g(x,s);\ s \in [t-\tfrac{1}{n}, t+\tfrac{1}{n}]\} \geq$$

$$\geq \int_I \frac{n}{2}\int_{t-\frac{1}{n}}^{t+\frac{1}{n}} g(x,s)ds = \text{ (Fubini) } \int_{a-\frac{1}{n}}^{b+\frac{1}{n}} \frac{n}{2}\int_{t_1(s)}^{t_2(s)} g(x,s)dtds =$$

$$= \int_{a-\frac{1}{n}}^{b+\frac{1}{n}} \frac{n}{2}\ (t_2(s)-t_1(s))\ g(x,s)ds =$$

$$= \int_a^b g(x,s)ds + \frac{n}{2}\int_{a-\frac{1}{n}}^{b+\frac{1}{n}}(s-\frac{1}{n}-a)g(x,s)ds+$$

$$+\frac{n}{2}\int_{b-\frac{1}{n}}^{b+\frac{1}{n}}(b-s-\frac{1}{n})g(x,s)ds \longrightarrow \int_a^b g(x,s)ds\,.$$

We have chosen n such that $\frac{2}{n} \leq b-a$, and

$$t_1(s) = \begin{cases} a, & \text{if } \ a-\frac{1}{n} \leq s \leq a+\frac{1}{n} \\ s-\frac{1}{n}, & \text{if } \ a+\frac{1}{n} \leq s \leq b+\frac{1}{n} \end{cases}$$

$$t_2(s) = \begin{cases} s+\frac{1}{n}, & \text{if } \ a-\frac{1}{n} \leq s \leq b-\frac{1}{n} \\ b, & \text{if } \ b-\frac{1}{n} \leq s \leq b+\frac{1}{n}\,. \end{cases}$$

□

Proof of Theorem 27 continued. If $e_1 > 0$ denotes the first eigenfunction of the operator $-\Delta$ in $H_0^1(\Omega)$, then, for t large enough,

$$\varphi(te_1) \leq \frac{\lambda_1 t^2}{2}\|e_1\|^2_{L^2(\Omega)} + \frac{t^2}{2}\int_\Omega ae_1^2 - \psi(te_1) \leq$$

$$\leq (\frac{\lambda_1}{2}\|e_1\|^2_{L^2(\Omega)} + \int_\Omega ae_1^2)t^2 + C_2 t\int_\Omega e_1 - C_1' t^\mu \int_\Omega e_1^\mu < 0$$

So, in order to obtain (5.14) it is enough to choose $v = te_1$, for t found above.

Verification of (5.15). Applying Poincaré's Inequality, the Sobolev embedding theorem and (5.12), we find that, for every $u \in H_0^1(\Omega)$,

$$\psi(u) \leq \frac{C_1}{2}\int_\Omega u^2 + \frac{C_2}{p+1}\int_\Omega |u|^{p+1} \leq \frac{C_1}{2\lambda_1}\|\nabla u\|^2_{L^2(\Omega)} + C'\|\nabla u\|^{p+1}_{L^2(\Omega)}\,.$$

Hence

$$\varphi(u) \geq \frac{1}{2}(1-\frac{C_1}{\lambda_1})\|\nabla u\|^2_{L^2(\Omega)} + \frac{1}{2}\int_\Omega au^2 - C'\|\nabla u\|^{p+1}_{L^2(\Omega)} \geq$$

$$\geq \frac{1}{2}(1-\frac{C_1}{\lambda_1} - \frac{1}{\alpha+\varepsilon})\|\nabla u\|^2_{L^2(\Omega)} - C'\|\nabla u\|^{p+1}_{L^2(\Omega)} \geq C > 0,$$

for $\varepsilon > 0$ sufficiently small, if $\|\nabla u\|_{L^2(\Omega)} = R$ is close to 0.

Verification of the Palais-Smale condition. Let (u_k) be a sequence in $H_0^1(\Omega)$ such that

$$\varphi(u_k) \quad \text{is bounded}$$

and

$$\lim_{k\to\infty} \lambda(u_k) = 0\,.$$

The definition of λ and (5.6) imply the existence of a sequence (w_k) such that

$$w_k \in \partial\psi_{|H_0^1(\Omega)} \subset L^q(\Omega)$$

and

$$-\Delta u_k + a(x)u_k - w_k \longrightarrow 0 \;\text{ in }\; H^{-1}(\Omega)\,.$$

Since, by (5.1), the mapping G is locally bounded with respect to the variable t and uniformly with respect to x, the hypothesis (5.11) yields

$$\mu G(x,u(x)) \leq \begin{cases} u(x)\underline{g}(x,u(x)) + C, & a.e. \text{ in } [u \geq 0] \\ u(x)\overline{g}(x,u(x)) + C, & a.e. \text{ in } [u \leq 0] \end{cases}$$

where u is a measurable function defined on Ω, while C is a positive constant not depending on u. It follows that, for every $u \in H_0^1(\Omega)$,

$$\psi(u) = \int_{[u\,\geq\,0]} G(x,u(x))dx + \int_{[u\,\leq\,0]} G(x,u(x))dx \leq$$

$$\leq \frac{1}{\mu}\int_{[u\,\geq\,0]} u(x)\underline{g}(x,u(x))dx + \frac{1}{\mu}\int_{[u\,\leq\,0]} u(x)\overline{g}(x,u(x))dx + C|\Omega|$$

This inequality and (5.7) show that, for every $u \in H_0^1(\Omega)$ and $w \in \partial\psi(u)$,

$$\psi(u) \leq \frac{1}{\mu}\int_\Omega u(x)w(x)dx + C'\,.$$

We first prove that the sequence (u_k) contains a subsequence which is weakly convergent in $H_0^1(\Omega)$. Indeed,

$$\varphi(u_k) = \frac{1}{2}\int_\Omega |\nabla u_k|^2 + \frac{1}{2}\int_\Omega a u_k^2 - \psi(u_k) =$$

$$= (\frac{1}{2} - \frac{1}{\mu})\int_\Omega (|\nabla u_k|^2 + a u_k^2) + \frac{1}{\mu}\langle -\Delta u_k + a u_k - w_k, u_k\rangle +$$

$$+ \frac{1}{\mu}\langle w_k, u_k\rangle - \psi(u_k) \geq$$

$$\geq (\frac{1}{2} - \frac{1}{\mu})\int_\Omega (|\nabla u_k|^2 + a u_k^2) + \frac{1}{\mu}\langle -\Delta u_k + a u_k - w_k, u_k\rangle - C' \geq$$

$$\geq C''\int_\Omega |\nabla u_k|^2 - \frac{1}{\mu}\,o(1)\sqrt{\int_\Omega |\nabla u_k|^2} - C'\,.$$

This implies easily that the sequence (u_k) is bounded in $H_0^1(\Omega)$. So, up to a subsequence, (u_k) is weakly convergent to $u \in H_0^1(\Omega)$. Since the embedding $H_0^1(\Omega) \subset L^{p+1}(\Omega)$ is compact, it follows that, up to a subsequence, (u_k) is strongly convergent in $L^{p+1}(\Omega)$. We remark that (u_k) is bounded in $L^q(\Omega)$. Since ψ is Lipschitz on the bounded subsets of $L^{p+1}(\Omega)$, it follows that (w_k) is bounded in $L^q(\Omega)$. From

$$\|\nabla u_k\|^2_{L^2(\Omega)} = \int_\Omega \nabla u_k \nabla u - \int_\Omega a u_k (u_k - u) +$$

$$+ \int_\Omega w_k (u_k - u) + \langle -\Delta u_k + a u_k - w_k, u_k - u \rangle_{H^{-1}, H_0^1(\Omega)}$$

it follows that

$$\|\nabla u_k\|_{L^2(\Omega)} \longrightarrow \|\nabla u\|_{L^2(\Omega)} .$$

Since $H_0^1(\Omega)$ is a Hilbert space, and

$$u_k \rightharpoonup u, \ \|u_k\|_{H_0^1(\Omega)} \to \|u\|_{H_0^1(\Omega)} ,$$

we deduce that (u_k) converges to u in $H_0^1(\Omega)$. □

A stronger variant of Theorem 27 is

Theorem 28. *Under the same hypotheses as in Theorem 27, let $b \in L^q(\Omega)$ such that there exists $\delta > 0$ so that*

$$\|b\|_{L^\infty(\Omega)} < \delta .$$

Then the problem

$$-\Delta u(x) + a(x)u(x) + b(x) \in [\underline{g}(x, u(x)), \overline{g}(x, u(x))], \quad a.e. \quad x \in \Omega \tag{5.21}$$

has a solution.

Proof. Define

$$\varphi(u) = \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)} + \frac{1}{2}\int_\Omega a u^2 + \int_\Omega b u - \int_\Omega \int_0^{u(x)} g(x,t)\,dt\,dx .$$

We have seen that if $b = 0$, then the problem (5.21) has a solution. For $\|b\|_{L^\infty(\Omega)}$ sufficiently small, the verification of the Palais-Smale condition and of the geometric hypotheses (5.14) and (5.15) follows the same lines as in the proof of Theorem 27. □

As a second application of the abstract theorems proved in the first two chapters, we shall study the multivalued pendulum problem

$$\begin{cases} x'' + f \in [\underline{g}(x), \overline{g}(x)] \\ x(0) = x(1) , \end{cases} \tag{5.22}$$

where

$$f \in L^p(0,1), \qquad \text{for some } \ p > 1 , \tag{5.23}$$

$$g \in L^\infty(\mathbb{R}) \quad \text{and there exists } \; T > 0 \; \text{ such that } g(x+T) = g(x), \; a.e. \; x \in \mathbb{R}\,, \tag{5.24}$$

$$\int_0^T g(t)dt = \int_0^1 f(t)dt = 0\,. \tag{5.25}$$

The smooth variant of the problem (5.22) was studied in Mawhin-Willem [56].

Theorem 29. *If f and g are as above then the problem (5.22) has at least two solutions in the space*

$$X := H^1_p(0,1) = \{x \in H^1(0,1);\; x(0) = x(1)\}\,.$$

Moreover, these solutions are distinct, in the sense that their difference is not an integer multiple of T.

Proof. As in the proof of Theorem 27, the critical points of the functional $\varphi : X \to \mathbb{R}$ defined by

$$\varphi(x) = -\frac{1}{2}\int_0^1 x'^2 + \int_0^1 fx - \int_0^1 G(x)$$

are solutions of the problem (5.22). The details of the proof are, essentially, the same as above.

Since $\varphi(x+T) = \varphi(x)$, we may apply Theorem 25. All we have to do is to prove that φ verifies the condition $(\mathrm{PS})_{Z,c}$, for any c.

In order to do this, let (x_n) be a sequence in X such that

$$\lim_{n\to\infty} \varphi(x_n) = c \tag{5.26}$$

$$\lim_{n\to\infty} \lambda(x_n) = 0\,. \tag{5.27}$$

Let

$$w_n \in \partial\varphi(x_n) \subset L^\infty(0,1) \tag{5.28}$$

be such that

$$\lambda(x_n) = x_n'' + f - w_n \to 0 \quad \text{in} \;\; H^{-1}(0,1)\,.$$

Observe that the last inclusion appearing in (5.28) is justified by the fact that

$$\underline{g} \circ x_n \le w_n \le \overline{g} \circ x_n,$$

and $\underline{g}, \overline{g} \in L^\infty(\mathbb{R})$.

By (5.27), after multiplication with x_n it follows that

$$\int_0^1 (x_n')^2 - \int_0^1 fx_n + \int_0^1 w_n x_n = o(1)\,\|x_n\|_{H^1_p}\,.$$

Then, by (5.26),

$$-\frac{1}{2}\int_0^1 (x_n')^2 + \int_0^1 fx_n - \int_0^1 G(x_n) \longrightarrow c\,.$$

So, there exist positive constants C_1 and C_2 such that

$$\int_0^1 (x_n')^2 \le C_1 + C_2\|x_n\|_{H^1_p}\,.$$

We observe that G is also T-periodic, so bounded.

For every n, replacing x_n with $x_n + k_n T$ for a convenable integer k_n, we can assume that

$$x_n(0) \in [0, T] .$$

We have obtained that the sequence (x_n) is bounded in $H_p^1(0,1)$.

Let $x \in H_p^1(0,1)$ be such that , up to a subsequence,

$$x_n \rightharpoonup x \quad \text{and} \quad x_n(0) \to x(0) .$$

Thus

$$\int_0^1 (x_n')^2 = \langle -x_n'' - f + w_n, x_n - x\rangle +$$

$$+ \int_0^1 w_n(x_n - x) - \int_0^1 f(x_n - x) + \int_0^1 x_n' x' \to \int_0^1 x'^2,$$

since $x_n \to x$ in $L^{p'}$, where p' is the conjugated exponent of p.

It follows that $x_n \to x$ in H_p^1, so $(\mathrm{PS})_{Z,c}$ is fulfilled. □

5.2 Multivalued problems of Landesman-Lazer type with strong resonance at infinity

In [51] Landesman and Lazer studied for the first time problems with resonance and they found sufficient conditions for the existence of a solution. We shall first recall the main definitions, in the framework of the singlevalued problems.

Let Ω be an open bounded open set in $\mathbb{R}^N$, and $f : \mathbb{R} \to \mathbb{R}$ a C^1 map. We consider the problem

$$\begin{cases} -\Delta u = f(u), & \text{in } \Omega \\ \quad u \in H_0^1(\Omega) . \end{cases} \tag{5.29}$$

For obtaining information on the existence of solutions, as well as estimates on the number of solutions, it is necessary to have further information f. In fact the solutions of (5.29) depend in an essential manner on the asymptotic behaviour of f. Let us assume, for example, that f is asymptotic linear, that is $\dfrac{f(t)}{t}$ has a finite limit as $|t| \to \infty$. Let

$$a = \lim_{|t|\to\infty} \frac{f(t)}{t} . \tag{5.30}$$

We write

$$f(t) = at - g(t),$$

where

$$\lim_{|t|\to\infty} \frac{g(t)}{t} = 0 .$$

Let $0 < \lambda_1 < \lambda_2 \leq ... \leq \lambda_n \leq ...$ the eigenvalues of the operator $(-\Delta)$ in $H_0^1(\Omega)$. The problem (5.29) is said to be with *resonance at infinity* if the number a from (5.30) is one of the eigenvalues of λ_k. With respect to the growth of g at infinity there are several "degrees" of resonance. If g has a "smaller" rate of increase at infinity, then its resonance is "stronger".

There are several situations:

a) $\lim\limits_{t\to\pm\infty} g(t) = l_\pm \in \mathbb{R}$ and $(l_+, l_-) \neq (0,0)$.

b) $\lim\limits_{|t|\to\infty} g(t) = 0$ and $\lim\limits_{|t|\to\infty} \int_0^t g(s)ds = \pm\infty$.

c) $\lim\limits_{|t|\to\infty} g(t) = 0$ and $\lim\limits_{|t|\to\infty} \int_0^t g(s)ds = \beta \in \mathbb{R}$.

The case c) is called as the case of a *strong resonance*.

For a treatment of these cases, we refer only to [51], [2], [14], [7], [8], [43], [83], [30], [56].

We shall study in what follows a multivalued variant of the Landesman-Lazer problem.

$$\begin{cases} -\Delta u(x) - \lambda_1 u(x) \in [\underline{f}(u(x)), \overline{f}(u(x)) \quad \text{a.e. } x \in \Omega \\ u \in H_0^1(\Omega) \setminus \{0\}\,, \end{cases} \tag{5.31}$$

where

i) $\Omega \subset \mathbb{R}^N$ is an open bounded set with sufficiently smooth boundary;

ii) λ_1 (respectively e_1) is the first eigenvalue (respectively eigenfunction) of the operator $(-\Delta)$ in $H_0^1(\Omega)$;

iii) $f \in L^\infty(\mathbb{R})$;

iv) $\underline{f}(t) = \lim\limits_{\varepsilon\searrow 0} \text{essinf}\{f(s);\ |t-s| < \varepsilon\}$, $\overline{f}(t) = \lim\limits_{\varepsilon\searrow 0} \text{esssup}\{f(s);\ |t-s| < \varepsilon\}$.

Consider in $H_0^1(\Omega)$ the functional $\varphi(u) = \varphi_1(u) - \varphi_2(u)$, where

$$\varphi_1(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 - \lambda_1 u^2) \quad \text{and} \quad \varphi_2(u) = \int_\Omega F(u)\,.$$

Observe first that φ is locally Lipschitz in $H_0^1(\Omega)$. Indeed, it is enough to show that φ_2 is locally Lipschitz in $H_0^1(\Omega)$, which follows from

$$|\varphi_1(u) - \varphi_2(u)| = |\int_\Omega (\int_{u(x)}^{v(x)} f(t)dt)dx| \leq$$

$$\leq \|f\|_{L^\infty} \cdot \|u - v\|_{L^1} \leq C_1\|u - v\|_{L^2} \leq C_2\|u - v\|_{H_0^1}\,.$$

We shall study the problem (5.31) under the hypothesis

$$f(+\infty) := \lim_{t\to+\infty} f(t) = 0, \quad F(+\infty) := \lim_{t\to+\infty} F(t) = 0\,. \tag{5.32}$$

Thus, by [51] and [14], the problem (5.31) becomes a Landesman-Lazer type problem, with strong resonance at $+\infty$.

As an application of Corollary 8 we shall prove the following sufficient condition for the existence of a solution to our problem.

Theorem 30. *Assume that f satisfies the condition $(f1)$, as well as*

$$-\infty \leq F(-\infty) \leq 0 \,. \tag{5.33}$$

If $F(-\infty)$ is finite, we assume further that

$$\text{there exists } \ \eta > 0 \ \text{ such that } \ F \ \text{ is nonnegative on } \ (0, \eta) \ \text{ or } \ (-\eta, 0) \,. \tag{5.34}$$

Under these hypotheses, the problem (5.31) has at least one solution.

We shall make use in the proof of the following auxiliary results:

Lemma 27. *Assume that $f \in L^\infty(\mathbb{R})$ and there exists $F(\pm\infty) \in \overline{\mathbb{R}}$. Moreover, assume that*
i) $f(+\infty) = 0$ if $F(+\infty)$ is finite.
ii) $f(-\infty) = 0$ if $F(-\infty)$ is finite.
Under these hypotheses,

$$\mathbb{R} \subset \ \{\alpha |\Omega| \ ; \ \ \alpha = -F(\pm\infty)\} \ \subset \ \{c \in \mathbb{R}; \ \varphi \textit{ satisface } (\mathrm{PS})_c\} \,.$$

Lemma 28. *Assume that f satisfies the condition (f1). Then φ satisfies the condition $(\mathrm{PS})_c$, for every $c \neq 0$ such that $c < -F(-\infty) \cdot |\Omega|$.*

Assume, for the moment, that these results have been proved.

Proof of Theorem 30 There are two distinct situations:
Case 1. $F(-\infty)$ is finite, that is $-\infty < F(-\infty) \leq 0$. In this case, φ is bounded from below, because

$$\varphi(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 - \lambda_1 u^2) - \int_\Omega F(u) \geq -\int_\Omega F(u)$$

and, in virtue of the hypothesis on $F(-\infty)$,

$$\sup_{u \in H_0^1(\Omega)} \int_\Omega F(u) < +\infty \,.$$

Hence

$$-\infty < a := \inf_{H_0^1(\Omega)} \varphi \leq 0 = \varphi(0) \,.$$

There exists a real number c, sufficiently small in absolute value and such that $F(ce_1) < 0$ (we observe that c may be chosen positive if $F > 0$ in $(0, \eta)$ and negative, if $F < 0$ in $(-\eta, 0)$). So, $\varphi(ce_1) < 0$, that is $a < 0$. By Lemma 28, it follows that φ satisfies the condition $(\mathrm{PS})_a$.
Case 2. $F(-\infty) = -\infty$. Then, by Lemma 27, φ satisfies the condition $(\mathrm{PS})_c$, for every $c \neq 0$.
Let V be the orthogonal complement with respect to $H_0^1(\Omega)$ of the space spanned by e_1, that is

$$H_0^1(\Omega) = \mathrm{Sp}\ \{e_1\} \oplus V \,.$$

For some fixed $t_0 > 0$, let

$$V_0 := \{t_0 e_1 + v;\ v \in V\}$$

$$a_0 := \inf_{V_0} \varphi \,.$$

We remark that φ is coercive on V. Indeed, for every $v \in V$,

$$\varphi(v) = \frac{1}{2}\int_\Omega (|\nabla v|^2 - \lambda_1 v^2) - \int_\Omega F(v) \geq \tag{3.36}$$

$$\geq \frac{\lambda_2 - \lambda_1}{2}\|v\|^2_{H_0^1} - \int_\Omega F(v) \to +\infty, \quad \text{if} \quad \|v\|_{H_0^1} \to +\infty\,, \tag{5.35}$$

because the first term in the right hand side of (5.35) has a quadratic growth at infinity (t_0 being fixed), while $\int_\Omega F(v)$ is uniformly bounded (with respect to v), by the behaviour of F near $\pm\infty$. So, a_0 is attained, because of the coercivity of φ in V. Taking into account the boundedness of φ in $H_0^1(\Omega)$, it follows that $-\infty < a \leq 0 = \varphi(0)$ and $a \leq a_0$.

At this stage, there are again two possibilities:

i) $a < 0$. Thus, by Lemma 28, φ satisfies $(PS)_a$. So, $a < 0$ is a critical value of φ.

ii) $a = 0 \leq a_0$. If $a_0 = 0$, then, by a preceding remark, a_0 is achieved. So, there exists $v \in V$ such that

$$0 = a_0 = \varphi(t_0 e_1 + v)\,.$$

Hence $u = t_0 e_1 + v \in H_0^1(\Omega) \setminus \{0\}$ is a critical point of φ, that is a solution of the problem (5.31).

If $a_0 > 0$, we observe that φ satisfies $(PS)_b$ for every $b \neq 0$. Since $\lim_{t\to+\infty} \varphi(te_1) = 0$, we can apply Corollary 8 for $X = H_0^1(\Omega),\ X_1 = V,\ X_2 = \text{Sp}\,\{e_1\},\ f = \varphi,\ z = t_0 e_1$. Therefore, φ has a critical value $c \geq a_0 > 0$. □

Proof of Lemma 27 We shall assume, without loss of generality, that $F(-\infty) \notin \mathbb{R}$ and $F(+\infty) \in \mathbb{R}$. In this case, if c is a critical value such that φ does not satisfy $(\text{PS})_c$, then it is enough to prove that $c = -F(+\infty)\cdot|\Omega|$. Since φ does not satisfy the condition $(\text{PS})_c$, there exist $t_n \in \mathbb{R}$ and $v_n \in V$ such that the sequence $(u_n) \subset H_0^1(\Omega)$, where $u_n = t_n e_1 + v_n$, has no convergent subsequence, while

$$\lim_{n\to\infty} \varphi(u_n) = c \tag{5.36}$$

$$\lim_{n\to\infty} \lambda(u_n) = 0\,. \tag{5.37}$$

<u>Step 1.</u> *The sequence (v_n) is bounded in $H_0^1(\Omega)$.*

By (5.37) and

$$\partial\varphi(u) = -\Delta u - \lambda_1 u - \partial\varphi_2(u)\,,$$

it follows that there exists $w_n \in \partial\varphi_2(u_n)$ such that

$$-\Delta u_n - \lambda_1 u_n - w_n \to 0 \quad \text{in} \quad H^{-1}(\Omega)\,.$$

So

$$\langle -\Delta u_n - \lambda_1 u_n - w_n, v_n\rangle = \int_\Omega |\nabla v_n|^2 - \lambda_1 \int_\Omega v_n^2 - \int_\Omega g_n(t_n e_1 + v_n) = o(\|v_n\|_{H_0^1}),$$

as $n \to \infty$, where $\underline{f} \le g_n \le \overline{f}$. Since f is bounded, it follows that

$$\|v_n\|^2_{H_0^1} - \lambda_1 \|v_n\|^2_{L^2} = O(\|v_n\|_{H_0^1})\,.$$

So, there exists $C > 0$ such that, for every $n \ge 1$, $\|v_n\|_{H_0^1} \le C$. Now, since (u_n) has no convergent subsequence, it follows that the sequence (u_n) has no convergent subsequence, too.

Step 2. $t_n \to +\infty$.

Since $\|v_n\|_{H_0^1} \le C$ and the sequence $(t_n e_1 + v_n)$ has no convergent subsequence, it follows that $|t_n| \to +\infty$.

On the other hand, by Lebourg's Mean Value Theorem, there exist $\theta \in (0,1)$ and $x^* \in \partial F(te_1(x) + \theta v(x))$ such that

$$\varphi_2(te_1 + v) - \varphi_2(te_1) = \int_\Omega \langle x^*, v(x)\rangle dx \le$$

$$\le \int_\Omega F^0(te_1(x) + v(x), v(x))dx =$$

$$= \int_\Omega \limsup_{\substack{y \to te_1(x)+v(x) \\ \lambda \searrow 0}} \frac{F(y + \lambda v(x)) - F(y)}{\lambda} dx \le$$

$$\le \|f\|_{L^\infty} \cdot \int_\Omega |v(x)| dx = \|f\|_{L^\infty} \cdot \|v\|_{L^1} \le C_1 \|v\|_{H_0^1}\,.$$

A similar computation for $\varphi_2(te_1) - \varphi_2(te_1 + v)$ together with the above inequality shows that, for every $t \in \mathbb{R}$ and $v \in V$,

$$|\varphi_2(te_1 + v) - \varphi_2(te_1)| \le C_2 \|v\|_{H_0^1}\,.$$

So, taking into account the boundedness of (v_n) in $H_0^1(\Omega)$, we find

$$|\varphi_2(t_n e_1 + v_n) - \varphi_2(t_n e_1)| \le C\,.$$

Therefore, since $F(-\infty) \notin \mathbb{R}$ and

$$\varphi(u_n) = \varphi_1(v_n) - \varphi_2(t_n e_1 + v_n) \to c,$$

it follows that $t_n \to +\infty$. In this argument we have also used the fact that $\varphi_1(v_n)$ is bounded.

Step 3. $\|v_n\|_{H_0^1} \to 0$ if $n \to \infty$.

By (f1) and Step 2 it follows that

$$\lim_{n\to\infty} \int_\Omega f(t_n e_1 + v_n) v_n = 0\,.$$

Using now (5.37) and Step 1 we find

$$\lim_{n\to\infty} \|v_n\|_{H_0^1} = 0\,.$$

Step 4.

$$\lim_{t\to+\infty}\varphi_2(te_1+v)=F(+\infty)\cdot|\Omega|\,,\tag{5.38}$$

uniformly on the bounded subsets of V. Assume the contrary. So, there exist $r>0$, $t_n\to+\infty$, $v_n\in V$ with $\|v_n\|\leq r$, such that (3.39) is not fulfilled. Thus, up to a subsequence, there exist $v\in H_0^1(\Omega)$ and $h\in L^2(\Omega)$ such that

$$v_n\rightharpoonup v\quad\text{weakly in}\quad H_0^1(\Omega)\,,\tag{5.39}$$

$$v_n\to v\quad\text{strongly in}\quad L^2(\Omega)\,,\tag{5.40}$$

$$v_n(x)\to v(x)\quad\text{a.e.}\quad x\in\Omega\,,\tag{5.41}$$

$$|v_n(x)|\leq h(x)\quad\text{a.e.}\quad x\in\Omega\,.\tag{5.42}$$

For any $n\geq 1$ we define

$$A_n=\{x\in\Omega;\ t_ne_1(x)+v_n(x)<0\}\,,$$

$$h_n(x)=F(t_ne_1+v_n)\chi_{A_n}\,,$$

where χ_A represents the characteristic function of the set A. By (5.42) and the choice of t_n it follows that $|A_n|\to 0$ if $n\to\infty$.

Using (5.41) we remark easily that

$$h_n(x)\to 0\quad\text{a.e.}\quad x\in\Omega\,.$$

Therefore

$$|h_n(x)|=\chi_{A_n}(x)\cdot|\int_0^{t_ne_1(x)+v_n(x)}f(s)ds|\leq$$

$$\leq\chi_{A_n}(x)\cdot\|f\|_{L^\infty}\cdot|t_ne_1(x)+v_n(x)|\leq C|v_n(x)|\leq Ch(x)\quad\text{a.e.}\quad x\in\Omega\,.$$

So, by Lebesgue's Dominated Convergence Theorem,

$$\lim_{n\to\infty}\int_{A_n}F(t_ne_1+v_n)=0\,.$$

On the other hand,

$$\lim_{n\to\infty}\int_{\Omega\setminus A_n}F(t_ne_1+v_n)=F(+\infty)\cdot|\Omega|\,.$$

So

$$\lim_{n\to\infty}\varphi_2(t_ne_1+v_n)=\lim_{n\to\infty}\int_\Omega F(t_ne_1+v_n)=F(+\infty)\cdot|\Omega|,$$

which contradicts our initial assumption.

Step 5. Taking into account the preceding step and the fact that $\varphi(te_1+v)=\varphi_1(v)-\varphi_2(te_1+v)$, we obtain

$$\lim_{n\to\infty}\varphi(t_ne_1+v_n)=\lim_{n\to\infty}\varphi_1(v_n)-\lim_{n\to\infty}\varphi_2(t_ne_1+v_n)=-F(+\infty)\cdot|\Omega|,$$

that is $c=-F(+\infty)\cdot|\Omega|$, which concludes our proof. $\square$

Proof of Lemma 28 It is enough to show that for every $c \neq 0$ and $(u_n) \subset H_0^1(\Omega)$ such that

$$
\begin{aligned}
&\varphi(u_n) \to c\,,\\
&\lambda(u_n) \to 0\,, \qquad\qquad (5.43)\\
&\|u_n\| \to \infty\,,
\end{aligned}
$$

we have $c \geq -F(-\infty)\cdot|\Omega|$.

Let $t_n \in \mathbb{R}$ and $v_n \in V$ be such that, for every $n \geq 1$,

$$u_n = t_n e_1 + v_n\,.$$

As we have already remarked,

$$\varphi(u_n) = \varphi_1(v_n) - \varphi_2(u_n)\,.$$

Moreover,

$$\varphi_1(v) = \frac{1}{2}\int_\Omega (|\nabla v|^2 - \lambda_1 v^2) \geq \frac{1}{2}(1-\frac{\lambda_1}{\lambda_2})\cdot \|v\|^2_{H_0^1} \to +\infty \quad \text{if} \ \ \|v\|_{H_0^1} \to \infty\,.$$

So, φ_1 is positive and coercive on V. We also have that φ_2 is bounded from below, by eqf1. So, again by (5.32), we conclude that the sequence (v_n) is bounded in $H_0^1(\Omega)$. Thus there exists $v \in H_0^1(\Omega)$ such that, up to a subsequence,

$$
\begin{aligned}
&v_n \rightharpoonup v \quad \text{weakly in} \ \ H_0^1(\Omega)\,,\\
&v_n \to v \quad \text{strongly in} \ \ L^2(\Omega)\,,\\
&v_n(x) \to v(x) \quad \text{a.e.} \ \ x \in \Omega\,,\\
&|v_n(x)| \leq h(x) \quad \text{a.e.} \ \ x \in \Omega\,,
\end{aligned}
$$

for some $h \in L^2(\Omega)$.

Since $\|u_n\|_{H_0^1} \to \infty$ and (v_n) is bounded in $H_0^1(\Omega)$, it follows that $|t_n| \to +\infty$.

Assume for the moment that we have already proved that $\|v_n\|_{H_0^1} \to 0$, if $t_n \to +\infty$. So,

$$\varphi(u_n) = \varphi_1(v_n) - \varphi_2(u_n) \to 0 \quad \text{if} \ \ n \to \infty\,.$$

Here, for proving that $\varphi_2(u_n) \to 0$, we have used (f1). The last relation yields a contradiction, since $\varphi(u_n) \to c \neq 0$. So, $t_n \to -\infty$.

Moreover, since $\varphi(u) \geq -\varphi_2(u)$ and F is bounded from below, it follows that

$$c = \liminf_{n\to\infty} \varphi(u_n) \geq \liminf_{n\to\infty}(-\varphi_2(u_n)) =$$

$$= -\limsup_{n\to\infty}\int_\Omega F(u_n) \geq -\int_\Omega \limsup_{n\to\infty} F(u_n) = -F(-\infty)\cdot|\Omega|,$$

which gives the desired contradiction.

So, for concluding the proof, we have to show that

$$\|v_n\|_{H_0^1} \to 0 \quad \text{if} \;\; t_n \to +\infty\,.$$

Since

$$\partial\varphi(u) = -\Delta u - \lambda_1 u - \partial\varphi_2(u),$$

it follows from (5.43) that there exists $w_n \in \partial\varphi_2(u_n)$ such that

$$-\Delta u_n - \lambda_1 u_n - w_n \to 0 \quad \text{in} \;\; H^{-1}(\Omega)\,.$$

Thus

$$\langle -\Delta u_n - \lambda_1 u_n - w_n, v_n\rangle = \int_\Omega |\nabla v_n|^2 - \lambda_1 \int_\Omega v_n^2 -$$
$$-\int_\Omega g_n(t_n e_1 + v_n) v_n = o(\|v_n\|) \quad \text{if} \;\; n \to \infty,$$

where $\underline{f} \leq g_n \leq \overline{f}$.

Now, for concluding the proof, it is sufficient to show that the last term tends to 0, as $n \to \infty$.

Let $\varepsilon > 0$. Because $f(+\infty) = 0$, it follows that there exists $T > 0$ such that

$$|f(t)| \leq \varepsilon \qquad \text{a.e.} \;\; t \geq T\,.$$

Set

$$A_n = \{x \in \Omega;\; t_n e_1(x) + v_n(x) \geq T\} \quad \text{and} \quad B_n = \Omega \setminus A_n\,.$$

We remark that for every $x \in B_n$,

$$|t_n e_1(x) + v_n(x)| \leq |v_n(x)| + T\,.$$

So, for every $x \in B_n$,

$$|g_n(t_n e_1(x) + v_n(x)) v_n(x)| \cdot \chi_{B_n}(x) \leq \|f\|_{L^\infty} \cdot h(x)\,.$$

By

$$\chi_{B_n}(x) \to 0 \quad \text{a.e.} \;\; x \in \Omega$$

and the Lebesgue Dominated Convergence Theorem it follows that

$$\int_{B_n} g_n(t_n e_1 + v_n) v_n \to 0 \quad \text{if} \;\; n \to \infty\,. \tag{5.44}$$

On the other hand, it is obvious that

$$|\int_{A_n} g_n(t_n e_1 + v_n) v_n| \leq \varepsilon \int_{A_n} |v_n| \leq \varepsilon\, \|h\|_{L^1}\,. \tag{5.45}$$

By (5.44) and (5.45) it follows that

$$\lim_{n\to\infty} \int_\Omega g_n(u_n) v_n = 0,$$

which concludes our proof. □

5.3 Multivalued problems of Landesman-Lazer type with mixed resonance

Let X be a Banach space. Assume that there exists $w \in X \setminus \{0\}$, which can be supposed to have the norm 1, and a linear subspace Y of X such that

$$X = \mathrm{Sp}\{w\} \oplus Y\,. \tag{5.46}$$

Definition 14. *A subset A of X is called to be w-bounded if there exists $r \in \mathbb{R}$ such that*

$$A \subset \{x = tw + y;\ t < r, y \in Y\}\,.$$

A functional $f : X \to \mathbb{R}$ is said to be w-coercive (or, coercive with respect to the decomposition (5.46)) if

$$\lim_{t\to\infty} f(tw + y) = +\infty,$$

uniformly with respect to $y \in Y$.

The functional f is called w-bounded from below if there exists $a \in \mathbb{R}$ such that the set $[f \le a]$ is w-bounded.

All the results from 1.3 can be extended in this more general framework. For example, we shall give the variant of Proposition 3:

Proposition 8. *Let f be a w-bounded from below locally Lipschitz functional. Assume that there exists $c \in \mathbb{R}$ such that f satisfies the condition (s-PS)$_c$, and the set $[f \le a]$ is w-bounded from below for every $a < c$.*

Then there exists $\alpha > 0$ such that the set $[f \le c + \alpha]$ is w-bounded.

Proof. Assume the contrary. So, for every $\alpha > 0$, the set $[f \le c + \alpha]$ is not w-bounded. Thus, for every $n \ge 1$, there exists $r_n \ge n$ such that

$$[f \le c - \frac{1}{n^2}] \subset A_n := \{x = tw + y;\ t < r_n,\ y \in Y\}\,.$$

Therefore

$$c_n := \inf_{X\setminus A_n} f \ge c - \frac{1}{n^2}\,. \tag{5.47}$$

Since the set $[f \le c + \frac{1}{n^2}]$ is not w-bounded, there exists a sequence (z_n) in X such that $z_n = t_n w + y_n$ and

$$t_n \ge r_n + 1 + \frac{1}{n}\,,$$

$$f(z_n) \le c + \frac{1}{n^2}\,. \tag{5.48}$$

It follows that $z_n \in X \setminus A_n$ and, by (5.47) and (5.48), we find

$$f(z_n) \le c_n + \frac{2}{n^2}\,.$$

We apply Ekeland's Variational Principle to f restricted to the set $X \setminus A_n$, provided $\varepsilon = \frac{2}{n^2}$ and $\lambda = \frac{1}{n}$. So, there exists $x_n = t'_n w + y'_n \in X \setminus A_n$ such that, for every $x \in X \setminus A_n$,

$$c \le f(x_n) \le f(z_n)\,,$$

$$f(x) \ge f(x_n) - \frac{2}{n}\,\|x - x_n\|\,, \tag{5.49}$$

$$\|x_n - z_n\| \le \frac{1}{n}\,. \tag{5.50}$$

Let P ($\|P\| = 1$) be the projection of X on Sp $\{w\}$. Using the continuity of P and the relation (5.50) we obtain

$$|t_n - t'_n| \le \frac{1}{n}\,.$$

So,

$$|t'_n| \ge r_n + 1\,.$$

Therefore x_n is an interior point of the set A_n. By (5.49) it follows that, for every $v \in X$,

$$f^0(x_n, v) \ge -\frac{2}{n}\,\|v\|\,.$$

This relation and the fact that f satisfies the condition (s-PS)$_c$ imply that the sequence (x_n) contains a convergent subsequence, contradiction, because this sequence is not w-bounded. □

As an application of these results we shall study the following multivalued Landesman-Lazer problem with mixed resonance.

Let $\Omega \subset \mathbb{R}^N$ be an open, bounded set with the boundary sufficiently smooth. Consider the problem

$$\begin{cases} -\Delta u(x) - \lambda_1 u(x) \in [\underline{f}(u(x)), \overline{f}(u(x))] \quad a.e. \;\; x \in \Omega \\ \qquad\qquad u \in H_0^1(\Omega)\,. \end{cases} \tag{5.51}$$

We shall study this problem under the following hypotheses:

(f1) $\quad f \in L^\infty(\mathbb{R})\,;$

(f2) $\quad$ if $F(t) = \int_0^1 f(s)ds$, then $\lim\limits_{t\to+\infty} F(t) = 0\,;$

(f3) $\quad \lim\limits_{t\to-\infty} F(t) = +\infty\,;$

(f4) $\quad$ there exists $\alpha < \frac{1}{2}(\lambda_2 - \lambda_1)$ such that, for every $t \in \mathbb{R}$, $F(t) \le \alpha t^2$.

Define on the space $H_0^1(\Omega)$ the functional $\varphi(u) = \varphi_1(u) - \varphi_2(u)$, where

$$\varphi_1(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 - \lambda_1 u^2)dx\,,$$

$$\varphi_2(u) = \int_\Omega F(u)dx\,.$$

As in the preceding section, we observe that the functional φ is locally Lipschitz.

Let Y be the orthogonal complement of the space spanned by e_1, that is

$$H_0^1(\Omega) = \mathrm{Sp}\{e_1\} \oplus Y\,.$$

Lemma 29. *With the above notations, The following hold:*

i) there exists $r_0 > 0$ such that, for every $t \in \mathbb{R}$ and $v \in Y$ with $\|v\|_{H_0^1} \geq r_0$,

$$\varphi(te_1 + v) \geq -\varphi_2(te_1)\,;$$

ii)

$$\lim_{t\to+\infty} \varphi(te_1 + v) = \varphi_1(v),$$

uniformly on the bounded subsets of Y.

Proof. i) We first observe that, for every $t \in \mathbb{R}$ and $v \in Y$,

$$\varphi_1(te_1 + v) = \varphi_1(v)\,.$$

So,

$$\varphi(te_1 + v) = \varphi_1(v) - \varphi_2(te_1 + v)\,. \tag{5.52}$$

On the other hand,

$$\varphi_2(te_1 + v) - \varphi_2(te_1) = \int_\Omega \Big(\int_{te_1(x)}^{te_1(x)+v(x)} f(s)ds\Big)dx = \tag{5.53}$$

$$= \int_\Omega (F(te_1(x) + v(x)) - F(te_1(x)))dx\,.$$

By the Lebourg Mean Value Theorem, there exist $\theta \in (0,1)$ and $x^* \in \partial F(te_1(x) + \theta v(x))$ such that

$$F(te_1(x) + v(x)) - F(te_1(x)) = \langle x^*, v(x)\rangle\,.$$

The relation (5.53) becomes

$$\begin{cases} \varphi_2(te_1 + v) - \varphi_2(te_1) = \displaystyle\int_\Omega \langle x^*, v(x)\rangle dx \leq \int_\Omega F^0(te_1(x) + \theta v(x), v(x))dx = \\ = \displaystyle\int_\Omega \limsup_{\substack{y\to te_1(x)+\theta v(x) \\ \lambda\searrow 0}} \frac{F(y + \lambda v(x)) - F(y)}{\lambda} dx \leq \\ \leq \|f\|_{L^\infty} \cdot \displaystyle\int_\Omega |v(x)|dx = \|f\|_{L^\infty} \cdot \|v\|_{L^1} \leq C\|v\|_{H_0^1}\,. \end{cases} \tag{5.54}$$

By (5.52) and (5.54) it follows that

$$\varphi(te_1 + v) \geq \varphi_1(v) - C\|v\|_{H_0^1} - \varphi_2(te_1)\,.$$

So, for concluding the proof, it is enough to choose $r_0 > 0$ such that

$$\varphi_1(v) - C\|v\|_{H_0^1} \geq 0,$$

for every $v \in Y$ cu $\|v\|_{H_0^1} \geq r_0$. This choice is possible if we take into account the variational characterization of λ_2 and the fact that λ_1 is a simple eigenvalue. Indeed,

$$\varphi_1(v) - C\|v\|_{H_0^1} = \frac{1}{2}(\|\nabla v\|_{L^2}^2 - \lambda_1\|v\|_{L^2}^2) - C\|\nabla v\|_{L^2} \geq$$

$$\geq (\frac{1}{2} - \varepsilon)\|\nabla v\|_{L^2}^2 - \frac{\lambda_1}{2}\|v\|_{L^2}^2 \geq$$

$$\geq (\frac{1}{2} - \varepsilon)\lambda_2\|v\|_{L^2}^2 - \frac{\lambda_1}{2}\|v\|_{L^2}^2 \geq 0,$$

for $\varepsilon > 0$ sufficiently small and $\|v\|_{L^2}$ (hence, $\|v\|_{H_0^1}$) large enough.

ii) By (5.52), our statement is equivalent with

$$\lim_{t\to+\infty} \varphi_2(te_1 + v) = 0\,, \tag{5.55}$$

uniformly with respect to v, in every closed ball.

Assume, by contradiction, that there exist $R > 0$, $t_n \to +\infty$ and $v_n \in Y$ with $\|v_n\| \leq R$ such that (5.55) does not hold. So, up to a subsequence, we may assume that there exist $v \in H_0^1(\Omega)$ and $h \in L^1(\Omega)$ such that

$$v_n \rightharpoonup v \quad \text{weakly in} \quad H_0^1(\Omega)\,, \tag{5.56}$$

$$v_n \to v \quad \text{strongly in} \quad L^2(\Omega)\,, \tag{5.57}$$

$$v_n(x) \to v(x) \quad \text{for a.e.} \quad x \in \Omega\,, \tag{5.58}$$

$$|v_n(x)| \leq h(x) \quad \text{a.e.} \quad x \in \Omega\,. \tag{5.59}$$

For every $n \geq 1$, denote

$$A_n = \{x \in \Omega;\ t_n e_1(x) + v_n(x) < 0\}\,,$$

$$g_n = F(t_n e_1 + v_n)\chi_{A_n}\,.$$

By (5.59) and the choice of t_n it follows that $|A_n| \to 0$.

Using now (5.58), it is easy to observe that

$$g_n(x) \to 0 \quad \text{a.e.} \quad x \in \Omega\,.$$

By (f1) and (5.59) it follows that

$$|g_n(x)| = \chi_{A_n}(x) \cdot |\int_0^{t_n e_1(x)+v_n(x)} f(s)ds| \leq \chi_{A_n}(x) \cdot \|f\|_{L^\infty} \cdot |t_n e_1(x) + v_n(x)| \leq$$

$$\leq C|v_n(x)| \leq Ch(x) \quad \text{a.e.} \quad x \in \Omega\,.$$

Thus, by Lebesgue's Dominated Convergence Theorem,

$$\lim_{n\to\infty}\int_{A_n} F(t_n e_1 + v_n)dx = 0\,.$$

By (f2) it follows that F is bounded on $[0,\infty)$. Using again (f2) we find

$$\lim_{n\to\infty}\int_{\Omega\setminus A_n} F(t_n e_1 + v_n)dx = 0\,.$$

So

$$\lim_{n\to\infty}\varphi_2(t_n e_1 + v_n) = \lim_{n\to\infty}\int_{\Omega} F(t_n e_1 + v_n)dx = 0,$$

which contradicts our initial assumption. □

Remark 10. *As a consequence of the above result,*

$$\lim_{t\to+\infty}\varphi(te_1) = 0\,,$$

$$\liminf_{t\to+\infty}\inf_{v\in Y}\varphi(te_1 + v) \geq 0\,.$$

Thus, the set $[\varphi \leq a]$ is e_1-bounded if $a < 0$ and is not e_1-bounded for $a > 0$. Moreover, φ is not bounded from below, because

$$\lim_{t\to-\infty}\varphi(te_1) = -\lim_{t\to-\infty}\varphi_2(te_1) = -\lim_{t\to-\infty}\int_{\Omega} F(te_1) = -\infty\,.$$

Thus, by Proposition 8, it follows that φ does not satisfy the condition $(s-PS)_0$.

Theorem 31. *Assume that f does not satisfy the conditions (f1)-(f4). If the functional φ has the strong Palais-Smale property $(s-PS)_a$, for every $a \neq 0$, then the multivalued problem (5.51) has at least a nontrivial solution.*

Proof. It is sufficient to show that φ has a critical point $u_0 \in H_0^1(\Omega)\setminus\{0\}$. It is obvious that

$$\partial\varphi(u) = -\Delta u - \lambda_1 u - \partial\varphi_2(u) \quad \text{in} \;\; H^{-1}(\Omega)\,.$$

If u_0 is a critical point of φ, then there exists $w \in \partial\varphi_2(u_0)$ such that

$$-\Delta u_0 - \lambda_1 u_0 = w \quad \text{in} \;\; H^{-1}(\Omega)\,.$$

Set

$$Y_1 = \{e_1 + v;\; v \in Y\}\,.$$

We first remark the coercivity of φ on Y_1:

$$\varphi(e_1 + v) = \frac{1}{2}(\|\nabla\|_{L^2}^2 - \lambda_1\|v\|_{L^2}^2) - \lambda_1\int_{\Omega} e_1 v dx - \int_{\Omega} G(e_1 + v)dx \geq$$

$$\geq \frac{1}{2}(\|\nabla v\|_{L^2}^2 - \lambda_1 \|v\|_{L^2}^2) - \lambda_1 \|v\|_{L^2} - \alpha(1 + \|v\|_{L^2}^2 + 2\int_\Omega e_1 v dx) \geq$$

$$\geq (\frac{\lambda_2 - \lambda_1}{2} - \alpha)\|v\|_{L^2}^2 - (\lambda_1 + 2|\alpha|)\|v\|_{L^2}^2 - \alpha,$$

which tends to $+\infty$ as $\|v\|_{L^2} \to \infty$, so, as $\|v\|_{H_0^1} \to \infty$.

Putting

$$m_1 = \inf_{Y_1} \varphi > -\infty,$$

it follows that m_1 is attained, by the coercivity of φ pe Y_1. Thus, there exists $u_0 = e_1 + v_0 \in Y_1$ such that $\varphi(u_0) = m_1$. There are two possibilities:

1) $m_1 > 0$. It follows from (f1) and (f3) that

$$\lim_{t \to +\infty} \varphi(te_1) = - \lim_{t \to +\infty} \varphi_2(te_1) = 0\,,$$

$$\lim_{t \to -\infty} \varphi(te_1) = - \lim_{t \to -\infty} \varphi_2(te_1) = -\infty\,.$$

Since $m_1 > 0 = \varphi(0)$ and φ has the property $(s - \mathrm{PS})_a$ for every $a > 0$, it follows from Corollary 8 that φ has a critical value $c \geq m_1 > 0$.

2) $m_1 \leq 0$. Let

$$W = \{te_1 + v;\ t \geq 0,\ v \in Y\}\,,$$

$$c = \inf_W \varphi\,.$$

So, $c \leq m_1 \leq 0$.

If $c < 0$, it follows by (s-PS)$_c$ and the Ekeland Principle that c is a critical value of φ.

If $c = m_1 = 0$, then u_0 is a local minimum point of φ, because

$$\varphi(u_0) = \varphi(e_1 + v_0) = 0 \leq \varphi(te_1 + v),$$

for every $t \geq 0$ and $v \in Y$.

So, $u_0 \neq 0$ is a critical point of φ. □

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