



ANISOTROPIC (p, q) -EQUATIONS WITH GRADIENT DEPENDENCE

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This paper is dedicated to the memory of Professor Giuseppe Da Prato

ABSTRACT. We consider a Dirichlet elliptic equation driven by the anisotropic (p, q) -Laplacian and with a reaction that has the combined effects of a parametric concave term $\lambda u^{\tau(z)-1}$ and of a convective Carathéodory perturbation $f(z, u, Du)$. The hypotheses on $x \rightarrow f(z, x, y)$ are local near zero. Using truncation and comparison techniques and the theory of operators of monotone type, we show the existence of a positive smooth solution when the parameter is small.

1. INTRODUCTION

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2 -boundary $\partial\Omega$. In this paper we study the following anisotropic Dirichlet problem with gradient dependence (convection):

$$\left\{ \begin{array}{l} -\Delta_{p(z)}u(z) - \Delta_{q(z)}u(z) = \lambda u(z)^{\tau(z)-1} + f(z, u(z), Du(z)) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, \ u > 0, \ \lambda > 0. \end{array} \right\} \quad (P_\lambda)$$

For $r \in C(\overline{\Omega})$ with $r(z) > 1$ for all $z \in \overline{\Omega}$, by $\Delta_{r(z)}$ we denote the anisotropic r -Laplace differential operator defined by

$$\Delta_{r(z)}u = \operatorname{div} \left(|Du|^{r(z)-2} Du \right) \text{ for all } u \in W_0^{1,r(z)}(\Omega).$$

In contrast to the isotropic r -Laplacian (that is, $r(\cdot)$ is a constant function), the anisotropic one is not homogeneous and this is a source of difficulties in the study of anisotropic equations. Problem (P_λ) is driven by the sum of two such operators with different variable exponents. In the reaction we have the combined effects of a parametric concave term $\lambda u^{\tau(z)-1}$ (we assume that $\tau(z) < q(z) < p(z)$ for all $z \in \overline{\Omega}$) and of a Carathéodory perturbation $f(z, x, y)$ (that is, $f(\cdot, x, y)$ is measurable on Ω and $f(z, \cdot, \cdot)$ is continuous on $\mathbb{R} \times \mathbb{R}^N$). So, the perturbation is gradient dependent. The special feature of problem (P_λ) , is that the conditions on $x \rightarrow f(z, x, y)$ are local near zero. We are looking for positive solutions. The gradient dependence of the perturbation f , means that problem (P_λ) is not variational and so our approach

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needs to be topological. Hence, in the analysis of (P_λ) we use truncation techniques and the theory of nonlinear operators of monotone type.

For isotropic convective (p, q) -equations, there have been recently several existence results. We mention the works of Bai [2], Bai-Gasiński-Papageorgiou [3], Candito-Gasiński-Papageorgiou [4], Faria-Miyagaki-Motreanu [8], Gasiński-Krech-Papageorgiou [9], Motreanu-Tanaka [11], Zeng-Papageorgiou [15]. There are no corresponding results for anisotropic (p, q) -equations with convection. There is only the very recent work of Albalawi-Alharthi-Vetro [1], for problems with no concave term and under global conditions on the function $(x, y) \rightarrow f(z, x, y)$.

2. MATHEMATICAL BACKGROUND AND HYPOTHESES

The study of anisotropic boundary value problems, uses variable Lebesgue and Sobolev spaces. The complete theory of these spaces, can be found in the book of Diening-Harjulehto-Hästö-Ruzicka [5].

Let $L^0(\Omega)$ be the space of all measurable functions $u : \Omega \rightarrow \mathbb{R}$. As usual, we identify two such functions which differ only on a Lebesgue-null set. Also, given $r \in C(\bar{\Omega})$, we set

$$r_- = \min_{\bar{\Omega}} r \quad \text{and} \quad r_+ = \max_{\bar{\Omega}} r.$$

We define $E_1 = \{r \in C(\bar{\Omega}) : 1 < r_-\}$. Then given $r \in E_1$, the variable Lebesgue space $L^{r(z)}(\Omega)$, is defined by

$$L^{r(z)}(\Omega) = \left\{ u \in L^0(\Omega) : \rho_r(u) = \int_{\Omega} |u|^{r(z)} dz < \infty \right\}.$$

We call $\rho_r(\cdot)$ the modular function corresponding to the variable exponent $r \in E_1$. We equip this space with the so-called “Luxemburg norm”, defined by

$$\|u\|_{r(z)} = \inf \left\{ \lambda > 0 : \rho_r\left(\frac{u}{\lambda}\right) \leq 1 \right\}.$$

With this norm, $L^{r(z)}(\Omega)$ becomes a Banach space which is separable and reflexive (in fact, it is uniformly convex since $x \rightarrow |x|^{r(z)}$ is a uniformly convex function). By $r'(\cdot)$ we denote the conjugate variable exponent corresponding to $r \in E_1$, defined by

$$r'(z) = \frac{r(z)}{r(z) - 1} \quad \left(\text{that is, } \frac{1}{r(z)} + \frac{1}{r'(z)} = 1 \right) \quad \text{for all } z \in \bar{\Omega}.$$

We have $L^{r(z)}(\Omega)^* = L^{r'(z)}(\Omega)$ and the following Hölder type inequality holds

$$\int_{\Omega} |uv| dz \leq \left(\frac{1}{r_-} + \frac{1}{r'_-} \right) \|u\|_{r(z)} \|v\|_{r'(z)}$$

for all $u \in L^{r(z)}(\Omega)$, all $v \in L^{r'(z)}(\Omega)$.

If $r_1, r_2 \in E_1$ and $r_1(z) \leq r_2(z)$ for all $z \in \bar{\Omega}$, then

$$L^{r_2(z)}(\Omega) \hookrightarrow L^{r_1(z)}(\Omega).$$

Here and in the sequel, the symbol $\hookrightarrow$ denotes continuous embedding.

Using the variable Lebesgue spaces, we can define the corresponding variable Sobolev spaces. So, let $r \in E_1$. We define

$$W^{1,r(z)}(\Omega) = \left\{ u \in L^{r(z)}(\Omega) : |Du| \in L^{r(z)}(\Omega) \right\}.$$

By Du we denote the weak gradient of u . This space is equipped with the following norm

$$\|u\|_{1,r(z)} = \|u\|_{r(z)} + \|Du\|_{r(z)} \quad \text{for all } u \in W^{1,r(z)}(\Omega).$$

Here $\|Du\|_{r(z)} = \| |Du| \|_{r(z)}$. Let $r \in C^{0,1}(\overline{\Omega}) \cap E_1$ (that is, $r(\cdot)$ is Lipschitz continuous and in E_1). We define

$$W_0^{1,r(z)}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{1,r(z)}}.$$

The spaces $W^{1,r(z)}$, $W_0^{1,r(z)}(\Omega)$ are separable, reflexive Banach spaces (in fact uniformly convex). Moreover, on $W_0^{1,r(z)}(\Omega)$, the Poincaré inequality holds, namely there exists $c > 0$ such that

$$\|u\|_{r(z)} \leq c \|Du\|_{r(z)} \quad \text{for all } u \in W_0^{1,r(z)}(\Omega).$$

So, on $W_0^{1,r(z)}(\Omega)$ we can use the equivalent norm

$$\|u\| = \|Du\|_{r(z)} \quad \text{for all } u \in W_0^{1,r(z)}(\Omega).$$

Given $r \in E_1$, by $r^*(z)$ we denote the critical Sobolev variable exponent defined by

$$r^*(z) = \begin{cases} \frac{Nr(z)}{N-r(z)} & \text{if } r(z) < N \\ +\infty & \text{if } N \leq r(z) \end{cases}, \quad z \in \overline{\Omega}.$$

There is the following “variable” version of the Sobolev embedding theorem.

Proposition 2.1. *If $r \in C^{0,1}(\overline{\Omega}) \cap E_1$, then*

- (a) $W_0^{1,r(z)}(\Omega) \hookrightarrow L^{s(z)}(\Omega)$ for all $s \in C(\overline{\Omega})$, $1 \leq s_- \leq s_+ < N$ and $s(z) \leq r^*(z)$ for all $z \in \overline{\Omega}$;
- (b) the embedding in (a) is compact if $s(z) < r^*(z)$ for all $z \in \overline{\Omega}$;
- (c) if $r_1, r_2 \in C^{0,1}(\overline{\Omega}) \cap E_1$ and $r_1(z) \leq r_2(z)$ for all $z \in \overline{\Omega}$, then $W_0^{1,r_2(z)}(\Omega) \hookrightarrow W_0^{1,r_1(z)}(\Omega)$.

There is a close relation between the modular function and the norm $\|\cdot\|$. This is illustrated in the proposition that follows.

Proposition 2.2. *If $r \in C^{0,1}(\overline{\Omega}) \cap E_1$ and $u \in W_0^{1,r(z)}(\Omega) \setminus \{0\}$, then*

- (a) $\|u\| = \lambda \Leftrightarrow \rho_r\left(\frac{Du}{\lambda}\right) = 1$;
- (b) $\|u\| < 1$ ($= 1, > 1$) $\Leftrightarrow \rho_r(Du) < 1$ ($= 1, > 1$);
- (c) $\|u\| < 1 \Rightarrow \|u\|^{r_+} \leq \rho_r(Du) \leq \|u\|^{r_-}$;
- (d) $\|u\| > 1 \Rightarrow \|u\|^{r_-} \leq \rho_r(Du) \leq \|u\|^{r_+}$;
- (e) $\|u\| \rightarrow 0$ (resp. $\rightarrow +\infty$) $\Leftrightarrow \rho_r(Du) \rightarrow 0$ (resp. $\rightarrow +\infty$).

We know that

$$W_0^{1,r(z)}(\Omega)^* = W^{-1,r'(z)}(\Omega).$$

Our hypotheses on the exponents p, q and τ , are the following

$H_0 : p, q \in C^{0,1}(\overline{\Omega}), \tau \in C(\overline{\Omega})$ and $1 < \tau_- \leq \tau_+ < q_- \leq q_+ < p_- \leq p_+ < N$.

We consider the nonlinear operator $V : W_0^{1,p(z)}(\Omega) \rightarrow W^{-1,p'(z)}(\Omega)$ defined by

$$\langle V(u), h \rangle = \int_{\Omega} (|Du|^{p(z)-2} + |Du|^{q(z)-2})(Du, Dh)_{\mathbb{R}^N} dz$$

for all $u, h \in W_0^{1,p(z)}(\Omega)$.

For this operator, we have the following properties (see Fan-Zhang [7]).

Proposition 2.3. *If hypotheses H_0 hold, then $V(\cdot)$ is bounded (that is, maps bounded sets to bounded sets), continuous, strictly monotone (thus maximal monotone too) and of type $(S)_+$, that is, “if $u_n \xrightarrow{w} u$ in $W_0^{1,p(z)}(\Omega)$ and $\limsup_{n \rightarrow \infty} \langle V(u_n), u_n - u \rangle \leq 0$, then $u_n \rightarrow u$ in $W_0^{1,p(z)}(\Omega)$ ”.*

The global regularity theory of Fan [6], will lead us to the space

$$C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0\}.$$

This is an ordered Banach space, with positive (order) cone

$$C_+ = \{u \in C_0^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\text{int}C_+ = \left\{ u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega, \left. \frac{\partial u}{\partial n} \right|_{\partial\Omega} < 0 \right\},$$

where $\frac{\partial u}{\partial n} = (Du, n)_{\mathbb{R}^N}$ with $n(\cdot)$ being the outward unit normal on $\partial\Omega$.

If $u \in L^0(\Omega)$, then we set

$$u^+(z) = \max\{u(z), 0\}, \quad u^-(z) = \max\{-u(z), 0\}, \quad z \in \Omega.$$

We have $u = u^+ - u^-$, $|u| = u^+ + u^-$ and if $u \in W_0^{1,p(z)}(\Omega)$, then $u^{\pm} \in W_0^{1,p(z)}(\Omega)$. If $u, v \in L^0(\Omega)$ and $u(z) \leq v(z)$ for a.a. $z \in \Omega$, then we define

$$[u, v] = \left\{ h \in W_0^{1,p(z)}(\Omega) : u(z) \leq h(z) \leq v(z) \text{ for a.a. } z \in \Omega \right\}.$$

The hypothesis on the convective term $f(z, x, y)$, are the following (recall that $\mathbb{R}_+ = [0, \infty)$):

$H_1 : f : \Omega \times \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function such that

(i) there exist $c > 0$ and $\vartheta > 0$ such that

$$c\vartheta^{\tau(z)-1} + f(z, \vartheta, 0) \leq 0 \quad \text{for a.a. } z \in \Omega;$$

(ii) $|f(z, x, y)| \leq \widehat{a}(z)(1 + |y|^{p_- - 1})$ for a.a. $z \in \Omega$, all $0 \leq x \leq \vartheta$, all $y \in \mathbb{R}^N$;

(iii) there exists $\delta \in (0, \vartheta)$ such that $0 \leq f(z, x, y)$ for a.a. $z \in \Omega$, all $0 \leq x \leq \delta$, all $y \in \mathbb{R}^N$.

Remark 2.4. We see that the conditions on $x \rightarrow f(z, x, y)$ are local and they imply an oscillatory behavior of the reaction near zero.

3. AN AUXILIARY DIRICHLET PROBLEM

In this section, we deal with the following parametric Dirichlet problem

$$-\Delta_{p(z)}u(z) - \Delta_{q(z)}u(z) = \lambda u^{\tau(z)-1} \text{ in } \Omega, \quad u|_{\partial\Omega} = 0, \quad \lambda > 0, \quad u > 0. \quad (Q_\lambda)$$

Proposition 3.1. *If hypotheses H_0 hold and $\lambda > 0$, then problem (Q_λ) has a unique solution $\bar{v}_\lambda \in \text{int}C_+$, $\{\bar{v}_\lambda\}_{\lambda>0}$ is nondecreasing, $\bar{v}_\lambda \rightarrow 0$ in $C_0^1(\bar{\Omega})$ as $\lambda \rightarrow 0^+$.*

Proof. We consider the C^1 -functional $\sigma_\lambda : W_0^{1,p(z)}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\sigma_\lambda(u) = \int_\Omega \frac{1}{p(z)} |Du|^{p(z)} dz + \int_\Omega \frac{1}{q(z)} |Du|^{q(z)} dz - \lambda \int_\Omega \frac{1}{\tau(z)} (u^+)^{\tau(z)} dz$$

for all $u \in W_0^{1,p(z)}(\Omega)$.

If $\|u\| \geq 1$ and $\|u\|_{\tau(z)} \geq 1$, then

$$\begin{aligned} \sigma_\lambda(u) &\geq \frac{1}{p_+} \rho_p(Du) - \frac{\lambda}{\tau_-} \rho_\tau(u) \\ &\geq \frac{1}{p_+} \|u\|^{p_-} - \frac{\lambda}{\tau_-} c_1 \|u\|^{\tau_+} \text{ for some } c_1 > 0 \\ &\quad (\text{ see Propositions 2.1 and 2.2}). \end{aligned}$$

But $\tau_+ < p_-$ (see hypotheses H_0). So, it follows that $\sigma_\lambda(\cdot)$ is coercive. Moreover using Proposition 2.1, we see that $\sigma_\lambda(\cdot)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem, we can find $\bar{v}_\lambda \in W_0^{1,p(z)}(\Omega)$ such that

$$(3.1) \quad \sigma_\lambda(\bar{v}_\lambda) = \inf \left\{ \sigma_\lambda(u) : u \in W_0^{1,p(z)}(\Omega) \right\}.$$

Let $v \in \text{int}C_+$ and $t \in (0, 1)$. We have

$$\begin{aligned} \sigma_\lambda(tv) &\leq \frac{t^{q_-}}{q_-} [\rho_p(Dv) + \rho_q(Dv)] - \frac{t^{\tau_+}}{\tau_+} \rho_\tau(v) \\ &= c_2 t^{q_-} - c_3 t^{\tau_+} \text{ for some } c_2 > 0, c_3 > 0. \end{aligned}$$

Since by hypotheses H_0 , $\tau_+ < q_-$, we see that by taking $t \in (0, 1)$ small, we have

$$\begin{aligned} \sigma_\lambda(tv) &< 0, \\ \Rightarrow \sigma_\lambda(\bar{v}_\lambda) &< 0 = \sigma_\lambda(0) \text{ (see (3.1))}, \\ \Rightarrow \bar{v}_\lambda &\neq 0. \end{aligned}$$

From (3.1) we have

$$\begin{aligned} \langle \sigma'_\lambda(\bar{v}_\lambda), h \rangle &= 0 \quad \text{for all } h \in W_0^{1,p(z)}(\Omega), \\ (3.2) \quad \Rightarrow \langle V(\bar{v}_\lambda), h \rangle &= \lambda \int_\Omega (\bar{v}_\lambda^+)^{\tau(z)-1} h dz \quad \text{for all } h \in W_0^{1,p(z)}(\Omega). \end{aligned}$$

In (3.2) we use the test function $h = -\bar{v}_\lambda^- \in W_0^{1,p(z)}(\Omega)$. Then

$$\begin{aligned} \rho_p(D\bar{v}_\lambda^-) &\leq 0, \\ \Rightarrow \bar{v}_\lambda &\geq 0, \quad \bar{v}_\lambda \neq 0 \text{ (see Proposition 2.2)}. \end{aligned}$$

From Proposition A1 of Papageorgiou-Rădulescu-Zhang [13], we know that $\bar{v}_\lambda \in L^\infty(\Omega)$. Then from the anisotropic regularity theory of Fan [6], we have that $\bar{v}_\lambda \in C_+ \setminus \{0\}$. Note that

$$-\Delta_{p(z)}\bar{v}_\lambda - \Delta_{q(z)}\bar{v}_\lambda \geq 0 \text{ in } \Omega.$$

Therefore from Proposition A2 of [13], we infer that

$$\bar{v}_\lambda \in \text{int}C_+.$$

Next we show the uniqueness of this positive solution. Suppose $\bar{w}_\lambda \in W_0^{1,p(z)}(\Omega)$ is another positive solution of problem (Q_λ) . Again we have $\bar{w}_\lambda \in \text{int}C_+$. So, using Proposition 4.1.22, p.274, of Papageorgiou-Rădulescu-Repovš [12], we have

$$(3.3) \quad \frac{\bar{v}_\lambda}{\bar{w}_\lambda} \in L^\infty(\Omega), \quad \frac{\bar{w}_\lambda}{\bar{v}_\lambda} \in L^\infty(\Omega).$$

Let $j : L^1(\Omega) \rightarrow \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ be the integral functional defined by

$$j(u) = \begin{cases} \int_\Omega \frac{1}{p(z)} \left| Du^{\frac{1}{q_-}} \right|^{p(z)} dz + \int_\Omega \frac{1}{q(z)} \left| Du^{\frac{1}{q_-}} \right|^{q(z)} dz & \text{if } u \geq 0, u^{\frac{1}{q_-}} \in W_0^{1,p(z)}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

From Theorem 2.2 of Takáč-Giacomoni [14], we know that $j(\cdot)$ is convex. Let $\text{dom}j = \{u \in L^1(\Omega) : j(u) < \infty\}$ (the effective domain of $j(\cdot)$) and consider $h = \bar{v}_\lambda^{q_-} - \bar{w}_\lambda^{q_-} \in C_0^1(\bar{\Omega})$. Using (3.3) we see that $t \in (0, 1)$ small

$$\bar{v}_\lambda^{q_-} + th \in \text{dom}j, \quad \bar{w}_\lambda^{q_-} + th \in \text{dom}j.$$

Therefore exploiting the convexity of $j(\cdot)$, we can compute the directional derivatives of $j(\cdot)$ at $\bar{v}_\lambda^{q_-}$ and at $\bar{w}_\lambda^{q_-}$ in the direction h . From Theorem 2.5 of [14], we have

$$\begin{aligned} j'(\bar{v}_\lambda^{q_-})(h) &= \frac{1}{q_-} \int_\Omega \frac{-\Delta_{p(z)}\bar{v}_\lambda - \Delta_{q(z)}\bar{v}_\lambda}{\bar{v}_\lambda^{q_- - 1}} h \, dz \\ &= \frac{1}{q_-} \int_\Omega \frac{1}{\bar{v}_\lambda^{q_- - \tau(z)}} h \, dz, \\ j'(\bar{w}_\lambda^{q_-})(h) &= \frac{1}{q_-} \int_\Omega \frac{-\Delta_{p(z)}\bar{w}_\lambda - \Delta_{q(z)}\bar{w}_\lambda}{\bar{w}_\lambda^{q_- - 1}} h \, dz \\ &= \frac{1}{q_-} \int_\Omega \frac{1}{\bar{w}_\lambda^{q_- - \tau(z)}} h \, dz. \end{aligned}$$

Recalling that the directional derivative of a convex function is monotone, we have

$$\begin{aligned} 0 &\leq \int_\Omega \left(\frac{1}{\bar{v}_\lambda^{q_- - \tau(z)}} - \frac{1}{\bar{w}_\lambda^{q_- - \tau(z)}} \right) (\bar{v}_\lambda^{q_-} - \bar{w}_\lambda^{q_-}) \, dz \leq 0, \\ &\Rightarrow \bar{v}_\lambda = \bar{w}_\lambda \quad (\text{recall that } \tau_+ < q_-). \end{aligned}$$

This proves the uniqueness of the positive solution $\bar{v}_\lambda \in \text{int}C_+$ of problem (Q_λ) .

Next we show that $\{\bar{v}_\lambda\}_{\lambda>0}$ is nondecreasing. Let $c < \mu < \lambda$ and introduce the Carathéodory function $g_\mu(z, x)$ defined by

$$(3.4) \quad g_\mu(z, x) = \begin{cases} \mu(x^+)^{\tau(z)-1} & \text{if } x \leq \bar{v}_\lambda(z) \\ \mu\bar{v}_\lambda(z)^{\tau(z)-1} & \text{if } \bar{v}_\lambda(z) < x. \end{cases}$$

We examine the following anisotropic Dirichlet problem

$$-\Delta_{p(z)}v(z) - \Delta_{q(z)}v(z) = g_\mu(z, v(z)) \text{ in } \Omega, \quad v|_{\partial\Omega} = 0.$$

We set $G_\mu(z, x) = \int_0^x g_\mu(z, s) ds$ and consider the C^1 -functional $\widehat{\sigma}_\mu : W_0^{1,p(z)}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\widehat{\sigma}_\mu(u) = \int_\Omega \frac{1}{p(z)} |Du|^{p(z)} dz + \int_\Omega \frac{1}{q(z)} |Du|^{q(z)} dz - \int_\Omega G_\mu(z, u) dz$$

for all $u \in W_0^{1,p(z)}(\Omega)$.

It is clear from (3.4) that $\widehat{\sigma}_\mu(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous (see Proposition 2.1). Therefore we can find $\widehat{v}_\mu \in W_0^{1,p(z)}(\Omega)$ such that

$$(3.5) \quad \widehat{\sigma}_\mu(\widehat{v}_\mu) = \inf \left\{ \widehat{\sigma}_\mu(v) : v \in W_0^{1,p(z)}(\Omega) \right\}.$$

Recall that $\bar{v}_\lambda \in \text{int}C_+$. So, if $u \in C_+ \setminus \{0\}$, then we can find $t \in (0, 1)$ small such that

$$0 \leq tu \leq \bar{v}_\lambda \text{ (see [12], Proposition 4.1.22, p.274).}$$

Then from (3.4) and as before using the hypothesis that $\tau_+ < q_-$, for $t \in (0, 1)$ small, we have

$$\begin{aligned} \widehat{\sigma}_\mu(tu) &< 0, \\ \Rightarrow \widehat{\sigma}_\mu(\widehat{v}_\mu) &< 0 = \widehat{\sigma}_\mu(0) \text{ (see (3.5))}, \\ \Rightarrow \widehat{v}_\mu &\neq 0. \end{aligned}$$

From (3.5), we have

$$(3.6) \quad \begin{aligned} \langle \widehat{\sigma}'_\mu(\widehat{v}_\mu), h \rangle &= 0 \quad \text{for all } h \in W_0^{1,p(z)}(\Omega), \\ \Rightarrow \langle V(\widehat{v}_\mu), h \rangle &= \int_\Omega g_\mu(z, \widehat{v}_\mu) h dz \quad \text{for all } h \in W_0^{1,p(z)}(\Omega). \end{aligned}$$

Using $h = -\widehat{v}_\mu^- \in W_0^{1,p(z)}(\Omega)$, we obtain

$$\begin{aligned} \rho_p(D\widehat{v}_\mu^-) &\leq 0 \quad \text{(see (3.4))}, \\ \Rightarrow \widehat{v}_\mu &\geq 0, \widehat{v}_\mu \neq 0 \quad \text{(see Proposition 2.2)}. \end{aligned}$$

Next in (3.6) we choose the test function $h = (\widehat{v}_\mu - \bar{v}_\lambda)^+ \in W_0^{1,p(z)}(\Omega)$. Then

$$\begin{aligned} & \langle V(\widehat{v}_\mu), (\widehat{v}_\mu - \bar{v}_\lambda)^+ \rangle \\ &= \int_{\Omega} \mu \bar{v}_\lambda^{\tau(z)-1} (\widehat{v}_\mu - \bar{v}_\lambda)^+ dz \quad (\text{see (3.4)}) \\ &\leq \int_{\Omega} \lambda \bar{v}_\lambda^{\tau(z)-1} (\widehat{v}_\mu - \bar{v}_\lambda)^+ dz \quad (\text{since } \mu < \lambda) \\ &= \langle V(\bar{v}_\lambda), (\widehat{v}_\mu - \bar{v}_\lambda)^+ \rangle, \\ &\Rightarrow \widehat{v}_\mu \leq \bar{v}_\lambda \quad (\text{see Proposition 2.3}). \end{aligned}$$

So, we have proved that

$$\widehat{v}_\mu \in [0, \bar{v}_\lambda], \widehat{v}_\mu \neq 0.$$

From (3.4) and (3.6) it follows that $\widehat{v}_\mu$ is a nontrivial solution of (Q_μ) . Therefore

$$\begin{aligned} & \widehat{v}_\mu = \bar{v}_\mu, \\ & \Rightarrow \bar{v}_\mu \leq \bar{v}_\lambda \text{ for } 0 < \mu < \lambda, \\ & \Rightarrow \{\bar{v}_\lambda\}_{\lambda>0} \text{ is nondecreasing.} \end{aligned}$$

Finally we show that $\bar{v}_\lambda \rightarrow 0$ in $C_0^1(\overline{\Omega})$ as $\lambda \rightarrow 0^+$. If $0 < \lambda \leq 1$, then

$$(3.7) \quad 0 \leq \bar{v}_\lambda(z) \leq \bar{v}_1(z) \quad \text{for all } z \in \overline{\Omega}.$$

Then from Fan [6] we know that there exist $\alpha \in (0, 1)$ and $c_4 > 0$ such that

$$\bar{v}_\lambda \in C_0^{1,\alpha}(\overline{\Omega}), \|\bar{v}_\lambda\|_{C_0^{1,\alpha}(\overline{\Omega})} \leq c_4 \quad \text{for all } 0 < \lambda \leq 1.$$

We know that $C_0^{1,\alpha}(\overline{\Omega}) \hookrightarrow C_0^1(\overline{\Omega})$ compactly (Arzela-Ascoli theorem). Therefore, we may assume that

$$\bar{v}_\lambda \rightarrow v^* \text{ in } C_0^1(\overline{\Omega}) \text{ as } \lambda \rightarrow 0^+.$$

But note that from (3.7) and since $\bar{v}_1 \in L^\infty(\Omega)$, we have

$$\begin{aligned} & \rho_p(D\bar{v}_\lambda) \leq \lambda c_5 \text{ for some } c_5 > 0, \text{ all } 0 < \lambda \leq 1, \\ & \Rightarrow \bar{v}_\lambda \rightarrow 0 \text{ in } C_0^1(\overline{\Omega}) \text{ as } \lambda \rightarrow 0^+. \end{aligned}$$

This completes the proof. □

4. POSITIVE SOLUTIONS

Using Proposition 3.1, we can find $\widehat{\lambda} \in (0, c]$ (here $c > 0$ is as in hypothesis $H_1(i)$) such that

$$0 \leq \bar{v}_\lambda(z) \leq \delta \text{ for a.a. } x \in \Omega, \text{ all } 0 < \lambda \leq \widehat{\lambda},$$

with $\delta > 0$ as postulated by hypothesis $H_1(iii)$. Then using that hypothesis, we have

$$(4.1) \quad 0 \leq f(z, \bar{v}_\lambda(z), D\bar{v}_\lambda(z)) \text{ for a.a. } z \in \Omega, \text{ all } 0 < \lambda \leq \widehat{\lambda}.$$

Recalling that $\delta \leq \vartheta$ (see hypothesis H_1 (iii)), we can introduce the truncation map $\tau_0 : L^{p(z)}(\Omega) \rightarrow L^{p(z)}(\Omega)$ defined by

$$(4.2) \quad \tau_0(u)(z) = \begin{cases} \bar{v}_\lambda(z) & \text{if } u(z) < \bar{v}_\lambda(z) \\ u(z) & \text{if } \bar{v}_\lambda(z) \leq u(z) \leq \vartheta \\ \vartheta & \text{if } \vartheta < u(z). \end{cases}$$

We have

$$(4.3) \quad D\tau_0(u)(z) = \begin{cases} D\bar{v}_\lambda(z) & \text{if } u(z) < \bar{v}_\lambda(z) \\ Du(z) & \text{if } \bar{v}_\lambda(z) \leq u(z) \leq \vartheta \\ 0 & \text{if } \vartheta < u(z) \end{cases} \quad (\text{see [5], p.244}).$$

From (4.2) and (4.3) it follows that $\tau_0 : W_0^{1,p(z)}(\Omega) \rightarrow W_0^{1,p(z)}(\Omega)$ is well-defined.

Proposition 4.1. *If hypotheses H_0 hold, then $\tau_0 : W_0^{1,p(z)}(\Omega) \rightarrow W_0^{1,p(z)}(\Omega)$ is continuous.*

Proof. Consider a sequence $\{u_n\}_{n \in \mathbb{N}} \subseteq W_0^{1,p(z)}(\Omega)$ such that

$$u_n \rightarrow u \text{ in } W_0^{1,p(z)}(\Omega).$$

We may assume that

$$(4.4) \quad \begin{aligned} Du_n(z) &\rightarrow Du(z) \text{ in } \mathbb{R}^N \text{ for a.a. } z \in \Omega, \\ \Rightarrow D\tau_0(u_n)(z) &\rightarrow D\tau_0(u)(z) \text{ in } \mathbb{R}^N \text{ for a.a. } z \in \Omega \text{ (see (4.3)).} \end{aligned}$$

Since $W_0^{1,p(z)}(\Omega) \hookrightarrow W_0^{1,p^-}(\Omega)$ (see Proposition 2.1), we see that

$$(4.5) \quad \begin{aligned} &\left\{ |D\tau_0(u_n)(\cdot) - D\tau_0(u)(\cdot)|^{p(\cdot)} \right\}_{n \in \mathbb{N}} \subseteq L^{p^-}(\Omega) \text{ is bounded,} \\ \Rightarrow &\left\{ |D\tau_0(u_n)(\cdot) - D\tau_0(u)(\cdot)|^{p(\cdot)} \right\}_{n \in \mathbb{N}} \subseteq L^1(\Omega) \text{ is uniformly integrable.} \end{aligned}$$

From (4.4), (4.5) and Vitali's theorem, it follows that

$$\begin{aligned} &\int_{\Omega} |D\tau_0(u_n) - D\tau_0(u)|^{p(z)} dz \rightarrow 0 \text{ as } n \rightarrow \infty, \\ \Rightarrow &\rho_p(D\tau_0(u_n) - D\tau_0(u)) \rightarrow 0 \text{ as } n \rightarrow \infty, \\ \Rightarrow &\|D\tau_0(u_n) - D\tau_0(u)\|_{p(z)} \rightarrow 0 \text{ as } n \rightarrow \infty \text{ (see Proposition 2.2),} \\ \Rightarrow &\tau_0(u_n) \rightarrow \tau_0(u) \text{ in } W_0^{1,p(z)}(\Omega) \text{ (by Poincaré's inequality),} \\ \Rightarrow &\tau_0(\cdot) \text{ is continuous on } W_0^{1,p(z)}(\Omega). \end{aligned}$$

The proof is now complete. □

On account of (4.2), (4.3) and hypothesis H_1 (ii), we have

$$f(\cdot, \tau_0(u)(\cdot), D\tau_0(u)(\cdot)) \in L^{p'(z)}(\Omega).$$

Also we know that

$$\begin{aligned} L^{p'(z)}(\Omega) &\hookrightarrow W_0^{1,p(z)}(\Omega)^* = W^{-1,p'(z)}(\Omega) \\ &\quad (\text{see Gasiński-Papageorgiou [10], p.141}). \end{aligned}$$

Therefore we can define the map $N : W_0^{1,p(z)}(\Omega) \rightarrow W^{-1,p'(z)}(\Omega)$ by

$$N(u)(\cdot) = f(\cdot, \tau_0(u)(\cdot), D\tau_0(u)(\cdot)) \text{ for all } u \in W_0^{1,p(z)}(\Omega).$$

For $0 < \lambda \leq \widehat{\lambda}$ and $u \in W_0^{1,p(z)}(\Omega)$, we set

$$K_\lambda(u) = V(u) - \lambda \tau_0(u)^{\tau(\cdot)-1} - N(u).$$

Then $K_\lambda : W_0^{1,p(z)}(\Omega) \rightarrow W^{-1,p'(z)}(\Omega)$ and clearly it is bounded and continuous.

Proposition 4.2. *If hypotheses H_0 and H_1 hold and $0 < \lambda \leq \widehat{\lambda}$, then $K_\lambda(\cdot)$ is pseudomonotone.*

Proof. Consider a sequence $\{u_n\}_{n \in \mathbb{N}} \subseteq W_0^{1,p(z)}(\Omega)$ such that

$$(4.6) \quad \left\{ \begin{array}{l} u_n \xrightarrow{w} u \text{ in } W_0^{1,p(z)}(\Omega), K_\lambda(u_n) \xrightarrow{w} u^* \text{ in } W^{-1,p'(z)}(\Omega), \\ \limsup_{n \rightarrow \infty} \langle K_\lambda(u_n), u_n - u \rangle \leq 0. \end{array} \right\}$$

From Proposition 2.1, we know $W_0^{1,p(z)}(\Omega) \hookrightarrow L^{p(z)}(\Omega)$ compactly. Therefore from the first convergence in (4.6), we have

$$(4.7) \quad \begin{aligned} u_n &\rightarrow u \text{ in } L^{p(z)}(\Omega) \text{ as } n \rightarrow \infty, \\ \Rightarrow \lambda \tau_0(u_n)^{\tau(\cdot)-1} &\rightarrow \lambda \tau_0(u)^{\tau(\cdot)-1} \text{ in } L^{\tau'(\cdot)}(\Omega) \hookrightarrow W^{-1,p'(z)}(\Omega) \\ &\quad (\text{recall that } \tau(\cdot) < p(\cdot)). \end{aligned}$$

Also using Hölder's inequality (see Section 2), we have

$$(4.8) \quad \int_{\Omega} N(u_n)(u_n - u) dz \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Therefore from (4.7), (4.8) and the inequality in (4.6), we infer that

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle V(u_n), u_n - u \rangle &\leq 0, \\ \Rightarrow u_n &\rightarrow u \text{ in } W_0^{1,p(z)}(\Omega) \text{ as } n \rightarrow \infty \text{ (see Proposition 2.3)}. \end{aligned}$$

The continuity of $K_\lambda(\cdot)$ implies that

$$\begin{aligned} K_\lambda(u_n) &\rightarrow K_\lambda(u) \text{ in } W^{-1,p'(z)}(\Omega), \\ \Rightarrow u^* &= K_\lambda(u) \text{ and } \langle K_\lambda(u_n), u_n \rangle \rightarrow \langle K_\lambda(u), u \rangle. \end{aligned}$$

Then by Proposition 2.10.3, p.151, of Papageorgiou-Rădulescu-Repovš [12], we have that $K_\lambda(\cdot)$ is pseudomonotone. $\square$

Proposition 4.3. *If hypotheses H_0 and H_1 hold and $0 < \lambda \leq \widehat{\lambda}$, then $K_\lambda(\cdot)$ is strongly coercive, that is,*

$$\frac{\langle K_\lambda(u), u \rangle}{\|u\|} \rightarrow +\infty \text{ as } \|u\| \rightarrow \infty.$$

Proof. Let $u \in W_0^{1,p(z)}(\Omega)$ with $\|u\| \geq 1$. Then

$$\begin{aligned}
 & \langle K_\lambda(u), u \rangle \\
 &= \langle V(u), u \rangle - \int_{\Omega} \lambda \tau_0(u)^{\tau(z)-1} u \, dz - \int_{\Omega} f(z, \tau_0(u), D\tau_0(u)) u \, dz \\
 (4.9) \quad & \geq \rho_p(Du) - \lambda c_6 \|u\| - c_7 \left[\int_{\Omega} |D\tau_0(u)|^{p_- - 1} |u| \, dz + \|u\| \right] \\
 & \quad \text{for some } c_6, c_7 > 0 \text{ (see (4.2) and hypothesis } H_1(ii)).
 \end{aligned}$$

We have

$$\begin{aligned}
 & \int_{\Omega} |D\tau_0(u)|^{p_- - 1} |u| \, dz \\
 & \leq \int_{\{u < \bar{v}_\lambda\}} |D\bar{v}_\lambda|^{p_- - 1} |u| \, dz + \int_{\{\bar{v}_\lambda \leq u \leq \vartheta\}} |Du|^{p_- - 1} \vartheta \, dz \text{ (see (4.3))} \\
 (4.10) \quad & \leq c_8 [\|u\| + \|u\|^{p_- - 1}] \text{ for some } c_8 > 0 \\
 & \quad \text{(recall that } W_0^{1,p(z)} \hookrightarrow W_0^{1,p_-}(\Omega)).
 \end{aligned}$$

Returning to (4.9) and using (4.10), we obtain

$$\begin{aligned}
 & \langle K_\lambda(u), u \rangle \geq \|u\|^{p_-} - (\lambda c_6 + c_9) \|u\| - c_8 \|u\|^{p_- - 1} \text{ for some } c_9 > 0 \\
 & \quad \text{(and recall that } \|u\| \geq 1) \\
 & \Rightarrow K_\lambda(\cdot) \text{ is strongly coercive (recall that } p_- > 1).
 \end{aligned}$$

This completes the proof. $\square$

Now we are ready for the existence theorem.

Theorem 4.4. *If hypotheses H_0 and H_1 hold and $0 < \lambda \leq \widehat{\lambda}$, then problem (P_λ) has a positive solution $u_\lambda^* \in \text{int}C_+$.*

Proof. A strongly coercive, pseudomonotone map is surjective (see Theorem 2.10.10, p.156, of [12]). So, we can find $u_\lambda^* \in W_0^{1,p(z)}(\Omega)$ such that

$$\begin{aligned}
 & K_\lambda(u_\lambda^*) = 0 \text{ in } W^{-1,p'(z)}(\Omega), \\
 (4.11) \quad & \Rightarrow \langle V(u_\lambda^*), h \rangle = \int_{\Omega} \left[\lambda \tau_0(u_\lambda^*)^{\tau(z)-1} + f(z, \tau_0(u_\lambda^*), D\tau_0(u_\lambda^*)) \right] h \, dz \\
 & \quad \text{for all } h \in W_0^{1,p(z)}(\Omega).
 \end{aligned}$$

In (4.11) first we choose the test function $h = (\bar{v}_\lambda - u_\lambda^*)^+ \in W_0^{1,p(z)}(\Omega)$. Then

$$\begin{aligned} & \langle V(u_\lambda^*), (\bar{v}_\lambda - u_\lambda^*)^+ \rangle \\ &= \int_\Omega \lambda \bar{v}_\lambda^{\tau(z)-1} (\bar{v}_\lambda - u_\lambda^*)^+ dz + \int_\Omega f(z, \bar{v}_\lambda, D\bar{v}_\lambda) (\bar{v}_\lambda - u_\lambda^*)^+ dz \\ & \hspace{15em} \text{(see (4.2), (4.3))} \\ &\geq \int_\Omega \lambda \bar{v}_\lambda^{\tau(z)-1} (\bar{v}_\lambda - u_\lambda^*)^+ dz \quad \text{(see (4.1))} \\ &= \langle V(\bar{v}_\lambda), (\bar{v}_\lambda - u_\lambda^*)^+ \rangle \quad \text{(see Proposition 3.1),} \\ &\Rightarrow \bar{v}_\lambda \leq u_\lambda^* \quad \text{(see Proposition 2.3).} \end{aligned}$$

Next in (4.11) we use $h = (u_\lambda^* - \vartheta)^+ \in W_0^{1,p(z)}(\Omega)$. Then

$$\begin{aligned} & \langle V(u_\lambda^*), (u_\lambda^* - \vartheta)^+ \rangle \\ &= \int_\Omega \left[\lambda \vartheta^{\tau(z)-1} + f(z, \vartheta, 0) \right] (u_\lambda^* - \vartheta)^+ dz \quad \text{(see (4.2), (4.3))} \\ &\leq 0 = \langle V(\vartheta), (u_\lambda^* - \vartheta)^+ \rangle \quad \text{(see hypothesis H}_1\text{(i) and recall that } \widehat{\lambda} \leq c), \\ &\Rightarrow u_\lambda^* \leq \vartheta. \end{aligned}$$

We have proved that

$$u_\lambda^* \in [\bar{v}_\lambda, \vartheta].$$

Then from (4.2), (4.3) and (4.11) it follows that

$$u_\lambda^* \text{ is a positive solution of } (P_\lambda).$$

Moreover, from Fan [6], we have that $u_\lambda^* \in \text{int } C_+$.

The proof is now complete. □

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