



Symmetry breaking of vectorial ground states for fractional Hartree–Fock systems with nonlinear coupling

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Abstract

This paper is concerned with the existence and symmetry breaking properties of vectorial ground states for the following fractional Hartree–Fock system with a general potential in $\mathbb{R}^2$:

$$\begin{cases} (-\Delta)^s u + u + l(x)\phi_{l,(u,v)}^t u = (|u|^{2p} + |v|^{p+1}|u|^{p-1})u & \text{in } \mathbb{R}^2, \\ (-\Delta)^s v + v + l(x)\phi_{l,(u,v)}^t v = (|v|^{2p} + |u|^{p+1}|v|^{p-1})v & \text{in } \mathbb{R}^2, \end{cases}$$

where $\frac{1}{2} < t \leq s < 1$ and $0 < p < \frac{1}{2}$. By exploring the interaction of double coupled terms and introducing some new estimates in combination with the multibump technique of Ruiz (*J. Funct. Anal.* 237 (2006) 655–674), we first establish the existence and multiplicity of positive radial solutions for the above system with a constant potential. Combining this result with the global compactness principle, the existence and nonexistence of non-radial ground state solutions are also obtained for the system without any symmetry on the potential function $l(x)$. When $l(x)$ is symmetric, we get a non-radial ground state solution for the above system with parameters, which implies that the symmetry breaking phenomenon can occur. Finally, by imposing some additional assumptions on $l(x)$, three positive solutions are obtained.

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1 Introduction

This paper is concerned with the existence of solutions to the following planar fractional Hartree–Fock (fHF) system

$$\begin{cases} (-\Delta)^s u + u + l(x)\phi_{l,(u,v)}^t u = (|u|^{2p} + |v|^{p+1}|u|^{p-1})u & \text{in } \mathbb{R}^2, \\ (-\Delta)^s v + v + l(x)\phi_{l,(u,v)}^t v = (|v|^{2p} + |u|^{p+1}|v|^{p-1})v & \text{in } \mathbb{R}^2, \end{cases} \quad (S_l)$$

where $\frac{1}{2} < t \leq s < 1$, $0 < p < \frac{1}{2}$ and the operator $(-\Delta)^s$ is the fractional Laplacian of order s , defined by

$$(-\Delta)^s u(x) = C_s P.V. \int_{\mathbb{R}^2} \frac{u(x) - u(y)}{|x - y|^{2+2s}} dy,$$

here $P.V.$ stands for the Cauchy principal value and C_s is called the normalized constant, that is, $C_s = \frac{2^{2s} \Gamma(1+s)}{\pi \Gamma(-s)}$, and

$$\phi_{l,(u,v)}^t = C_t \int_{\mathbb{R}^2} \frac{l(y) (u^2(y) + v^2(y))}{|x - y|^{2-2t}} dy \in D^{t,2}(\mathbb{R}^2) \quad (1)$$

is the unique solution of

$$(-\Delta)^t \phi = l(x) (u^2 + v^2) \quad \text{in } \mathbb{R}^2. \quad (2)$$

Coupled elliptic systems (S_l) arise in various branches of mathematical physics and nonlinear optics. Moreover, solutions of system (S_l) are related with standing wave solutions of a two-component system of nonlinear Schrödinger equations. For more details on the physical background, we refer the readers to [4, 11, 20, 21, 37] and references therein.

Remark 1.1 From the expressions of $\phi_{l,(u,v)}^t$, it is easy to know that the fractional planar Hartree–Fock system is similar to that in $\mathbb{R}^N$ ($N > 2s$). The reason why we consider $N = 2$ in this paper is mainly to obtain a comparison with the classical planar Hartree–Fock system below.

For a solution of system (S_I) , we introduce some concepts of its triviality and positiveness.

Definition 1.2 A vector function (u, v) is said to be

- (i) nontrivial if either $u \neq 0$ or $v \neq 0$;
- (ii) semitrivial if it is nontrivial but either $u = 0$ or $v = 0$;
- (iii) vectorial if both u and v are not zero;
- (iv) nonnegative if $u \geq 0$ and $v \geq 0$;
- (v) positive if $u > 0$ and $v > 0$.

The fractional Schrödinger equation:

$$i \frac{\partial \psi}{\partial t} = (-\Delta)^s \psi + V(x)\psi - f(x, \psi) \quad \text{in } \mathbb{R}^N \quad (3)$$

is a fundamental equation of fractional quantum mechanics in the study of particles on stochastic fields modeled by Lévy process [20, 21], where $0 < s < 1$, $(x, t) \in \mathbb{R}^N \times (0, +\infty)$, f satisfies $f(x, e^{i\theta}) = e^{i\theta} f(x, \psi)$ and $V : \mathbb{R}^N \rightarrow \mathbb{R}$ is an external function. By using a similar argument to that of the classical case $s = 1$, the standing wave solution to Eq.(3) is of the form

$$\psi(x, t) = e^{-i\omega t} u(x),$$

where u solves the following fractional elliptic equation:

$$(-\Delta)^s u + (V(x) - \omega)u = f(x, u) \quad \text{in } \mathbb{R}^N. \quad (4)$$

Unlike the classical Laplacian operator $-\Delta$, the usual analysis tools for the elliptic PDEs cannot be directly applied to Eq.(4) since $(-\Delta)^s$ ($0 < s < 1$) is a nonlocal operator. In [2], Cafferelli and Silvestre developed a powerful extension method, which transferred the nonlocal Eq.(4) into a local one settled on a half space. Later, the authors in [7, 13] gave a detailed introduction on the fractional Sobolev spaces and variational methods for the nonlocal fractional problems. Among other studies involving variational methods, there have been a considerable amount of effort to look for ground states of Eq.(4), see [16, 30] and the references therein. For instance, Frank, Lenzmann and Silvestre [16] studied the following single fractional Schrödinger equation:

$$\begin{cases} (-\Delta)^s u + u = |u|^{2p} u, & \text{in } \mathbb{R}^N, \\ u \in H^s(\mathbb{R}^N), & u > 0, \text{ in } \mathbb{R}^N, \end{cases} \quad (5)$$

and obtained the following properties of the ground state solution for Eq.(5).

Proposition 1.3 [16] Let $N \geq 1$, $s \in (0, 1)$ and $0 < p < \frac{2^*_s - 2}{2}$, where

$$2^*_s = \begin{cases} \frac{2N}{N-2s}, & \text{if } N > 2s, \\ +\infty, & \text{if } N \leq 2s. \end{cases}$$

Then the following hold:

- (i) (Uniqueness) The ground state solution $U \in H^s(\mathbb{R}^N)$ for Eq.(5) is unique.
- (ii) (Symmetry, regularity and decay) $U(x)$ is radial, positive and strictly decreasing in $|x|$. Moreover, the function $U(x)$ belongs to $H^{2s+1}(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N)$ and satisfies

$$\frac{C_1}{1 + |x|^{N+2s}} \leq U(x) \leq \frac{C_2}{1 + |x|^{N+2s}}, \quad \forall x \in \mathbb{R}^N,$$

with some constants $C_2 \geq C_1 > 0$.

(iii) (Non-degeneracy) The linearized operator $L_0 = (-\Delta)^s + 1 - (2p + 1)U^{2p}$ is non-degenerate, i.e., its kernel is given by

$$\ker L_0 = \text{span}\{\partial_{x_1} U, \dots, \partial_{x_N} U\}.$$

In [17], Guo and He studied the following fractional Schrödinger system with nonconstant potentials:

$$\begin{cases} \varepsilon^{2s}(-\Delta)^s u + P_1(x)u = (|u|^{2p} + \beta|u|^{p-1}|v|^{p+1})u, & \text{in } \mathbb{R}^N, \\ \varepsilon^{2s}(-\Delta)^s v + P_2(x)v = (|v|^{2p} + \beta|v|^{p-1}|u|^{p+1})v, & \text{in } \mathbb{R}^N, \end{cases} \quad (6)$$

where $s \in (0, 1)$, $0 < p < \frac{2s}{N-2s}$, $\varepsilon > 0$ is a small parameter and the coupling constant $\beta > 0$. Under some appropriate hypotheses on the potentials $P_1(x)$ and $P_2(x)$, the authors obtained the existence of nontrivial nonnegative solutions for the system (6) which concentrate around local minima of the potentials. When $\varepsilon = 1$, $P_1(x) = 1$ and $P_2(x) = \omega^{2s}$, $\omega > 0$, Guo and He [18] obtained the existence of a least energy solution for the system (6) with $0 < p < \frac{2s}{N-2s}$ on the Nehari manifold. Furthermore, the existence of the least energy positive solution with both nontrivial components was also established. Recently, in [22], by finding a new kind of solution to the system (6) with $\beta > 0$ and $1 < p < 1 + 2s/N$, and comparing its energy with those of other kinds of solutions, Luan and Luo observed that each component of the least energy solution to the system (6) is nontrivial, which complements the results of [18]. For more results about the fractional Schrödinger system, one can refer to [1, 10–12, 19] and the references therein.

Recently, the following nonlocal elliptic system

$$\begin{cases} -\Delta u + V_1(x)u + l(x)\phi_{l,(u,v)}u = (|u|^{2p} + \beta|v|^{p+1}|u|^{p-1})u & \text{in } \mathbb{R}^N, \\ -\Delta v + V_2(x)v + l(x)\phi_{l,(u,v)}v = (|v|^{2p} + \beta|u|^{p+1}|v|^{p-1})v & \text{in } \mathbb{R}^N, \end{cases} \quad (7)$$

has been studied by many papers, where

$$\phi_{l,(u,v)} = \begin{cases} \int_{\mathbb{R}^2} l(y) \log(|x-y|) (u^2(y) + v^2(y)) dy, & \text{if } N = 2, \\ \int_{\mathbb{R}^N} \frac{l(y)(u^2(y) + v^2(y))}{|x-y|^{N-2}} dy, & \text{if } N \geq 3 \end{cases}$$

is the unique solution of

$$-\Delta \phi = l(x) (u^2 + v^2) \quad \text{in } \mathbb{R}^N.$$

When $N = 2$, $p \geq 2$ and $V_i(x) = l(x) \equiv 1$ ($i = 1, 2$), by using the Nehari manifold technique and some delicate estimates, Carvalho et al. [3] studied the existence of the least energy solution for the system (7) in the space $H_r^1(\mathbb{R}^N)$, the radially symmetric subspace of $H^s(\mathbb{R}^2)$. When $\frac{3}{2} < p < 2$ and $\beta \geq p - 1$, or $2 \leq p < 3$ and $\beta \geq 2^{p-1} - 1$, by using the Nehari–Pohozaev manifold technique defined in $H_r^1(\mathbb{R}^N) \times H_r^1(\mathbb{R}^N)$ developed by Ruiz [28], d’Avenia et al. [9] studied the existence of radial vectorial solutions for the system (7) with $N = 3$ and $V_i(x) = l(x) \equiv 1$ ($i = 1, 2$). Later, by developing a novel constraint method, Sun and Wu [31] obtained the existence of vectorial ground states to the system (7) with $1 < p < \frac{3}{2}$, $N = 3$ and $V_i(x) = l(x) \equiv 1$ ($i = 1, 2$). Very recently, for the case that $V(x)$ is not identically equal to a constant, Sun and Wu [32] proved the existence of a global minimizer with negative energy for the system (7) with $N = 3$ and $1 < p < \frac{3}{2}$, and found vectorial ground states with $\frac{3}{2} < p < 2$. Moreover, the phenomenon of symmetry breaking of ground states with $1 < p < \frac{3}{2}$ was also explored. In [36], Sun et al. studied the existence of ground normalized solutions for the system (7) with $N = 3$ and $1 < p < 2$. Besides, the local (global) well-posedness theory and the blow up behavior of the solutions were also

studied. For more results about the system (7), we refer the interested reader to [6, 24] and the reference therein.

It is worth pointing out that despite lots of work has been obtained about the Hartree–Fock system, there are few results about the fractional Hartree–Fock system. In the present paper, we focus our attention on the existence, multiplicity and symmetry breaking of positive solutions for the planar fHF system (S_l) , extending the results in [3] to the fractional case. In addition, we assume that the function $l(x)$ satisfies the following condition:

(D_1) $l \in C(\mathbb{R}^2, \mathbb{R})$ is positive on $\mathbb{R}^2$ and $\lim_{|x| \rightarrow \infty} l(x) = l_\infty > 0$;

Observe that the case of $1 < p < \frac{2^*_s-1}{2}$ is trivial, so we only consider the case of $0 < p < \frac{1}{2}$.

Let $\mathbf{H} = H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2)$ and $\mathbf{H}_r = H_r^s(\mathbb{R}^2) \times H_r^s(\mathbb{R}^2)$. From a variational point of view, solutions of system (S_l) are critical points of

$$J_l(u, v) = \frac{1}{2} \|(u, v)\|_{\mathbf{H}}^2 + \frac{1}{4} \int_{\mathbb{R}^2} l(x) \phi_{l, (u, v)}^t (u^2 + v^2) dx - \frac{1}{2p+2} \int_{\mathbb{R}^2} F(u, v) dx,$$

where

$$F(u, v) = |u|^{2p+2} + |v|^{2p+2} + 2|u|^{p+1}|v|^{p+1},$$

and

$$\|(u, v)\|_{\mathbf{H}}^2 = \int_{\mathbb{R}^2} \left(|(-\Delta)^{\frac{s}{2}} u|^2 + u^2 \right) dx + \int_{\mathbb{R}^2} \left(|(-\Delta)^{\frac{s}{2}} v|^2 + v^2 \right) dx.$$

Compared with the existing results, there seems to be more challenging in our study under the effect of the nonlocal operator $(-\Delta)^s$ ($\frac{1}{2} < s < 1$) and the nonlocal term $\phi_{l, (u, v)}^t$. On one hand, when $0 < p < \frac{1}{2}$, neither Nehari and Nehari–Pohozaev manifolds nor the mountain pass geometry of the functional J_l can be used to construct bounded (PS)-sequences in finding multiple positive solutions, even without the classical (AR)-condition or other similar conditions. On the other hand, since we consider system (S_l) in $\mathbf{H}$, it is also tough to prove the convergence of the Palais Smale sequence without any radial symmetry setting on the potential function $l(x)$. In order to overcome these difficulties, as an important step, we shall develop a different approach inspired by Ruiz [28] (or [8]) to find a global minimizer by establishing new estimates for the fractional Laplacian in $H_r^s(\mathbb{R}^2)$. Secondly, in order to obtain a ground state solution for the system (S_l) , we need to establish a version of global decomposition of solution sequences (such solutions maybe not positive), which seems new for the fractional Hartree–Fock system. This decomposition is crucial in using the Brezis–Lieb Lemma [33] and some delicate estimates of solutions to prove the compactness of solution sequences. Thirdly, when we study the symmetry breaking phenomenon, the usual approach (see [8, 35]) is only used to deal with the case of $\frac{1}{2} < t < \frac{2}{3}$ due to complexity of the fractional Laplacian. We thus introduce a new approach to solve the case of $\frac{2}{3} \leq t < 1$ by controlling the potential $l(x)$. Indeed, such approach is also valid for the case of $\frac{1}{2} < t < \frac{2}{3}$.

1.1 Main results

Before presenting our main results, we need to state the following maximization problems:

$$\Lambda_0 := \sup_{(u, v) \in \mathbf{H}_r \setminus \{(0, 0)\}} \frac{\frac{1}{2p+2} \int_{\mathbb{R}^2} F(u, v) dx - \frac{1}{2} \|(u, v)\|_{\mathbf{H}}^2}{\frac{1}{4} \int_{\mathbb{R}^2} \phi_{u, v}^t (u^2 + v^2) dx}$$

and

$$\Lambda_1 := \sup_{(u,v) \in \mathbf{H}_r \setminus \{(0,0)\}} \frac{\int_{\mathbb{R}^2} F(u, v) dx - \frac{1}{2} \|(u, v)\|_{\mathbf{H}}^2}{\int_{\mathbb{R}^2} \phi_{l, (u,v)}^t (u^2 + v^2) dx},$$

where

$$\int_{\mathbb{R}^2} \phi_{u,v}^t (u^2 + v^2) dx = \int_{\mathbb{R}^2} l(x) \phi_{l, (u,v)}^t (u^2 + v^2) dx$$

with $l(x) \equiv 1$. By using similar arguments with [8, Appendix], we have

$$0 < \Lambda_0 < \infty \text{ and } 0 < \Lambda_1 < \infty.$$

We now summarize our main results. First of all, we consider the autonomous case, i.e., $l(x) \equiv \mu > 0$ as a parameter, then system (S_l) becomes the following system:

$$\begin{cases} (-\Delta)^s u + u + \mu \phi_{l, (u,v)}^t u = (|u|^{2p} + |v|^{p+1} |u|^{p-1}) u & \text{in } \mathbb{R}^2, \\ (-\Delta)^s v + v + \mu \phi_{l, (u,v)}^t v = (|v|^{2p} + |u|^{p+1} |v|^{p-1}) v & \text{in } \mathbb{R}^2, \end{cases} \quad (S_\mu)$$

and the following theorem holds.

Theorem 1.4 Suppose that $\frac{1}{2} < t \leq s < 1$ and the condition (D_1) holds. Then the following statements are true.

- (i) For every $0 < \mu < \Lambda_0$, system (S_μ) admits two positive radial solutions $(u_\mu^{(1)}, v_\mu^{(1)})$, $(u_\mu^{(2)}, v_\mu^{(2)}) \in \mathbf{H}_r$ with $J_\mu(u_\mu^{(1)}, v_\mu^{(1)}) < 0 < J_\mu(u_\mu^{(2)}, v_\mu^{(2)})$, where $J_\mu(u, v) = J_l(u, v)$ with $l(x) \equiv \mu$.
- (ii) For every $\mu > \Lambda_1$, $(u, v) = (0, 0)$ is the unique solution of system (S_μ) .

With Theorem 1.4 in hand, we have the following results for the non-autonomous system (S_l) .

Theorem 1.5 Suppose that $\frac{1}{2} < t \leq s < 1$ and the condition (D_1) holds. In addition, the following conditions are also satisfied:

- (D_2) $l_{\min} := \inf_{x \in \mathbb{R}^2} l(x) < b_0 < l_\infty$, where

$$b_0 := \left(\frac{B_0^{2p} p^{2p} 2^{3-p} (1-2p)^{1-2p}}{p+1} \right)^{1/2p} > 0, \quad (8)$$

here $B_0 > 0$ is the best constant of fractional Gagliardo–Nirenberg interpolation inequality [29] as follows:

$$\int_{\mathbb{R}^2} (|(-\Delta)^{\frac{r}{2}} u|^2 + u^2) dx \leq B_0 \left(\int_{\mathbb{R}^2} u^2 dx \right)^{2(1-r)} \left[\int_{\mathbb{R}^2} (|(-\Delta)^{\frac{s}{2}} u|^2 + u^2) dx \right]^{2r}$$

for $0 < r = \frac{t}{s} < 1$.

- (D_3) there exists $0 < \mu < \Lambda_0$ such that

$$\begin{aligned} & \int_{\mathbb{R}^2} l(x) \phi_{l, (u_\mu^{(1)}(x-x_0), v_\mu^{(1)}(x-x_0))}^t \left[(u_\mu^{(1)}(x-x_0))^2 + (v_\mu^{(1)}(x-x_0))^2 \right] dx \\ & < \mu \int_{\mathbb{R}^2} \phi_{u_\mu^{(1)}, v_\mu^{(1)}}^t \left[(u_\mu^{(1)})^2 + (v_\mu^{(1)})^2 \right] dx \end{aligned}$$

for some $x_0 \in \mathbb{R}^2$, where $(u_\mu^{(1)}, v_\mu^{(1)})$ is the radial solution of system (S_μ) obtained in Theorem 1.4.

Then system (S_l) admits a vectorial ground state solution $(u_l, v_l) \in \mathbf{H}$ such that

$$J_l(u_l, v_l) < J_\mu(u_\mu^{(1)}, v_\mu^{(1)}) < 0.$$

Theorem 1.6 Suppose that $\frac{1}{2} < t \leq s < 1$ and the condition (D_1) holds. If $l_{\min} > \sqrt{\Lambda_1}$, then $(u, v) = (0, 0)$ is the unique solution of system (S_l) .

Remark 1.7 (i) For the autonomous planar Hartree Fock system, a positive solution with $\beta > 2^{p-1} - 1$, ($p \geq 1$) is obtained in [3], while two positive solutions can be found for the autonomous planar fHF system in Theorem 1.4, which the energy of a positive solution is negative. For the non-autonomous case, as far as we are concerned, there seems no any information on the ground state solution for the planar fHF system. However, for the planar fHF system, we can find a ground state solution with negative energy in Theorem 1.5. We think these are some of the main differences between two types of systems.

(ii) The asymptotic behavior of positive solutions as the order $t \rightarrow 1$ will be an interesting issue in future.

To study the existence of non-radial ground state solutions, we consider the following system:

$$\begin{cases} (-\Delta)^s u + u + l_\varepsilon(x) \phi_{l_\varepsilon}^t(u, v) u = (|u|^{2p} + |v|^{p+1}|u|^{p-1}) u & \text{in } \mathbb{R}^2, \\ (-\Delta)^s v + v + l_\varepsilon(x) \phi_{l_\varepsilon}^t(u, v) v = (|v|^{2p} + |u|^{p+1}|v|^{p-1}) v & \text{in } \mathbb{R}^2, \end{cases} \quad (S_{l_\varepsilon})$$

where $l_\varepsilon(x) = l(\varepsilon x)$ for any $\varepsilon > 0$. Then we have the following results.

Theorem 1.8 Suppose that $\frac{1}{2} < t < \frac{2}{3}$, $t \leq s < 1$ and conditions (D_1) , (D_2) hold. In addition, the following conditions are also satisfied.

(D_4) $l(x) = l(|x|)$;

(D_5) the sublevel set $\{l(x) < \sqrt{\Lambda_0}\} := \{x \in \mathbb{R}^2 \mid l(x) < \sqrt{\Lambda_0}\}$ is a nonempty open set.

Then system (S_{l_ε}) admits a positive solution $(u_{l_\varepsilon}, v_{l_\varepsilon}) \in \mathbf{H}$ such that

$$J_{l_\varepsilon}(u_{l_\varepsilon}, v_{l_\varepsilon}) < -K \leq \inf_{(u,v) \in \mathbf{H}_r} J_{l_\varepsilon}(u, v) \text{ for some constant } K > 0 \text{ and } \varepsilon > 0 \text{ sufficiently small.}$$

Furthermore, $(u_{l_\varepsilon}, v_{l_\varepsilon})$ is a non-radial ground state solution of system (S_{l_ε}) .

Theorem 1.9 Assume that $\frac{2}{3} \leq t \leq s < 1$ and conditions (D_1) , (D_2) , (D_4) and (D_5) hold. In addition, we assume that

(D_6) $0 \notin \{l(x) < \sqrt{\Lambda_0}\}$.

Then system (S_{l_ε}) admits a positive solution $(u_{l_\varepsilon}, v_{l_\varepsilon}) \in \mathbf{H}$ such that

$$J_{l_\varepsilon}(u_{l_\varepsilon}, v_{l_\varepsilon}) < -K \leq \inf_{(u,v) \in \mathbf{H}_r} J_{l_\varepsilon}(u, v) \text{ for some constant } K > 0 \text{ and } \varepsilon > 0 \text{ sufficiently small.}$$

Furthermore, $(u_{l_\varepsilon}, v_{l_\varepsilon})$ is a non-radial ground state solution of system (S_{l_ε}) .

Remark 1.10 (i) Assume that conditions (D_1) , (D_2) and (D_5) hold. Then there exist $x_0 \in \{l(x) < \sqrt{\Lambda_0}\}$ and $0 < \mu_0 < \Lambda_0$ such that $l(x_0) < \sqrt{\mu_0}$. Let $(u_{\mu_0}^{(1)}, v_{\mu_0}^{(1)})$ be the

radial positive solution of equation (S_{μ_0}) obtained in Theorem 1.4. Define $(u_\varepsilon(x), v_\varepsilon(x)) = (u_{\mu_0}^{(1)}(x - \frac{x_0}{\varepsilon}), v_{\mu_0}^{(1)}(x - \frac{x_0}{\varepsilon}))$. Then by the condition (D_2) , one has

$$\begin{aligned} & \int_{\mathbb{R}^2} l_\varepsilon(x) \phi_{l_\varepsilon, (u_\varepsilon, v_\varepsilon)}^t (u_\varepsilon^2 + v_\varepsilon^2) dx \\ &= \int_{\mathbb{R}^2} l(\varepsilon x + x_0) \phi_{l(\varepsilon x + x_0), (u_{\mu_0}^{(1)}, v_{\mu_0}^{(1)})}^t \left[\left(u_{\mu_0}^{(1)} \right)^2 + \left(v_{\mu_0}^{(1)} \right)^2 \right] dx \\ &= l^2(x_0) \int_{\mathbb{R}^2} \phi_{u_{\mu_0}^{(1)}, v_{\mu_0}^{(1)}}^t \left[\left(u_{\mu_0}^{(1)} \right)^2 + \left(v_{\mu_0}^{(1)} \right)^2 \right] dx + o(\varepsilon) \\ &< \mu_0 \int_{\mathbb{R}^2} \phi_{u_{\mu_0}^{(1)}, v_{\mu_0}^{(1)}}^t \left[\left(u_{\mu_0}^{(1)} \right)^2 + \left(v_{\mu_0}^{(1)} \right)^2 \right] dx + o(\varepsilon) \text{ for } \varepsilon > 0 \text{ sufficiently small,} \end{aligned}$$

which implies that when $l(x)$ is replaced by $l(\varepsilon x)$, the condition (D_3) still holds for $\varepsilon > 0$ sufficiently small. Therefore, by Theorem 1.5, system (S_{l_ε}) admits a positive ground state solution $(u_{l_\varepsilon}, v_{l_\varepsilon}) \in \mathbf{H}$ such that $J_{l_\varepsilon}(u_{l_\varepsilon}, v_{l_\varepsilon}) < J_{\mu_0}(u_{\mu_0}^{(1)}, v_{\mu_0}^{(1)}) < 0$.

(ii) Assume that all conditions hold in Theorem 1.8. If $0 \in \{l(x) < \sqrt{\Lambda_0}\}$, then similar to the argument in part (i), there exists $l^2(0) < \mu < \Lambda_0$, for $\varepsilon > 0$ small enough, there holds

$$\int_{\mathbb{R}^2} l(\varepsilon x) \phi_{l_\varepsilon, (u_\mu^{(1)}, v_\mu^{(1)})}^t \left[\left(u_\mu^{(1)} \right)^2 + \left(v_\mu^{(1)} \right)^2 \right] dx < \mu \int_{\mathbb{R}^2} \phi_{u_\mu^{(1)}, v_\mu^{(1)}}^t \left[\left(u_\mu^{(1)} \right)^2 + \left(v_\mu^{(1)} \right)^2 \right] dx,$$

which implies that the symmetric case still holds in Theorem 1.5. Thus, system (S_{l_ε}) admits a radial positive solution $(u_{l_\varepsilon}^{(1)}, v_{l_\varepsilon}^{(1)}) \in \mathbf{H}_r$ satisfying

$$J_{l_\varepsilon}(u_{l_\varepsilon}, v_{l_\varepsilon}) \leq J_{l_\varepsilon}(u_{l_\varepsilon}^{(1)}, v_{l_\varepsilon}^{(1)}) < 0 \text{ for } \varepsilon > 0 \text{ sufficiently small.}$$

Theorem 1.11 Assume that $\frac{1}{2} < t < \frac{2}{3}$, $t \leq s < 1$ and all the conditions in Theorem 1.8 hold. If

$$0 \in \{l(x) < \sqrt{\Lambda_0}\},$$

then system (S_{l_ε}) admits three positive solutions $(u_{l_\varepsilon}, v_{l_\varepsilon}) \in \mathbf{H}$ and $(\bar{u}_{l_\varepsilon}, \bar{v}_{l_\varepsilon}), (\tilde{u}_{l_\varepsilon}, \tilde{v}_{l_\varepsilon}) \in \mathbf{H}_r$ such that

$$J_{l_\varepsilon}(u_{l_\varepsilon}, v_{l_\varepsilon}) < J_{l_\varepsilon}(\tilde{u}_{l_\varepsilon}, \tilde{v}_{l_\varepsilon}) < 0 < J_{l_\varepsilon}(\bar{u}_{l_\varepsilon}, \bar{v}_{l_\varepsilon}) \text{ for } \varepsilon > 0 \text{ sufficiently small.}$$

Theorem 1.12 Suppose that $\frac{2}{3} \leq t \leq s < 1$ and all the conditions in Theorem 1.9 hold. In addition, we assume that

(D_7) there exists $0 < \mu < \Lambda_0$ such that

$$\int_{\mathbb{R}^2} l(\varepsilon x) \phi_{l_\varepsilon, (u_\mu^{(1)}, v_\mu^{(1)})}^t \left[\left(u_\mu^{(1)} \right)^2 + \left(v_\mu^{(1)} \right)^2 \right] dx < \mu \int_{\mathbb{R}^2} \phi_{u_\mu^{(1)}, v_\mu^{(1)}}^t \left[\left(u_\mu^{(1)} \right)^2 + \left(v_\mu^{(1)} \right)^2 \right] dx,$$

where $(u_\mu^{(1)}, v_\mu^{(1)})$ is the radial solution of system (S_l) obtained in Theorem 1.4.

Then system (S_{l_ε}) admits three positive solutions $(u_{l_\varepsilon}, v_{l_\varepsilon}) \in \mathbf{H}$ and $(\bar{u}_{l_\varepsilon}, \bar{v}_{l_\varepsilon}), (\tilde{u}_{l_\varepsilon}, \tilde{v}_{l_\varepsilon}) \in \mathbf{H}_r$ such that

$$J_{l_\varepsilon}(u_{l_\varepsilon}, v_{l_\varepsilon}) < J_{l_\varepsilon}(\tilde{u}_{l_\varepsilon}, \tilde{v}_{l_\varepsilon}) < 0 < J_{l_\varepsilon}(\bar{u}_{l_\varepsilon}, \bar{v}_{l_\varepsilon}) \text{ for } \varepsilon > 0 \text{ sufficiently small.}$$

Remark 1.13 Note that the condition $(D_6) : 0 \notin \{l(x) < \sqrt{\Lambda_0}\}$ is required in Theorem 1.12, therefore, the condition (D_7) is required, since we cannot get the similar result as what Remark 1.10(ii) stated.

Remark 1.14 By Theorems 1.8 and 1.9, we know that the symmetry breaking phenomenon for the planar fHF system can occur under the joint actions of the fractional Laplacian and the potential $l(x)$. However, we have not found any results on the symmetry breaking for the planar HF system so far.

Remark 1.15 Note that from the expressions of

$$\phi_{l,(u,v)}^t = C_t \int_{\mathbb{R}^2} \frac{l(y) (u^2(y) + v^2(y))}{|x - y|^{2-2t}} dy$$

and

$$\phi_{l,(u,v)} = \int_{\mathbb{R}^2} l(y) \log(|x - y|) (u^2(y) + v^2(y)) dy,$$

it is easy to see that system (S_l) is totally different from the planar Hartree-Fock system. Moreover, Theorems 1.4–1.12 still hold when the space dimension $N \in (2s, 6s)$ with $\frac{1}{2} < s < 1$. The arguments are similar to those of $N = 2$ in the present paper.

Notation Throughout this paper, we denote that $B_d(y) = \{x \in \mathbb{R}^2 : |x - y| \leq d\}$ for each $d \in [0, +\infty)$. $(H^s(\mathbb{R}^2))'$ is the dual space of $H^s(\mathbb{R}^2)$. S_r is the best Sobolev constant of the embedding from $H^s(\mathbb{R}^2)$ into $L^r(\mathbb{R}^2)$ ($2 \leq r \leq 2_s^* = \frac{2}{1-s}$) and S represents the best Sobolev constant of the embedding from $D^{t,2}(\mathbb{R}^2)$ into $L^{2_i^*}(\mathbb{R}^2)$. In particular, we shall also denote by $|\cdot|_r$ the L^r -norm for any $1 \leq r \leq +\infty$, $o_n(1)$ is the quantity which tends to zero as $n \rightarrow \infty$ and C represents various constants which may vary from line to line. “ $\rightarrow$ ” and “ $\rightharpoonup$ ” represent the strong and weak convergence. We may denote it again by $\{(u_n, v_n)\}$ if we take a subsequence of a sequence $\{(u_n, v_n)\}$.

This paper is organized as follows. In Sect. 2, we give some preliminaries which will be used in the sequel. After proving Theorem 1.4 in Sec. 3, we can directly prove Theorems 1.5–1.6 in Sec. 4, respectively. In Sects. 5 and 6, by studying the symmetry breaking of ground state solution, we give the proofs of Theorems 1.8–1.9 and 1.11–1.12, respectively. Finally, the global compactness principle will be proved in Sect. 7.

2 Preliminary results

Lemma 2.1 [13] *Let $s \in (0, 1)$ and $N > 2s$. Then $\mathbf{H}$ is continuously embedded into $L^r(\mathbb{R}^2) \times L^r(\mathbb{R}^2)$ for any $r \in [2, 2_s^*]$ and compactly embedded into $L_{loc}^r(\mathbb{R}^2) \times L_{loc}^r(\mathbb{R}^2)$ for any $r \in [1, 2_s^*)$.*

Lemma 2.2 *The function $\phi_{l,(u,v)}^t$ defined in (1) possesses the following properties:*

- (i) $\phi_{l,(u,v)}^t \geq 0$;
- (ii) $\int_{\mathbb{R}^2} l(x) \phi_{l,(u,v)}^t (u^2 + v^2) dx \leq l_{\max}^2 S^{-1} S_{\frac{4}{t+1}}^{-2} \|(u, v)\|_{\mathbf{H}}^4$;
- (iii) if $(u_n, v_n) \rightharpoonup (u, v)$ in $\mathbf{H}$, then $\phi_{l,(u_n,v_n)}^t \rightharpoonup \phi_{l,(u,v)}^t$ in $D^{t,2}(\mathbb{R}^2)$;
- (iv) if $(u_n, v_n) \rightarrow (u, v)$ in $\mathbf{H}$, then $\phi_{l,(u_n,v_n)}^t \rightarrow \phi_{l,(u,v)}^t$ in $D^{t,2}(\mathbb{R}^2)$.

Proof (i) It is obvious by the condition (D_1) and the expression of $\phi_{l,(u,v)}^t$ as in (1).

(ii) Multiplying the equation (1) by $\phi_{l,(u,v)}^t$, and together with the Hölder inequality, there holds

$$\|\phi_{l,(u,v)}^t\|_{D_{t,2}}^2 = \int_{\mathbb{R}^2} |(-\Delta)^{\frac{t}{2}} \phi_{l,(u,v)}^t|^2 dx$$

$$\begin{aligned}
&= \int_{\mathbb{R}^2} l(x) \phi_{l,(u,v)}^t (u^2 + v^2) dx \\
&\leq l_{\max}^2 S^{-1} \|\phi_{l,(u,v)}^t\|_{D_{t,2}} \left(|u|^{\frac{4}{t+1}} + |v|^{\frac{4}{t+1}} \right)^{\frac{t+1}{2}} \\
&\leq l_{\max}^2 S^{-1} S_{\frac{4}{t+1}}^{-2} \|\phi_{l,(u,v)}^t\|_{D_{t,2}} \|(u, v)\|_{\mathbf{H}}^2,
\end{aligned}$$

which implies that

$$\|\phi_{l,(u,v)}^t\|_{D_{t,2}} \leq l_{\max}^2 S^{-1} S_{\frac{4}{t+1}}^{-2} \|(u, v)\|_{\mathbf{H}}^2,$$

and

$$\int_{\mathbb{R}^2} l(x) \phi_{l,(u,v)}^t (u^2 + v^2) dx = \|\phi_{l,(u,v)}^t\|_{D_{t,2}}^2 \leq l_{\max}^4 S^{-2} S_{\frac{4}{t+1}}^{-4} \|(u, v)\|_{\mathbf{H}}^4.$$

(iii) Let us define the linear and continuous operators T, T_n in $D^{t,2}(\mathbb{R}^2)$ by

$$T(\omega) = \int_{\mathbb{R}^2} (-\Delta)^{\frac{t}{2}} \phi_{l,(u,v)}^t (-\Delta)^{\frac{t}{2}} \omega dx \left(= \int_{\mathbb{R}^2} l(x) (u^2 + v^2) \omega dx \right),$$

and

$$T_n(\omega) = \int_{\mathbb{R}^2} (-\Delta)^{\frac{t}{2}} \phi_{l,(u_n,v_n)}^t (-\Delta)^{\frac{t}{2}} \omega dx \left(= \int_{\mathbb{R}^2} l(x) (u_n^2 + v_n^2) \omega dx \right).$$

Then from the condition (D_1) and the Hölder inequality, for any $\varphi \in C_0^\infty(\mathbb{R}^2)$, there holds

$$\begin{aligned}
&|T_n(\varphi) - T(\varphi)| \\
&= \left| \int_{\mathbb{R}^2} \left((-\Delta)^{\frac{t}{2}} \phi_{l,(u_n,v_n)}^t - (-\Delta)^{\frac{t}{2}} \phi_{l,(u,v)}^t \right) (-\Delta)^{\frac{t}{2}} \varphi dx \right| \\
&= \left| \int_{\mathbb{R}^2} l(x) (u_n^2 - u^2 + v_n^2 - v^2) \varphi dx \right| \\
&\leq l_{\max} S^{-1} \|\varphi\|_{D^{t,2}} |u_n + u|_{\frac{4}{t+1}} \left(\int_{\text{supp} \varphi} |u_n - u|^{\frac{4}{t+1}} dx \right)^{\frac{t+1}{4}} \\
&\quad + l_{\max} S^{-1} \|\varphi\|_{D^{t,2}} |v_n + v|_{\frac{4}{t+1}} \left(\int_{\text{supp} \varphi} |v_n - v|^{\frac{4}{t+1}} dx \right)^{\frac{t+1}{4}} \\
&\leq o_n(1) \|\varphi\|_{D^{t,2}},
\end{aligned}$$

since $\frac{1}{2} < t \leq s < 1$, $(u_n, v_n) \rightarrow (u, v)$ in $L_{loc}^r(\mathbb{R}^2) \times L_{loc}^r(\mathbb{R}^2)$ ($1 \leq r < 2^*$) by Lemma 2.1 and $C_0^\infty(\mathbb{R}^2)$ is dense in $D^{t,2}(\mathbb{R}^2)$. Hence, $(T_n - T) \rightarrow 0$ as operators on $D^{t,2}(\mathbb{R}^2)$, and the Riesz representation theorem implies that $\phi_{l,(u_n,v_n)}^t \rightharpoonup \phi_{l,(u,v)}^t$ in $D^{t,2}(\mathbb{R}^2)$.

(iv) To prove (iv), it suffices to show that

$$\begin{aligned}
\|\phi_{l,(u_n,v_n)}^t\|_{D_{t,2}}^2 &= \int_{\mathbb{R}^2} l(x) \phi_{l,(u_n,v_n)}^t (u_n^2 + v_n^2) dx \\
&\rightarrow \|\phi_{l,(u,v)}^t\|_{D_{t,2}}^2 = \int_{\mathbb{R}^2} l(x) \phi_{l,(u,v)}^t (u^2 + v^2) dx
\end{aligned} \tag{9}$$

as $n \rightarrow \infty$. In fact, since $(u_n, v_n) \rightarrow (u, v)$ in $\mathbf{H}$, then $(u_n, v_n) \rightharpoonup (u, v)$ in $\mathbf{H}$. Thus, by the previous point (iii), $\phi_{l,(u_n,v_n)}^t \rightharpoonup \phi_{l,(u,v)}^t$ in $D^{t,2}(\mathbb{R}^2)$ as $n \rightarrow \infty$. The Sobolev embedding inequality implies that

$$\phi_{l,(u_n,v_n)}^t \rightharpoonup \phi_{l,(u,v)}^t \text{ in } L^{2^*}(\mathbb{R}^2),$$

and then

$$\int_{\mathbb{R}^2} l(x) \left(\phi'_{l,(u_n,v_n)} - \phi'_{l,(u,v)} \right) (u^2 + v^2) dx = o_n(1) \quad (10)$$

by the condition (H_1) and $(u^2 + v^2) \in L^{\frac{2}{i+1}}(\mathbb{R}^2)$ by Lemma 2.1. Then it follows from the Hölder inequality, Lemma 2.1 and (10) that

$$\begin{aligned} & \left| \int_{\mathbb{R}^2} l(x) \left[\phi'_{l,(u_n,v_n)} (u_n^2 + v_n^2) - \phi'_{l,(u,v)} (u^2 + v^2) \right] dx \right| \\ & \leq l_{\max}^2 \int_{\mathbb{R}^2} |\phi'_{u_n,v_n} (u_n^2 - u^2 + v_n^2 - v^2)| dx \\ & \quad + l_{\max}^2 \int_{\mathbb{R}^2} |(\phi'_{u_n,v_n} - \phi'_{u,v}) (u^2 + v^2)| dx \\ & \leq l_{\max} S^{-1} |\varphi|_{D^{i,2}} |u_n + u|^{\frac{4}{i+1}} |u_n - u|^{\frac{4}{i+1}} \\ & \quad + l_{\max} S^{-1} |\varphi|_{D^{i,2}} |v_n + v|^{\frac{4}{i+1}} |v_n - v|^{\frac{4}{i+1}} \\ & = o_n(1), \end{aligned}$$

which implies that (9) holds. The proof is complete. $\square$

Let

$$\Gamma(u, v) = \int_{\mathbb{R}^2} \phi'_{u,v} (v^2 + u^2) dx.$$

Lemma 2.3 *If $(u_n, v_n) \rightarrow (u, v)$ in $\mathbf{H}$, then*

$$\Gamma(u_n, v_n) = \Gamma(u, v) + \Gamma(u_n - u, v_n - v) + o_n(1),$$

and

$$\int_{\mathbb{R}^2} |u_n|^{p+1} |v_n|^{p+1} dx = \int_{\mathbb{R}^2} |u|^{p+1} |v|^{p+1} dx + \int_{\mathbb{R}^2} |u_n - u|^{p+1} |v_n - v|^{p+1} dx + o_n(1).$$

Proof The proof is similar to [36, Lemma 2.6], and we omit it here. $\square$

Now we establish the following estimates on the nonlinearity.

Lemma 2.4 *Suppose that $b > 0$ and $0 < p < \frac{1}{2}$. Let*

$$h_b(\theta) = \frac{1}{8} + \frac{b}{\sqrt{8}B_0} \theta - \frac{1}{p+1} \theta^{2p} \quad \text{for } \theta > 0,$$

where $B_0 > 0$ is as in (8). Then there exist

$$b_0 := \left(\frac{B_0^{2p} p^{2p} 2^{3-p} (1-2p)^{1-2p}}{p+1} \right)^{1/2p} > 0 \quad \text{and} \quad \theta_0(b) := \left(\frac{4\sqrt{2}B_0 p}{b(p+1)} \right)^{1/(1-2p)},$$

such that

- (i) $h'_b(\theta_0(b)) = 0$ and $h_{b_0}(\theta_0(b_0)) = 0$;
- (ii) For each $b < b_0$, there exist $\alpha_b, \beta_b > 0$ such that $\alpha_b < \theta_0(b) < \beta_b$, and $h_b(\theta) < 0$ for every $\theta \in (\alpha_b, \beta_b)$;
- (iii) for each $b > b_0$, $h_b(\theta) > 0$ for every $\theta > 0$.

Proof The proof is straightforward, and we omit it here. $\square$

Lemma 2.5 Let $0 < p < \frac{1}{2}$ and

$$g(\vartheta) = \vartheta^{p+1} + (1 - \vartheta)^{p+1} + 2\vartheta^{\frac{p+1}{2}}(1 - \vartheta)^{\frac{p+1}{2}} \text{ for } 0 \leq \vartheta \leq 1.$$

Then there exists $\vartheta_0 \in (0, 1)$ such that

$$g(\vartheta_0) = \max_{\vartheta \in [0, 1]} g(\vartheta) > 1.$$

Proof The proof is similar to [9, Lemma 2.4], and we omit it here. $\square$

Lemma 2.6 Let $0 < p < \frac{1}{2}$ and the condition (D_1) holds. Then for each $z \in H^s(\mathbb{R}^2) \setminus \{0\}$, there exists $\vartheta_z \in (0, 1)$ such that

$$J_l(\sqrt{\vartheta_z z}, \sqrt{1 - \vartheta_z z}) < J_l(z, 0) = J_l(0, z) = I_0(z) < 0,$$

where

$$I_0(z) = \frac{1}{2} \int_{\mathbb{R}^2} \left(|(-\Delta)^{\frac{s}{2}} z|^2 + z^2 \right) dx + \frac{1}{4} \int_{\mathbb{R}^2} l(x) \phi_{l,z}^t z^2 dx - \frac{1}{2p+2} \int_{\mathbb{R}^2} |z|^{2p+2} dx.$$

Proof The proof is similar to [31, Lemma 2.4], and we omit it here. $\square$

Remark 2.7 From [5, Theorem 1.2], we know that $\inf_{z \in \mathbb{R}^2} I_0(z) < 0$.

By virtue of Lemma 2.6, we may obtain the following result.

Theorem 2.8 Assume that $\frac{1}{2} < t \leq s < 1$ and the condition (D_1) holds. Let $(u_0, v_0) \in \mathbf{H}$ with

$$J_l(u_0, v_0) = \inf_{(u,v) \in \mathbf{H}} J_l(u, v) < \inf_{z \in H^s(\mathbb{R}^2)} I_0(z) < 0.$$

Then $u_0 \neq 0$ and $v_0 \neq 0$.

Proof Argue by contradiction that $v_0 = 0$. Let $(u, v) = (\sqrt{\vartheta} u_0, \sqrt{1 - \vartheta} u_0)$. By a straightforward calculation, we know that

$$\|(u, v)\|_{\mathbf{H}}^2 = \vartheta \|u_0\|_H^2 + (1 - \vartheta) \|u_0\|_H^2 = \|u_0\|_H^2$$

and

$$\int_{\mathbb{R}^2} l(x) \phi_{l,(u,v)}^t (u_0^2 + v_0^2) dx = \int_{\mathbb{R}^2} l(x) \phi_{l,(u,v)}^t (\vartheta u_0^2 + (1 - \vartheta) u_0^2) dx = \int_{\mathbb{R}^2} l(x) \phi_{l,u_0}^t u_0^2 dx.$$

In view of Lemma 2.6, there exists $\vartheta_0 \in (0, 1)$ such that

$$\begin{aligned} & \int_{\mathbb{R}^2} (|u|^{2p+2} + |v|^{2p+2} + 2|u|^{p+1}|v|^{p+1}) dx \\ &= \left[\vartheta_0^{p+1} + (1 - \vartheta_0)^{p+1} + 2\vartheta_0^{\frac{p+1}{2}}(1 - \vartheta_0)^{\frac{p+1}{2}} \right] \int_{\mathbb{R}^2} |u_0|^{2p+2} dx \\ &> \int_{\mathbb{R}^2} |u_0|^{2p+2} dx. \end{aligned}$$

Therefore,

$$J_l(\sqrt{\vartheta_0 u_0}, \sqrt{1 - \vartheta_0 u_0}) \leq \frac{1}{2} \|u_0\|_H^2 + \frac{1}{4} \int_{\mathbb{R}^2} l(x) \phi_{l,u_0}^t u_0^2 dx$$

$$\begin{aligned}
& -\frac{1}{2p+2} \left[\vartheta_0^{p+1} + (1-\vartheta_0)^{p+1} + 2\vartheta_0^{\frac{p+1}{2}} (1-\vartheta_0)^{\frac{p+1}{2}} \right] \int_{\mathbb{R}^2} |u_0|^{2p+2} dx \\
& < \frac{1}{2} \|u_0\|_H^2 + \frac{1}{4} \int_{\mathbb{R}^2} l(x) \phi_{l,u_0}^t u_0^2 dx - \frac{1}{2p+2} \int_{\mathbb{R}^2} |u_0|^{2p+2} dx \\
& = J_l(u_0, 0) = \inf_{(u,v) \in \mathbf{H}} J_l(u, v),
\end{aligned}$$

which is a contradiction. The proof is complete. $\square$

Lemma 2.9 Suppose that $\frac{1}{2} < t \leq s < 1$ and conditions (D_1) , (D_2) hold. Then the energy functional J_l is coercive and bounded below on $\mathbf{H}$.

Proof From the fractional Gagliardo–Nirenberg interpolation inequality, we obtain

$$\begin{aligned}
& \frac{1}{\sqrt{8}B_0} \int_{\mathbb{R}^2} l(x) (|u|^3 + v^2|u|) dx = \frac{1}{\sqrt{8}B_0} \int_{\mathbb{R}^2} ((-\Delta)^t \phi_{l,(u,v)}) |u| dx \\
& = \frac{1}{\sqrt{8}B_0} \int_{\mathbb{R}^2} \left\langle (-\Delta)^{\frac{t}{2}} \phi_{l,(u,v)}^t, (-\Delta)^{\frac{t}{2}} |u| \right\rangle dx \\
& \leq \frac{1}{4B_0^2} \int_{\mathbb{R}^2} |(-\Delta)^{\frac{t}{2}} u|^2 dx + \frac{1}{8} \int_{\mathbb{R}^2} |(-\Delta)^{\frac{t}{2}} \phi_{l,(u,v)}^t|^2 dx \\
& = \frac{1}{4B_0} \int_{\mathbb{R}^2} |(-\Delta)^{\frac{t}{2}} u|^2 dx + \frac{1}{8} \int_{\mathbb{R}^2} l(x) \phi_{l,(u,v)}^t (u^2 + v^2) dx \\
& \leq \frac{1}{4B_0} \|u\|_{H^t}^2 + \frac{1}{8} \int_{\mathbb{R}^2} l(x) \phi_{l,(u,v)}^t (u^2 + v^2) dx \\
& \leq \frac{1}{4} \|u\|_{H^s}^2 + \frac{1}{8} \int_{\mathbb{R}^2} l(x) \phi_{l,(u,v)}^t (u^2 + v^2) dx \quad \text{for all } (u, v) \in \mathbf{H}.
\end{aligned} \tag{11}$$

Similarly,

$$\frac{1}{\sqrt{8}B_0} \int_{\mathbb{R}^2} l(x) (|v|^3 + u^2|v|) dx \leq \frac{1}{4} \|v\|_{H^s}^2 + \frac{1}{8} \int_{\mathbb{R}^2} l(x) \phi_{l,(u,v)}^t (u^2 + v^2) dx \tag{12}$$

for all $(u, v) \in \mathbf{H}$. Thus, by (11)–(12), there holds

$$\begin{aligned}
& \frac{1}{4} \int_{\mathbb{R}^2} l(x) \phi_{l,(u,v)}^t (u^2 + v^2) dx \geq \frac{1}{\sqrt{8}B_0} \int_{\mathbb{R}^2} l(x) (|u|^3 + |v|^3) dx \\
& + \frac{1}{\sqrt{8}B_0} \int_{\mathbb{R}^2} l(x) (u^2|v| + v^2|u|) dx - \frac{1}{4} \|(u, v)\|_{\mathbf{H}}^2
\end{aligned} \tag{13}$$

for all $(u, v) \in \mathbf{H}$. Then by (13), the Young inequality and the Hölder inequality, we obtain

$$\begin{aligned}
J_l(u, v) & = \frac{1}{2} \|(u, v)\|_{\mathbf{H}}^2 + \frac{1}{4} \int_{\mathbb{R}^2} l(x) \phi_{l,(u,v)}^t (u^2 + v^2) dx \\
& - \frac{1}{2p+2} \int_{\mathbb{R}^2} (|u|^{2p+2} + |v|^{2p+2}) dx - \frac{1}{p+1} \int_{\mathbb{R}^2} |u|^{p+1} |v|^{p+1} dx \\
& \geq \frac{1}{4} \|(u, v)\|_{\mathbf{H}}^2 + \frac{1}{\sqrt{8}B_0} \int_{\mathbb{R}^2} l(x) (|u|^3 + |v|^3) dx \\
& + \frac{1}{\sqrt{8}B_0} \int_{\mathbb{R}^2} l(x) (u^2|v| + v^2|u|) dx \\
& - \frac{1}{2p+2} \int_{\mathbb{R}^2} (|u|^{2p+2} + |v|^{2p+2}) dx - \frac{1}{p+1} \int_{\mathbb{R}^2} |u|^{p+1} |v|^{p+1} dx
\end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{8} \|(u, v)\|_{\mathbf{H}}^2 + \int_{\mathbb{R}^2} u^2 \left(\frac{1}{8} + \frac{1}{\sqrt{8}B_0} l(x)|u| - \frac{1}{p+1} |u|^{2p} \right) dx \\
&\quad + \int_{\mathbb{R}^2} v^2 \left(\frac{1}{8} + \frac{1}{\sqrt{8}B_0} l(x)|v| - \frac{1}{p+1} |v|^{2p} \right) dx \\
&\geq \frac{1}{8} \|(u, v)\|_{\mathbf{H}}^2 + \int_{\{l(x) < b_0\}} u^2 \left(\frac{1}{8} + \frac{1}{\sqrt{8}B_0} l(x)|u| - \frac{1}{p+1} |u|^{2p} \right) dx \\
&\quad + \int_{\{l(x) < b_0\}} v^2 \left(\frac{1}{8} + \frac{1}{\sqrt{8}B_0} l(x)|v| - \frac{1}{p+1} |v|^{2p} \right) dx \\
&\quad + \int_{\{l(x) \geq b_0\}} u^2 \left(\frac{1}{8} + \frac{1}{\sqrt{8}B_0} l(x)|u| - \frac{1}{p+1} |u|^{2p} \right) dx \\
&\quad + \int_{\{l(x) \geq b_0\}} v^2 \left(\frac{1}{8} + \frac{1}{\sqrt{8}B_0} l(x)|v| - \frac{1}{p+1} |v|^{2p} \right) dx,
\end{aligned} \tag{14}$$

and together with Lemma 2.4, we further have

$$\begin{aligned}
J_l(u, v) &\geq \frac{1}{8} \|(u, v)\|_{\mathbf{H}}^2 + \int_{\{l(x) < b_0\}} u^2 \left(\frac{1}{8} + \frac{1}{\sqrt{8}B_0} l(x)|u| - \frac{1}{p+1} |u|^{2p} \right) dx \\
&\quad + \int_{\{l(x) < b_0\}} v^2 \left(\frac{1}{8} + \frac{1}{\sqrt{8}B_0} l(x)|v| - \frac{1}{p+1} |v|^{2p} \right) dx \\
&\geq \frac{1}{8} \|(u, v)\|_{\mathbf{H}}^2 + 2 \int_{\{l(x) < b_0\}} m_l(x) dx,
\end{aligned} \tag{15}$$

where

$$m_l(x) := \inf_{\theta \geq 0} \left(\frac{1}{8} \theta^2 + \frac{1}{\sqrt{8}B_0} l(x) \theta^3 - \frac{1}{p+1} \theta^{2p+2} \right) < 0 \quad \text{for all } x \in \{l(x) < b_0\}.$$

Note that

$$\inf_{x \in \{l(x) < b_0\}} m_l(x) \leq \frac{1}{8} \theta_0^2(l_{\min}) + \frac{l_{\min}}{\sqrt{8}B_0} \theta_0^3(l_{\min}) - \frac{1}{p+1} \theta_0^{2p+2}(l_{\min}) < 0, \tag{16}$$

where $\theta_0(l_{\min}) = \left(\frac{\sqrt{32}pB_0}{(p+1)l_{\min}} \right)^{1/(1-2p)} > 0$.

On the other hand, from the conditions (D_1) , (D_2) , we know that

$$0 < |\{l(x) < b_0\}| < \infty. \tag{17}$$

Then by the definition of $m_l(x)$ and (16)–(17), we obtain

$$0 > \int_{\{l(x) < b_0\}} m_l(x) dx \geq \inf_{x \in \{l(x) < b_0\}} m_l(x) |l(x) < b_0| > -\infty,$$

and together with (13), we know that the energy functional J_l is coercive and bounded below on $\mathbf{H}$. The proof is complete. $\square$

Lemma 2.10 Suppose that $\frac{1}{2} < t \leq s < 1$ and conditions (D_1) , (D_2) hold. Then

$$\inf_{(u,v) \in \mathbf{H}} J_{l_\infty}(u, v) \geq 0,$$

where $J_{l_\infty} = J_l$ with $l \equiv l_\infty$.

Proof By Lemma 2.4, (11)–(12) and $l_\infty > b_0$, for any $(u, v) \in \mathbf{H}$, we have

$$J_{l_\infty}(u, v) \geq \frac{1}{8} \|(u, v)\|_{\mathbf{H}}^2 + \int_{\mathbb{R}^2} u^2 \left(\frac{1}{8} + \frac{1}{\sqrt{8}B_0} l(x)|u| - \frac{1}{p+1} |u|^{2p} \right) dx \\ + \int_{\mathbb{R}^2} v^2 \left(\frac{1}{8} + \frac{1}{\sqrt{8}B_0} l(x)|v| - \frac{1}{p+1} |v|^{2p} \right) dx > 0.$$

The proof is complete. $\square$

Now we introduce the global concentration–compactness principle by means of a profile decomposition type result for bounded Palais–Smale sequences of J_l in $\mathbf{H}$. We point out that this result is one of the main tools to proceed with our argument, in order to search our critical points of J_l .

Proposition 2.11 *Assume that $\frac{1}{2} < t \leq s < 1$ and the condition (D_1) holds. Let $\{(u_n, v_n)\}$ be a bounded $(PS)_\alpha$ -sequence in H for J_l , where α is an arbitrary constant. Then there exist a subsequence of $\{(u_n, v_n)\}$, still denoted $\{(u_n, v_n)\}$ and an integer $\ell \in \mathbb{N} \cup \{0\}$, a sequence $\{y_n^{(k)}\} \subset \mathbb{R}^2$, a function $(u_0, v_0) \in \mathbf{H}$, and $(w^{(k)}, z^{(k)}) \in \mathbf{H}$ for $1 \leq k \leq \ell$ such that*

- (i) $|y_n^{(k)}| \rightarrow +\infty$ and $|y_n^{(k)} - y_n^{(k')}| \rightarrow +\infty$ for $1 \leq k \neq k' \leq \ell$;
(ii)

$$\begin{cases} (-\Delta)^s u_0 + u_0 + l(x)\phi_{l_\infty, (u_0, v_0)}^t u_0 = (|u_0|^{2p} + |v_0|^{p+1}|u_0|^{p-1}) u_0 & \text{in } \mathbb{R}^2, \\ (-\Delta)^s v_0 + v_0 + l(x)\phi_{l_\infty, (u_0, v_0)}^t v_0 = (|v_0|^{2p} + |u_0|^{p+1}|v_0|^{p-1}) v_0 & \text{in } \mathbb{R}^2; \end{cases}$$

(iii)

$$\begin{cases} (-\Delta)^s w^{(k)} + w^{(k)} + l_\infty \phi_{l_\infty, (w^{(k)}, z^{(k)})}^t w^{(k)} = (|w^{(k)}|^{2p} + |z^{(k)}|^{p+1}|w^{(k)}|^{p-1}) w^{(k)} & \text{in } \mathbb{R}^2, \\ (-\Delta)^s z^{(k)} + z^{(k)} + l_\infty \phi_{l_\infty, (w^{(k)}, z^{(k)})}^t z^{(k)} = (|z^{(k)}|^{2p} + |w^{(k)}|^{p+1}|z^{(k)}|^{p-1}) z^{(k)} & \text{in } \mathbb{R}^2; \end{cases}$$

(iv)

$$\begin{cases} u_n = u_0 + \sum_{k=1}^{\ell} w^{(k)}(\cdot - y_n^k) + o_n(1) & \text{in } \mathbb{R}^2, \\ v_n = v_0 + \sum_{k=1}^{\ell} z^{(k)}(\cdot - y_n^k) + o_n(1) & \text{in } \mathbb{R}^2; \end{cases}$$

(v) $J_l(u_n, v_n) = J_l(u_0, v_0) + \sum_{k=1}^{\ell} J_{l_\infty}(w^{(k)}, z^{(k)}).$

Here we agree that in the case that $\ell = 0$, the above results hold without $\{w^{(k)}\}$, $\{z^{(k)}\}$ and $\{y_n^{(k)}\}$.

Since the proof of proposition 2.11 is technical, so we postpone it in the Appendix of Section 7.

From Lemma 2.9, we may define

$$\beta_l := \inf_{(u, v) \in \mathbf{H}} J_l(u, v).$$

Then we may get the following conclusion from Proposition 2.11.

Corollary 2.12 *Suppose that $\frac{1}{2} < t \leq s < 1$ and conditions (D_1) , (D_2) hold. Let $\{(u_n, v_n)\}$ be a $(PS)_{\beta_l}$ -sequence in $\mathbf{H}$ for J_l . Then there exist a subsequence $\{(u_n, v_n)\}$ and a nonzero function (u_0, v_0) in $\mathbf{H}$ such that $(u_n, v_n) \rightarrow (u_0, v_0)$ strongly in $\mathbf{H}$ and $J_l(u_0, v_0) = \beta_l$. Moreover, (u_0, v_0) is a solution of system (S_l) .*

Proof Let $\{(u_n, v_n)\}$ be a $(PS)_{\beta_l}$ -sequence in $\mathbf{H}$ for J_l . It follows from Lemma 2.1 that $\{(u_n, v_n)\}$ is bounded in $\mathbf{H}$, then there exist a subsequence $\{(u_n, v_n)\}$ and a nonzero function (u_0, v_0) in $\mathbf{H}$ such that $(u_n, v_n) \rightharpoonup (u_0, v_0)$ weakly in $\mathbf{H}$ and $J'_l(u_0, v_0) = 0$. Furthermore, according to Lemma 2.10 and Proposition 2.11 (i) $v) - (v)$, $(u_n, v_n) \rightarrow (u_0, v_0)$ strongly in $\mathbf{H}$ and $J_l(u_0, v_0) = \beta_l$. Then (u_0, v_0) is a solution of system (S_l) . The proof is complete. $\square$

3 The existence and nonexistence of radial solutions

Lemma 3.1 ([25, Theorem 3.2]) *Let $N \geq 2$ and $\frac{1}{2} < s < \frac{N}{2}$. Then there holds*

$$\sup_{x \in \mathbb{R}^N \setminus \{0\}} |x|^{\frac{1}{2}} |u(x)| \leq C(N, s) \|u\|_{H^s} \text{ for all } u \in H^s_r(\mathbb{R}^N).$$

We are now ready to prove Theorem 1.4. (i) According to the definition of Θ_0 , there exists $(u_0, v_0) \in \mathbf{H}_r \setminus \{(0, 0)\}$ such that for each $\mu < \Theta_0$, there holds

$$\frac{\frac{1}{2p+2} \int_{\mathbb{R}^2} F(u_0, v_0) dx - \frac{1}{2} \|(u_0, v_0)\|_{\mathbf{H}}^2}{\frac{1}{4} \int_{\mathbb{R}^2} \phi_{u_0, v_0}^t (u_0^2 + v_0^2) dx} > \mu,$$

which implies that

$$J_\mu(u_0, v_0) = \frac{1}{2} \|(u_0, v_0)\|_{\mathbf{H}}^2 + \frac{\mu}{4} \int_{\mathbb{R}^2} \phi_{u_0, v_0}^t (u_0^2 + v_0^2) dx - \frac{1}{2p+2} \int_{\mathbb{R}^2} F(u_0, v_0) dx < 0. \quad (18)$$

Furthermore, for any $(u, v) \in \mathbf{H}$, we derive

$$\begin{aligned} J_\mu(u, v) &= \frac{1}{2} \|(u, v)\|_{\mathbf{H}}^2 + \frac{\mu}{4} \int_{\mathbb{R}^2} \phi_{u, v}^t (u^2 + v^2) dx - \frac{1}{2p+2} \int_{\mathbb{R}^2} F(u, v) dx \\ &\geq \frac{1}{2} \|(u, v)\|_{\mathbf{H}}^2 + \frac{\mu}{4} \int_{\mathbb{R}^2} \phi_{u, v}^t (u^2 + v^2) dx - \frac{1}{p+1} \int_{\mathbb{R}^2} (|u|^{2p+2} + |v|^{2p+2}) dx, \end{aligned}$$

where $0 < p < \frac{1}{2}$. Thus, by Lemma 3.1 and the same argument of [28, Theorem 4.3] (or [8, Theorem 1.1]), there exists $M > 0$ such that

$$\begin{aligned} J_\mu(u, v) &\geq \frac{1}{2} \|(u, v)\|_{\mathbf{H}}^2 + \frac{\mu}{4} \int_{\mathbb{R}^2} \phi_{u, v}^t (u^2 + v^2) dx - \frac{1}{p+1} \int_{\mathbb{R}^2} (|u|^{2p+2} + |v|^{2p+2}) dx \\ &\geq \frac{1}{4} \|(u, v)\|_{\mathbf{H}}^2 - M, \text{ for all } (u, v) \in \mathbf{H}_r. \end{aligned} \quad (19)$$

Therefore, J_μ is coercive and bounded below on $\mathbf{H}_r$. In view of (18) and (19), there holds

$$-\infty < \beta_\mu := \inf_{(u, v) \in \mathbf{H}_r} J_\mu(u, v) < 0.$$

From the Ekeland variational principle [14] and Palais criticality principle [26], there exists a sequence $\{(u_n, v_n)\} \subset \mathbf{H}_r$ such that

$$J_\mu(u_n, v_n) = \beta_\mu + o(1) \text{ and } (J_\mu)'(u_n, v_n) = o(1) \text{ in } \mathbf{H}_r.$$

Again, using the similar argument of [28, Theorem 4.3], there exist a subsequence $\{(u_n, v_n)\}$ and a function $(u_\mu^{(1)}, v_\mu^{(1)}) \in \mathbf{H}_r \setminus \{(0, 0)\}$ such that $(u_n, v_n) \rightarrow (u_\mu^{(1)}, v_\mu^{(1)})$ strongly in $\mathbf{H}_r$ and $(u_\mu^{(1)}, v_\mu^{(1)})$ is a nontrivial solution of system (S_μ) . This indicates that $J_\mu(u_\mu^{(1)}, v_\mu^{(1)}) = \beta_\mu < 0$. Furthermore, it follows from Theorem 2.8 that $u_\mu^{(1)} \neq 0$ and $v_\mu^{(1)} \neq 0$, and is so

$(|u_\mu^{(1)}|, |v_\mu^{(1)}|)$. Thus, $(u_\mu^{(1)}, v_\mu^{(1)})$ is a vectorial positive ground state solution of system (S_μ) . Besides, from the Sobolev inequality, we obtain

$$\begin{aligned} J_\mu(u, v) &\geq \frac{1}{2} \|(u, v)\|_{\mathbf{H}}^2 + \frac{\mu}{4} \int_{\mathbb{R}^2} \phi_{u,v}^t (u^2 + v^2) dx - \frac{1}{2p+2} \int_{\mathbb{R}^2} F(u, v) dx \\ &\geq \frac{1}{2} \|(u, v)\|_{\mathbf{H}}^2 - \frac{1}{(p+1)S_{2p+2}^{2p+2}} \|(u, v)\|_{\mathbf{H}}^{2p+2} \text{ for all } (u, v) \in \mathbf{H}. \end{aligned} \quad (20)$$

Therefore, it follows from (20) that there exist $\eta, \kappa > 0$ such that

$$\|(u_\mu^{(1)}, v_\mu^{(1)})\|_{\mathbf{H}} > \eta \text{ and } J_\mu(u_\mu^{(1)}, v_\mu^{(1)}) < 0 < \kappa \leq \inf_{\|(u,v)\|_{\mathbf{H}}=\eta} J_\mu(u, v).$$

Define

$$\sigma_\mu := \inf_{\gamma \in \Gamma_\mu} \max_{0 \leq \tau \leq 1} J_\mu(\gamma(\tau)) < 0,$$

where

$$\Gamma_\mu := \left\{ \gamma \in C([0, 1], \mathbf{H}_r) \mid \gamma(0) = 0, \gamma(1) = (u_\mu^{(1)}, v_\mu^{(1)}) \right\}.$$

Using the mountain pass theorem [15, 27] and Palais criticality principle [26], there exists a sequence $\{(u_n, v_n)\} \subset \mathbf{H}_r$ such that

$$J_\mu(u_n, v_n) = \sigma_\mu \geq \kappa \text{ and } (J_\mu)'(u_n, v_n) \rightarrow 0 \text{ as } n \rightarrow \infty,$$

and using the similar argument of [28, Theorem 4.3], there exist a subsequence $\{(u_n, v_n)\}$ and a function $(u_\mu^{(2)}, v_\mu^{(2)}) \in \mathbf{H}_r \setminus \{(0, 0)\}$ such that $(u_n, v_n) \rightarrow (u_\mu^{(2)}, v_\mu^{(2)})$ strongly in $\mathbf{H}_r$. Then $(J_\mu)'(u_\mu^{(2)}, v_\mu^{(2)}) = 0$ and $J_\mu(u_\mu^{(2)}, v_\mu^{(2)}) = \sigma_\mu > 0$. Thus, we know that $(u_\mu^{(2)}, v_\mu^{(2)})$ is a nontrivial solution of system (S_μ) .

(ii) Arguing by the contrary that (u_0, v_0) is a nontrivial solution of system (S_μ) with $\mu > \Lambda_1$. It follows from the definition of Λ_1 , we obtain

$$\begin{aligned} 0 &= \|(u_0, v_0)\|_{\mathbf{H}}^2 + \mu \int_{\mathbb{R}^2} \phi_{u_0, v_0}^t (u_0^2 + v_0^2) dx - \int_{\mathbb{R}^2} F(u_0, v_0) dx \\ &> \|(u_0, v_0)\|_{\mathbf{H}}^2 + \Theta_1 \int_{\mathbb{R}^2} \phi_{u_0, v_0}^t (u_0^2 + v_0^2) dx - \int_{\mathbb{R}^2} F(u_0, v_0) dx \\ &\geq \|(u_0, v_0)\|_{\mathbf{H}}^2 + \frac{\int_{\mathbb{R}^2} F(u_0, v_0) dx - \frac{1}{2} \|(u_0, v_0)\|_{\mathbf{H}}^2}{\int_{\mathbb{R}^2} \phi_{u_0, v_0}^t (u_0^2 + v_0^2) dx} \int_{\mathbb{R}^2} \phi_{u_0, v_0}^t (u_0^2 + v_0^2) dx \\ &\quad - \int_{\mathbb{R}^2} F(u_0, v_0) dx = \frac{1}{2} \|(u_0, v_0)\|_{\mathbf{H}}^2 \geq 0, \end{aligned}$$

which is a contradiction. The proof is complete. $\square$

4 The existence and nonexistence of ground state solutions

Lemma 4.1 Assume that $\frac{1}{2} < t \leq s < 1$ and conditions (D_1) , (D_2) and (D_3) hold. Then there holds

$$-\infty < \beta_l := \inf_{(u,v) \in \mathbf{H}} J_l(u, v) < 0.$$

Proof Let $\{(u_\mu^{(1)}, v_\mu^{(1)})\}$ be the radial positive solution of system (S_μ) obtained in Theorem 1.4. It follows from the condition (D_3) that

$$\begin{aligned} J_l(u_\mu^{(1)}(x - x_0), v_\mu^{(1)}(x - x_0)) &= \frac{1}{2} \| (u_\mu^{(1)}, v_\mu^{(1)}) \|_{\mathbf{H}}^2 - \frac{1}{2p+2} \int_{\mathbb{R}^2} F(u_\mu^{(1)}, v_\mu^{(1)}) dx \\ &\quad + \frac{1}{4} \int_{\mathbb{R}^2} l(x) \phi_{l, (u_\mu^{(1)}(x-x_0), v_\mu^{(1)}(x-x_0))}^t \left[(u_\mu^{(1)}(x - x_0))^2 + (v_\mu^{(1)}(x - x_0))^2 \right] dx \\ &\leq \frac{1}{2} \| (u_\mu^{(1)}, v_\mu^{(1)}) \|_{\mathbf{H}}^2 + \frac{\mu}{4} \int_{\mathbb{R}^2} \phi_{u_\mu^{(1)}, v_\mu^{(1)}}^t \left[(u_\mu^{(1)})^2 + (v_\mu^{(1)})^2 \right] dx \\ &\quad - \frac{1}{2p+2} \int_{\mathbb{R}^2} F(u_\mu^{(1)}, v_\mu^{(1)}) dx < 0. \end{aligned} \quad (21)$$

Then by Lemma 2.9 and (21), we have

$$-\infty < \beta_l := \inf_{(u,v) \in \mathbf{H}} J_l(u, v) < 0.$$

The proof is complete. $\square$

We are now ready to prove Theorem 1.5. By virtue of Lemma 2.9 and (21), the functional J_l is coercive on $\mathbf{H}$ and $-\infty < \beta_l < 0$. Then by the Ekeland variational principle [14], there exists a sequence $\{(u_n, v_n)\} \subset \mathbf{H}$ such that

$$J_l(u_n, v_n) = \beta_l + o(1) \quad \text{and} \quad (J_l)'(u_n, v_n) = o_n(1) \quad \text{in} \quad H^{-s}(\mathbb{R}^2) \times H^{-s}(\mathbb{R}^2).$$

Hence, by Lemma 2.9 and Corollary 2.12, there exist a subsequence $\{(u_n, v_n)\} \subset \mathbf{H}$ and a function $(u_l, v_l) \in \mathbf{H} \setminus \{(0, 0)\}$ such that $(u_n, v_n) \rightarrow (u_l, v_l)$ strongly in $\mathbf{H}$ and (u_l, v_l) is a nontrivial solution of system (S_l) with $J_l(u_l, v_l) = \beta_l < 0$. Besides, it follows from Theorem 2.8 that $u_l \neq 0$ and $v_l \neq 0$, i.e., (u_l, v_l) is a vectorial ground state solution of system (S_l) . $\square$

We are now ready to prove Theorem 1.6. Assume on the contrary that (u_0, v_0) is a nontrivial solution of system (S_l) with $l_{\min} > \sqrt{\Lambda_1}$. It follows from the definition of Λ_1 that

$$\begin{aligned} 0 &= \|(u_0, v_0)\|_{\mathbf{H}}^2 + \int_{\mathbb{R}^2} l(x) \phi_{l, (u_0, v_0)}^t (u_0^2 + v_0^2) dx - \frac{1}{2p+2} \int_{\mathbb{R}^2} F(u_0, v_0) dx \\ &\geq \|(u_0, v_0)\|_{\mathbf{H}}^2 + l_{\min}^2 \int_{\mathbb{R}^2} \phi_{u_0, v_0}^t (u_0^2 + v_0^2) dx - \frac{1}{2p+2} \int_{\mathbb{R}^2} F(u_0, v_0) dx \\ &> \|(u_0, v_0)\|_{\mathbf{H}}^2 + \Lambda_1 \int_{\mathbb{R}^2} \phi_{u_0, v_0}^t (u_0^2 + v_0^2) dx - \frac{1}{2p+2} \int_{\mathbb{R}^2} F(u_0, v_0) dx \geq 0, \end{aligned}$$

which is a contradiction.

5 Symmetry breaking of ground state solutions

Lemma 5.1 Suppose that $\frac{1}{2} < t \leq s < 1$, conditions (D_1) , (D_4) and (D_5) hold. Then the following statements are true:

- (i) There exists $K > 0$ such that $\inf_{(u,v) \in \mathbf{H}_r} J_{l_\varepsilon}(u, v) \geq -K$ for all $\varepsilon > 0$.
- (ii) If $0 \in \{l(x) < \sqrt{\Lambda_0}\}$, then we have $\inf_{(u,v) \in \mathbf{H}_r} J_{l_\varepsilon}(u, v) < 0$ for $\varepsilon > 0$ small enough.

Proof (i) By the condition (H_1) , Lemma 3.1 and the same argument of [28, Theorem 4.3] (or [8, Theorem 1.1]), there exists $K > 0$ such that

$$J_{l_\varepsilon}(u, v) \geq \frac{1}{2} \|(u, v)\|_{\mathbf{H}}^2 + \frac{l_{\min}^2}{4} \int_{\mathbb{R}^2} \phi_{u,v}^t (u^2 + v^2) dx - \frac{1}{2p+2} \int_{\mathbb{R}^2} F(u, v) dx$$

$$\begin{aligned} &\geq \frac{1}{2} \|(u, v)\|_{\mathbf{H}}^2 + \frac{\mu}{4} \int_{\mathbb{R}^2} \phi_{u,v}^t (u^2 + v^2) dx - \frac{1}{p+1} \int_{\mathbb{R}^2} (|u|^{2p+2} + |v|^{2p+2}) dx \\ &> -K \quad \text{for all } (u, v) \in \mathbf{H}_r, \end{aligned}$$

which implies that $\inf_{(u,v) \in \mathbf{H}_r} J_{l_\varepsilon}(u, v) \geq -K$.

(ii) It follows from Remark 1.10 (ii) that there exists $l^2(0) < \mu < \Theta_0$ such that

$$\begin{aligned} \inf_{(u,v) \in \mathbf{H}_r} J_{l_\varepsilon}(u, v) &\leq \frac{1}{2} \|(u_\mu^{(1)}, v_\mu^{(1)})\|_{H^s}^2 + \frac{1}{4} \int_{\mathbb{R}^2} l(\varepsilon x) \phi_{l_\varepsilon, (u_\mu^{(1)}, v_\mu^{(1)})}^t \left[(u_\mu^{(1)})^2 + (v_\mu^{(1)})^2 \right] dx \\ &\quad - \frac{1}{2p+2} \int_{\mathbb{R}^2} F(u_\mu^{(1)}, v_\mu^{(1)}) dx \\ &\leq \|(u_\mu^{(1)}, v_\mu^{(1)})\|_{H^s}^2 + \frac{\mu}{4} \int_{\mathbb{R}^2} \phi_{u_\mu^{(1)}, v_\mu^{(1)}}^t \left[(u_\mu^{(1)})^2 + (v_\mu^{(1)})^2 \right] \\ &\quad - \frac{1}{2p+2} \int_{\mathbb{R}^2} F(u_\mu^{(1)}, v_\mu^{(1)}) dx \\ &< 0 \quad \text{for } \varepsilon > 0 \text{ sufficiently small,} \end{aligned}$$

where $(u_\mu^{(1)}, v_\mu^{(1)})$ is the nontrivial radial solution of system (S_μ) with $J_\mu(u_\mu^{(1)}, v_\mu^{(1)}) < 0$. The proof is complete. $\square$

5.1 The case of $\frac{1}{2} < t < \frac{2}{3}$ and $t \leq s < 1$

Now, we adopt the idea of multibump technique by Ruiz [28] (also see [23]) to obtain the following lemma.

Lemma 5.2 Assume that $\frac{1}{2} < t < \frac{2}{3}$, $t \leq s < 1$ and conditions (D_1) and (D_5) hold. Then we have

$$\beta_{l_\varepsilon} := \inf_{(u,v) \in \mathbf{H}} J_{l_\varepsilon}(u, v) \rightarrow -\infty \quad \text{as } \varepsilon \rightarrow 0^+.$$

Proof It follows from conditions (D_1) and (D_5) that there exists $x_0 \in \{l(x) < \sqrt{\Lambda_0}\}$ and $0 < \mu_0 < \Lambda_0$ such that $l(x_0) < \sqrt{\mu_0}$ and $\{l(x) < \sqrt{\mu_0}\}$ is an open set. Without loss of generality, we may assume that

$$B^2(x_0, 1) \subset \{x \in \mathbb{R}^2 \mid l(x) < \sqrt{\mu_0}\},$$

where $B^2(x_0, 1)$ is a ball centered at x_0 with radius 1 in $\mathbb{R}^2$. This implies that

$$B^2\left(\frac{x_0}{\varepsilon}, \frac{1}{\varepsilon}\right) \subset \Omega_\varepsilon := \{x \in \mathbb{R}^2 \mid l(\varepsilon x) < \sqrt{\mu_0}\} \quad \text{for all } \varepsilon > 0. \quad (22)$$

For any $R > 0$, we define a cut-off function $\psi_R \in C^1(\mathbb{R}^2, [0, 1])$ as

$$\psi_R(x) = \begin{cases} 1, & \text{if } |x| < \frac{R}{2}, \\ 0, & \text{if } |x| > R, \end{cases}$$

and $|\nabla \psi_R| \leq 1$ in $\mathbb{R}^2$. Let

$$(u_R(x), v_R(x)) = \left(\psi_R(x) u_{\mu_0}^{(1)}(x), \psi_R(x) v_{\mu_0}^{(1)}(x) \right) \quad \text{for any } x \in \mathbb{R}^2,$$

where $(u_{\mu_0}^{(1)}, v_{\mu_0}^{(1)})$ is the radial positive solution of system (S_{μ_0}) obtained in Theorem 1.4 with

$$J_{\mu_0}(u_{\mu_0}^{(1)}, v_{\mu_0}^{(1)}) < 0.$$

Evidently,

$$\int_{\mathbb{R}^2} F(u_R, v_R) dx \rightarrow \int_{\mathbb{R}^2} F(u_{\mu_0}^{(1)}, v_{\mu_0}^{(1)}) dx \text{ as } n \rightarrow \infty, \quad (23)$$

$$\|(u_R, v_R)\|_{\mathbf{H}}^2 \rightarrow \|(u_{\mu_0}^{(1)}, v_{\mu_0}^{(1)})\|_{\mathbf{H}}^2 \text{ as } n \rightarrow \infty, \quad (24)$$

and

$$\int_{\mathbb{R}^2} \phi_{u_R, v_R}^t (u_R^2 + v_R^2) dx \rightarrow \int_{\mathbb{R}^2} \phi_{u_{\mu_0}^{(1)}, v_{\mu_0}^{(1)}}^t \left[(u_{\mu_0}^{(1)})^2 + (v_{\mu_0}^{(1)})^2 \right] dx \text{ as } n \rightarrow \infty, \quad (25)$$

where $\int_{\mathbb{R}^2} \phi_{u, v}^t (u^2 + v^2) dx = \int_{\mathbb{R}^2} l(x) \phi_{l, (u, v)}^t (u^2 + v^2) dx$ with $l(x) \equiv 1$ on $\mathbb{R}^2$. Since $J_{\mu_0} \in C^1(\mathbf{H}, \mathbb{R})$ and $J_{\mu_0}(u_{\mu_0}^{(1)}, v_{\mu_0}^{(1)}) < 0$, by (23) – (25), there exists $R_0 > 0$ such that

$$J_{\mu_0}(u_{R_0}, v_{R_0}) < 0. \quad (26)$$

Let $0 < \varepsilon_N < \frac{1}{N^4 + R_0}$ and

$$u_{R_0, N}^{(i)}(x) = u_{R_0} \left(x + iN^3 e - \frac{x_0}{\varepsilon_N} \right) \text{ and } v_{R_0, N}^{(i)}(x) = v_{R_0} \left(x + iN^3 e - \frac{x_0}{\varepsilon_N} \right),$$

for $e \in S^1$ and $i = 1, 2, \dots, m$, where $N^3 > 2R_0$. A straightforward calculation shows that

$$\text{supp } u_{R_0, N}^{(i)}(x) \subset B^2 \left(\frac{x_0}{\varepsilon_N}, \frac{1}{\varepsilon_N} \right) \text{ and } \text{supp } v_{R_0, N}^{(i)}(x) \subset B^2 \left(\frac{x_0}{\varepsilon_N}, \frac{1}{\varepsilon_N} \right) \text{ for all } i = 1, 2, \dots, N.$$

Moreover, by (23), we deduce that

$$\begin{aligned} \|(u_{R_0, N}^{(i)}, v_{R_0, N}^{(i)})\|_{\mathbf{H}}^2 &= \|(u_{R_0}, v_{R_0})\|_{\mathbf{H}}^2 \text{ for all } N, \\ \int_{\mathbb{R}^2} F(u_{R_0, N}^{(i)}, v_{R_0, N}^{(i)}) dx &= \int_{\mathbb{R}^2} F(u_{R_0}, v_{R_0}) dx \text{ for all } N, \end{aligned}$$

and

$$\begin{aligned} &\int_{\mathbb{R}^2} l(\varepsilon_N x) \phi_{\varepsilon_N, (u_{R_0, N}^{(i)}, v_{R_0, N}^{(i)})}^t \left[(u_{R_0, N}^{(i)})^2 + (v_{R_0, N}^{(i)})^2 \right] dx \\ &\leq \mu_0 \int_{\mathbb{R}^2} \phi_{u_{R_0}, v_{R_0}}^t \left[(u_{R_0, N}^{(i)})^2 + (v_{R_0, N}^{(i)})^2 \right] dx \end{aligned}$$

for all $e \in S^1$ and $i = 1, 2, \dots, N$. Let

$$w_{R_0, N}(x) = \sum_{i=1}^N u_{R_0, N}^{(i)}(x) \text{ and } z_{R_0, N}(x) = \sum_{i=1}^N v_{R_0, N}^{(i)}(x) \text{ for any } x \in \mathbb{R}^2.$$

It is obvious that $(w_{R_0, N}, z_{R_0, N})$ is a sum of translation of (u_{R_0}, v_{R_0}) . When $N^3 > 2R_0$, the summands have disjoint support and

$$\text{supp } w_{R_0, N}(x) \subset B^2 \left(\frac{x_0}{\varepsilon_N}, \frac{1}{\varepsilon_N} \right). \quad (27)$$

In this case, we have

$$\|(w_{R_0, N}, z_{R_0, N})\|_{\mathbf{H}}^2 = N \|(u_{R_0}, v_{R_0})\|_{\mathbf{H}}^2, \quad (28)$$

$$\int_{\mathbb{R}^2} F(w_{R_0, N}, z_{R_0, N}) dx = \sum_{i=1}^N \int_{\mathbb{R}^2} F(u_{R_0, N}^{(i)}, v_{R_0, N}^{(i)}) dx = N \int_{\mathbb{R}^2} F(u_{R_0}, v_{R_0}) dx \quad (29)$$

and

$$\begin{aligned} & \int_{\mathbb{R}^2} l(\varepsilon_m x) \phi_{l_{\varepsilon_N}, (w_{R_0, N}, z_{R_0, N})}^t \left[(w_{R_0, N})^2 + (z_{R_0, N})^2 \right] dx \\ & < \mu_0 \int_{\mathbb{R}^2} \phi_{w_{R_0, N}, z_{R_0, N}}^t \left[(w_{R_0, N})^2 + (z_{R_0, N})^2 \right] dx. \end{aligned}$$

Hence, together with the Fubini Theorem, we obtain

$$\begin{aligned} & \int_{\mathbb{R}^2} \phi_{w_{R_0, N}, z_{R_0, N}}^t \left[(w_{R_0, N})^2 + (z_{R_0, N})^2 \right] dx \\ &= C_t \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\left[(w_{R_0, N}(x))^2 + (z_{R_0, N}(x))^2 \right] \left[(w_{R_0, N}(y))^2 + (z_{R_0, N}(y))^2 \right]}{|x - y|^{2-2t}} dx dy \\ &= C_t \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\left[w_{R_0, N}(x) \right]^2 \left[w_{R_0, N}(y) \right]^2}{|x - y|^{2-2t}} dx dy \\ & \quad + C_t \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\left[z_{R_0, N}(x) \right]^2 \left[z_{R_0, N}(y) \right]^2}{|x - y|^{2-2t}} dx dy \\ & \quad + 2C_t \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\left[w_{R_0, N}(x) \right]^2 \left[z_{R_0, N}(y) \right]^2}{|x - y|^{2-2t}} dx dy \\ &= C_t \sum_{i=1}^N \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\left[u_{R_0, N}^{(i)}(x) \right]^2 \left[u_{R_0, N}^{(i)}(y) \right]^2}{|x - y|^{2-2t}} dx dy \tag{30} \\ & \quad + C_t \sum_{i=1}^N \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\left[v_{R_0, N}^{(i)}(x) \right]^2 \left[v_{R_0, N}^{(i)}(y) \right]^2}{|x - y|^{2-2t}} dx dy \\ & \quad + C_t \sum_{i \neq j}^N \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\left[u_{R_0, N}^{(i)}(x) \right]^2 \left[u_{R_0, N}^{(j)}(y) \right]^2}{|x - y|^{2-2t}} dx dy \\ & \quad + C_t \sum_{i \neq j}^N \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\left[v_{R_0, N}^{(i)}(x) \right]^2 \left[v_{R_0, N}^{(j)}(y) \right]^2}{|x - y|^{2-2t}} dx dy \\ & \quad + 2C_t \sum_{i=1}^N \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\left[u_{R_0, N}^{(i)}(x) \right]^2 \left[v_{R_0, N}^{(i)}(y) \right]^2}{|x - y|^{2-2t}} dx dy \\ & \quad + 2C_t \sum_{i \neq j}^N \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\left[u_{R_0, N}^{(i)}(x) \right]^2 \left[v_{R_0, N}^{(j)}(y) \right]^2}{|x - y|^{2-2t}} dx dy \\ &= N \int_{\mathbb{R}^2} \phi_{u_{R_0}, v_{R_0}}^t \left[(u_{R_0})^2 + (v_{R_0})^2 \right] dx \\ & \quad + C_t \sum_{i \neq j}^N \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\left[u_{R_0, N}^{(i)}(x) \right]^2 \left[u_{R_0, N}^{(j)}(y) \right]^2}{|x - y|^{2-2t}} dx dy \end{aligned}$$

$$\begin{aligned}
& + C_t \sum_{i \neq j}^N \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{[v_{R_0,N}^{(i)}(x)]^2 [v_{R_0,N}^{(j)}(y)]^2}{|x-y|^{2-2t}} dx dy \\
& + 2C_t \sum_{i \neq j}^N \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{[u_{R_0,N}^{(i)}(x)]^2 [v_{R_0,N}^{(j)}(y)]^2}{|x-y|^{2-2t}} dx dy.
\end{aligned}$$

By a direct calculation, we obtain

$$\begin{aligned}
0 & < C_t \sum_{i \neq j}^N \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{[u_{R_0,N}^{(i)}(x)]^2 [u_{R_0,N}^{(j)}(y)]^2}{|x-y|^{2-2t}} dx dy \\
& \leq \frac{C_t N(N-1)}{(N^3 - 2R_0)^{2-2t}} \left[\int_{\mathbb{R}^2} (u_{\mu}^{(1)})^2 dx \right]^2.
\end{aligned} \quad (31)$$

If $\frac{1}{2} < t < \frac{2}{3}$ and $N^3 > 2R_0$, then by (31), we deduce that

$$C_t \sum_{i \neq j}^N \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{[u_{R_0,N}^{(i)}(x)]^2 [u_{R_0,N}^{(j)}(y)]^2}{|x-y|^{2-2t}} dx dy \rightarrow 0 \text{ as } N \rightarrow \infty. \quad (32)$$

Similarly, when $\frac{1}{2} < t < \frac{2}{3}$ and $N^3 > 2R_0$, there hold

$$C_t \sum_{i \neq j}^N \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{[v_{R_0,N}^{(i)}(x)]^2 [v_{R_0,N}^{(j)}(y)]^2}{|x-y|^{2-2t}} dx dy \rightarrow 0, \quad (33)$$

and

$$2C_t \sum_{i \neq j}^N \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{[u_{R_0,N}^{(i)}(x)]^2 [v_{R_0,N}^{(j)}(y)]^2}{|x-y|^{2-2t}} dx dy \rightarrow 0 \quad (34)$$

as $N \rightarrow \infty$.

In view of (28) – (31) and (32) – (34), we derive

$$\begin{aligned}
J_{l_{\varepsilon_N}}(w_{R_0,N}, z_{R_0,N}) &= \frac{1}{2} \|(w_{R_0,N}, z_{R_0,N})\|_{\mathbf{H}}^2 - \frac{1}{2p+2} \int_{\mathbb{R}^2} F(w_{R_0,N}, z_{R_0,N}) dx \\
&+ \frac{\mu_0}{4} \int_{\mathbb{R}^2} l(\varepsilon_N x) \phi_{l_{\varepsilon_N}, (w_{R_0,N}, z_{R_0,N})}^t \left[(w_{R_0,N})^2 + (z_{R_0,N})^2 \right] dx \\
&\leq \frac{N}{2} \|(u_{R_0}, v_{R_0})\|_{\mathbf{H}}^2 - N \int_{\mathbb{R}^2} F(u_{R_0}, v_{R_0}) dx \\
&+ \frac{N\mu_0}{4} \int_{\mathbb{R}^2} \phi_{u_{R_0}, v_{R_0}}^t \left[(u_{R_0})^2 + (v_{R_0})^2 \right] dx \\
&+ \frac{\mu_0 C_t}{4} \sum_{i \neq j}^N \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{[u_{R_0,N}^{(i)}(x)]^2 [u_{R_0,N}^{(j)}(y)]^2}{|x-y|^{2-2t}} dx dy \\
&+ \frac{\mu_0 C_t}{4} \sum_{i \neq j}^N \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{[v_{R_0,N}^{(i)}(x)]^2 [v_{R_0,N}^{(j)}(y)]^2}{|x-y|^{2-2t}} dx dy
\end{aligned} \quad (35)$$

$$\begin{aligned}
 & + \frac{\mu_0 C_t}{2} \sum_{i \neq j}^N \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\left[u_{R_0, N}^{(i)}(x) \right]^2 \left[v_{R_0, N}^{(j)}(y) \right]^2}{|x - y|^{2-2t}} dx dy \\
 & \leq N J_{\mu_0}(u_{R_0}, v_{R_0}) + C_0 \text{ for some } C_0 > 0.
 \end{aligned}$$

Thus, it follows from (26) and (35) that

$$J_{l_{\varepsilon_N}}(w_{R_0, N}, z_{R_0, N}) \rightarrow -\infty \text{ as } N \rightarrow +\infty,$$

and together with the fact that $\varepsilon_N \rightarrow 0$ as $N \rightarrow \infty$. The proof is complete. $\square$

Define

$$\delta_{l_{\varepsilon}} := \inf_{(u, v) \in \mathbf{H}_r} J_{l_{\varepsilon}}(u, v).$$

Then by Lemmas 5.1 and 5.2, we have the following result.

Lemma 5.3 Assume that $\frac{1}{2} < t < \frac{2}{3}$, $t \leq s < 1$, conditions (D_1) , (D_2) and (D_4) and (D_5) hold. Then there exists $K > 0$ such that

$$\beta_{l_{\varepsilon}} < -K \leq \delta_{l_{\varepsilon}} \text{ for } \varepsilon > 0 \text{ sufficiently small.}$$

We are now ready to prove Theorem 1.8. It follows from conditions (D_1) , (D_2) and (D_5) that there exist $x_0 \in \{l(x) < \sqrt{\Lambda_0}\}$ and $0 < \mu_0 < \Lambda_0$ such that $l(x_0) < \mu_0$. Let $(u_{\mu_0}^{(1)}, v_{\mu_0}^{(1)})$ be the radial solution of system (S_{μ_0}) obtained in Theorem 1.4. Denote $(u_{\varepsilon}(x), v_{\varepsilon}(x)) = (u_{\mu_0}^{(1)}(x - \frac{x_0}{\varepsilon}), v_{\mu_0}^{(1)}(x - \frac{x_0}{\varepsilon}))$. Then by the condition (D_2) , for ε small enough, we deduce that

$$\begin{aligned}
 & \int_{\mathbb{R}^2} l_{\varepsilon}(x) \phi_{l_{\varepsilon}, (u_{\varepsilon}, v_{\varepsilon})}^t (u_{\varepsilon}^2 + v_{\varepsilon}^2) dx \\
 & = \int_{\mathbb{R}^2} l(\varepsilon x + x_0) \phi_{l(\varepsilon x + x_0), (u_{\mu_0}^{(1)}, v_{\mu_0}^{(1)})}^t \left[\left(u_{\mu_0}^{(1)} \right)^2 + \left(v_{\mu_0}^{(1)} \right)^2 \right] dx \\
 & = l^2(x_0) \int_{\mathbb{R}^2} \phi_{u_{\mu_0}^{(1)}, v_{\mu_0}^{(1)}}^t \left[\left(u_{\mu_0}^{(1)} \right)^2 + \left(v_{\mu_0}^{(1)} \right)^2 \right] dx + o(\varepsilon) \\
 & < \mu_0 \int_{\mathbb{R}^2} \phi_{u_{\mu_0}^{(1)}, v_{\mu_0}^{(1)}}^t \left[\left(u_{\mu_0}^{(1)} \right)^2 + \left(v_{\mu_0}^{(1)} \right)^2 \right] dx,
 \end{aligned}$$

where $o(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Moreover, by Theorem 1.5 and Lemma 5.3, system $(S_{l_{\varepsilon}})$ possesses a ground state solution $(u_{l_{\varepsilon}}, v_{l_{\varepsilon}}) \in \mathbf{H}$ such that

$$J_{l_{\varepsilon}}(u_{l_{\varepsilon}}, v_{l_{\varepsilon}}) = \beta_{l_{\varepsilon}} < -K \leq \delta_{l_{\varepsilon}} \text{ for } \varepsilon > 0 \text{ sufficiently small,}$$

which implies that $(u_{l_{\varepsilon}}, v_{l_{\varepsilon}})$ is a non-radial ground state solution of system $(S_{l_{\varepsilon}})$. Besides, in view of Theorem 2.8, $(u_{l_{\varepsilon}}, v_{l_{\varepsilon}})$ is also vectorial.

5.2 The case of $\frac{2}{3} \leq t \leq s < 1$

Lemma 5.4 Suppose that $\frac{2}{3} \leq t \leq s < 1$, conditions (D_1) , (D_2) , (D_4) and (D_5) hold. Then we have $\beta_{l_{\varepsilon}} < \delta_{l_{\varepsilon}}$ for $\varepsilon > 0$ sufficiently small.

Proof By Remark 1.10 (i) and Lemma 5.1 (i), there exists $0 < \mu_0 < \Lambda_0$ such that system $(S_{l_{\varepsilon}})$ possesses a ground state solution $(u_{l_{\varepsilon}}, v_{l_{\varepsilon}}) \in \mathbf{H}$ such that $J_{l_{\varepsilon}}(u_{l_{\varepsilon}}, v_{l_{\varepsilon}}) = \beta_{l_{\varepsilon}} <$

$J_{\mu_0}(u_{\mu_0}^{(1)}, v_{\mu_0}^{(1)}) < 0$ and $\delta_{l_\varepsilon} \geq -K$ for some $K > 0$, where $(u_{\mu_0}^{(1)}, v_{\mu_0}^{(1)})$ is the radial solution of system (S_{μ_0}) obtained in Theorem 1.4. Now, we claim that

$$J_{l_\varepsilon}(u_{l_\varepsilon}, v_{l_\varepsilon}) = \beta_{l_\varepsilon} < J_{\mu_0}(u_{\mu_0}^{(1)}, v_{\mu_0}^{(1)}) \leq \delta_{l_\varepsilon} \text{ for } \varepsilon > 0 \text{ sufficiently small.}$$

Otherwise, there exists a sequence $\{\varepsilon_m\} \subset \mathbb{R}^+$ with $\varepsilon_m \rightarrow 0$ as $m \rightarrow \infty$ such that

$$-K \leq \delta_{l_{\varepsilon_m}} = \inf_{(u,v) \in \mathbf{H}_r} J_{l_{\varepsilon_m}}(u, v) < J_{\mu_0}(u_{\mu_0}^{(1)}, v_{\mu_0}^{(1)}) < 0.$$

For each $m = 1, 2, \dots$, by the Ekeland variational principle [14] and Palais criticality principle [26], there exists a sequence $\{(u_n, v_n)\} \subset \mathbf{H}_r$ such that

$$J_{l_{\varepsilon_m}}(u_n, v_n) = \delta_{l_{\varepsilon_m}} + o_n(1) \quad \text{and} \quad (J_{l_{\varepsilon_m}})'(u_n, v_n) = o_n(1) \quad \text{in } \mathbf{H}^{-1}.$$

It follows from Lemma 2.9 that $\{(u_n, v_n)\}$ is bounded in $\mathbf{H}_r$. Then there exist a subsequence $\{(u_n, v_n)\}$ and a function $(u_{l_{\varepsilon_m}}, v_{l_{\varepsilon_m}}) \in \mathbf{H}_r$ such that $(u_n, v_n) \rightharpoonup (u_{l_{\varepsilon_m}}, v_{l_{\varepsilon_m}})$ as $n \rightarrow \infty$ weakly in $\mathbf{H}_r$. Furthermore, using a similar argument with that of [28, Theorem 4.3], we get $J'_{l_{\varepsilon_m}}(u_{l_{\varepsilon_m}}, v_{l_{\varepsilon_m}}) = 0$ and $(u_n, v_n) \rightarrow (u_{l_{\varepsilon_m}}, v_{l_{\varepsilon_m}})$ strongly in $\mathbf{H}_r$ as $n \rightarrow \infty$. This implies that $J_{l_{\varepsilon_m}}(u_{l_{\varepsilon_m}}, v_{l_{\varepsilon_m}}) = \delta_{l_{\varepsilon_m}}$. Since $J'_{l_{\varepsilon_m}}(u_{l_{\varepsilon_m}}, v_{l_{\varepsilon_m}}) = 0$ and

$$-K \leq J_{l_{\varepsilon_m}}(u_{l_{\varepsilon_m}}, v_{l_{\varepsilon_m}}) = \delta_{l_{\varepsilon_m}} < J_{\mu_0}(u_{\mu_0}^{(1)}, v_{\mu_0}^{(1)}) < 0 \text{ for all } m = 1, 2, \dots \quad (36)$$

Again, by Lemma 2.9, we know that $\{(u_{l_{\varepsilon_m}}, v_{l_{\varepsilon_m}})\}$ is bounded in $\mathbf{H}_r$. Hence, there exist a subsequence $\{(u_{l_{\varepsilon_m}}, v_{l_{\varepsilon_m}})\}$ and a function $(u_{l_0}, v_{l_0}) \in \mathbf{H}_r$ such that $(u_{l_{\varepsilon_m}}, v_{l_{\varepsilon_m}}) \rightarrow (u_{l_0}, v_{l_0})$ strongly in $\mathbf{H}_r$ as $m \rightarrow \infty$. This implies that

$$\begin{aligned} J_{l_{\varepsilon_m}}(u_{l_{\varepsilon_m}}, v_{l_{\varepsilon_m}}) &= \frac{1}{2} \|(u_{l_{\varepsilon_m}}, v_{l_{\varepsilon_m}})\|_{\mathbf{H}}^2 + \frac{1}{4} \int_{\mathbb{R}^2} l(\varepsilon_m x) \phi_{l_{\varepsilon_m}, (u_{l_{\varepsilon_m}}, v_{l_{\varepsilon_m}})}^t [u_{l_{\varepsilon_m}}^2 + v_{l_{\varepsilon_m}}^2] dx \\ &\quad - \frac{1}{2p+2} \int_{\mathbb{R}^2} F(u_{l_{\varepsilon_m}}, v_{l_{\varepsilon_m}}) dx \\ &= \frac{1}{2} \|(u_{l_0}, v_{l_0})\|_{H^s}^2 + \frac{l^2(0)}{4} \int_{\mathbb{R}^2} \phi_{u_{l_0}, v_{l_0}}^t [u_{l_0}^2 + v_{l_0}^2] dx \\ &\quad - \frac{1}{2p+2} \int_{\mathbb{R}^2} F(u_0, v_0) dx + o_m(1), \end{aligned}$$

where $o_m(1) \rightarrow 0$ as $m \rightarrow \infty$. By the fact that $0 \notin \{l(x) < \sqrt{\Lambda_0}\}$, there holds

$$\begin{aligned} &\frac{1}{2} \|(u_{l_0}, v_{l_0})\|_{H^s}^2 + \frac{l^2(0)}{4} \int_{\mathbb{R}^2} \phi_{u_{l_0}, v_{l_0}}^t (u_{l_0}^2 + v_{l_0}^2) dx - \frac{1}{2p+2} \int_{\mathbb{R}^2} F(u_0, v_0) dx \\ &\geq \frac{1}{2} \|(u_{l_0}, v_{l_0})\|_{H^s}^2 + \frac{\Lambda_0}{4} \int_{\mathbb{R}^2} \phi_{u_{l_0}, v_{l_0}}^t (u_{l_0}^2 + v_{l_0}^2) dx - \frac{1}{2p+2} \int_{\mathbb{R}^2} F(u_0, v_0) dx \geq 0, \end{aligned}$$

which is a contradiction with (36). Hence, we obtain

$$J_{l_\varepsilon}(u_{l_\varepsilon}, v_{l_\varepsilon}) = \beta_{l_\varepsilon} < J_{\mu_0}(u_{\mu_0}^{(1)}, v_{\mu_0}^{(1)}) \leq \delta_{l_\varepsilon} \text{ for } \varepsilon > 0 \text{ sufficiently small.}$$

The proof is complete. $\square$

We are now ready to prove Theorem 1.9. From Theorem 1.5 and Lemma 5.4, similarly to the argument of Theorem 1.8, system (S_{l_ε}) possesses a vectorial ground state solution $(u_{l_\varepsilon}, v_{l_\varepsilon}) \in \mathbf{H}$ such that

$$J_{l_\varepsilon}(u_{l_\varepsilon}, v_{l_\varepsilon}) < \delta_{l_\varepsilon} \text{ for } \varepsilon > 0 \text{ sufficiently small,}$$

which implies that $(u_{l_\varepsilon}, v_{l_\varepsilon})$ is a non-radial vectorial ground state solution of system (S_{l_ε}) . $\square$

6 Multiplicity of positive solutions

We are now ready to prove Theorem 1.11. In view of $0 \in \{l(x) < \sqrt{\Lambda_0}\}$, Lemmas 5.1 and 5.3, we obtain

$$\beta_{l_\varepsilon} < \delta_{l_\varepsilon} < 0 \text{ for } \varepsilon > 0 \text{ sufficiently small.} \quad (37)$$

From the Ekeland variational principle [14] and Palais criticality principle [26], for $\varepsilon > 0$ sufficiently small, there exists a sequence $\{(u_n, v_n)\} \subset \mathbf{H}_r$ such that

$$J_{l_\varepsilon}(u_n, v_n) = \delta_{l_\varepsilon} + o_n(1) \text{ and } (J_{l_\varepsilon})'(u_n, v_n) = o_n(1) \text{ in } \mathbf{H}_r^{-1}.$$

It follows from Lemma 2.9 that $\{(u_n, v_n)\}$ is bounded in $\mathbf{H}_r$. Then there exist a subsequence $\{(u_n, v_n)\}$ and a function $(\tilde{u}_{l_\varepsilon}, \tilde{v}_{l_\varepsilon}) \in \mathbf{H}_r$ such that $(u_n, v_n) \rightharpoonup (\tilde{u}_{l_\varepsilon}, \tilde{v}_{l_\varepsilon})$ weakly in $\mathbf{H}_r$. Moreover, using similar argument with that of [28, Theorem 4.3], we know that $(J_{l_\varepsilon})'(\tilde{u}_{l_\varepsilon}, \tilde{v}_{l_\varepsilon}) = 0$ and $(u_n, v_n) \rightarrow (\tilde{u}_{l_\varepsilon}, \tilde{v}_{l_\varepsilon})$ strongly in $\mathbf{H}_r$. Thus, $J_{l_\varepsilon}(\tilde{u}_{l_\varepsilon}, \tilde{v}_{l_\varepsilon}) = \delta_{l_\varepsilon}$. Therefore, $(\tilde{u}_{l_\varepsilon}, \tilde{v}_{l_\varepsilon})$ is a radial ground state solution of system (S_{l_ε}) . Hence, by (37) and Theorem 1.8, system (S_{l_ε}) admits two different positive solutions $(u_{l_\varepsilon}, v_{l_\varepsilon}) \in \mathbf{H}$ and $(\tilde{u}_{l_\varepsilon}, \tilde{v}_{l_\varepsilon}) \in \mathbf{H}_r$ satisfying

$$\beta_{l_\varepsilon} = J_{l_\varepsilon}(u_{l_\varepsilon}, v_{l_\varepsilon}) < \delta_{l_\varepsilon} = J_{l_\varepsilon}(\tilde{u}_{l_\varepsilon}, \tilde{v}_{l_\varepsilon}) < 0 \text{ for } \varepsilon > 0 \text{ sufficiently small.}$$

Since $(\tilde{u}_{l_\varepsilon}, \tilde{v}_{l_\varepsilon})$ is a radial ground state solution of system (S_{l_ε}) and

$$\beta_{l_\varepsilon} := \inf_{(u,v) \in \mathbf{H}} J_{l_\varepsilon}(u, v) < \delta_{l_\varepsilon} = \inf_{(u,v) \in \mathbf{H}_r} J_{l_\varepsilon}(u, v) \text{ for } \varepsilon > 0 \text{ sufficiently small,}$$

we may conclude that $(u_{l_\varepsilon}, v_{l_\varepsilon})$ is a non-radial ground state solution of system (S_{l_ε}) . Besides, we have an isolated minimum at 0 and also an absolute minimum at $(\tilde{u}_{l_\varepsilon}, \tilde{v}_{l_\varepsilon})$. Indeed, it follows from the Sobolev inequality that

$$\begin{aligned} J_{l_\varepsilon}(u, v) &\geq \frac{1}{2} \|(u, v)\|_{\mathbf{H}}^2 + \frac{\mu}{4} \int_{\mathbb{R}^2} \phi_{u,v}^t (u^2 + v^2) dx - \frac{1}{2p+2} \int_{\mathbb{R}^2} F(u, v) dx \\ &\geq \frac{1}{2} \|(u, v)\|_{\mathbf{H}}^2 - \frac{1}{(p+1)S_{2p+2}^{2p+2}} \|(u, v)\|_{\mathbf{H}}^{2p+2} \text{ for all } (u, v) \in \mathbf{H}, \end{aligned}$$

where $0 < p < \frac{1}{2}$. Thus, there exist $\eta, \kappa > 0$ such that

$$\|(u_{l_\varepsilon}, v_{l_\varepsilon})\|_{H^s} > \eta \text{ and } J_{l_\varepsilon}(\tilde{u}_{l_\varepsilon}, \tilde{v}_{l_\varepsilon}) < 0 < \kappa \leq \inf_{\|(u,v)\|_{\mathbf{H}}=\eta} J_{l_\varepsilon}(u, v).$$

Define

$$\sigma_{l_\varepsilon} := \inf_{\gamma \in \Gamma} \max_{0 \leq \tau \leq 1} J_{l_\varepsilon}(\gamma(\tau)) > 0,$$

where

$$\Gamma := \{\gamma \in C([0, 1], \mathbf{H}_r) \mid \gamma(0) = (0, 0), \gamma(1) = (\tilde{u}_{l_\varepsilon}, \tilde{v}_{l_\varepsilon})\}.$$

Then by the mountain pass theorem [15, 27] and Palais criticality principle [26], there exists a sequence $\{(u_n, v_n)\} \subset \mathbf{H}_r$ such that

$$J_{l_\varepsilon}(u_n, v_n) = \sigma_{l_\varepsilon} \geq \kappa > 0 \text{ and } (J_{l_\varepsilon})'(u_n, v_n) \rightarrow 0 \text{ as } n \rightarrow \infty,$$

and using the similar argument of [28, Theorem 4.3], there exist a subsequence $\{(u_n, v_n)\}$ and a function $(\bar{u}_{l_\varepsilon}, \bar{v}_{l_\varepsilon}) \in \mathbf{H}_r \setminus \{(0, 0)\}$ such that $(u_n, v_n) \rightarrow (\bar{u}_{l_\varepsilon}, \bar{v}_{l_\varepsilon})$ strongly in $\mathbf{H}_r$. Then $(J_\mu)'(\bar{u}_{l_\varepsilon}, \bar{v}_{l_\varepsilon}) = 0$ and $J_{l_\varepsilon}(\bar{u}_{l_\varepsilon}, \bar{v}_{l_\varepsilon}) = \sigma_{l_\varepsilon} > 0$. Thus, $(\bar{u}_{l_\varepsilon}, \bar{v}_{l_\varepsilon})$ is a nontrivial radial solution

of system (S_{l_ε}) . Hence, we may conclude that system (S_{l_ε}) possesses three nontrivial solutions $(u_{l_\varepsilon}, v_{l_\varepsilon}) \in \mathbf{H}$ and $(\tilde{u}_{l_\varepsilon}, \tilde{v}_{l_\varepsilon}), (\bar{u}_{l_\varepsilon}, \bar{v}_{l_\varepsilon}) \in \mathbf{H}_r$ satisfying

$$\beta_{l_\varepsilon} = J_{l_\varepsilon}(u_{l_\varepsilon}, v_{l_\varepsilon}) < \delta_{l_\varepsilon} = J_{l_\varepsilon}(\tilde{u}_{l_\varepsilon}, \tilde{v}_{l_\varepsilon}) < 0 < \sigma_{l_\varepsilon} = J_{l_\varepsilon}(\bar{u}_{l_\varepsilon}, \bar{v}_{l_\varepsilon}) \text{ for } \varepsilon > 0 \text{ sufficiently small.}$$

The proof is complete. $\square$

We are now ready to prove Theorem 1.12. Since

$$\int_{\mathbb{R}^2} l(\varepsilon x) \phi_{l_\varepsilon, (u_\mu^{(1)}, v_\mu^{(1)})}^t \left[\left(u_\mu^{(1)} \right)^2 + \left(v_\mu^{(1)} \right)^2 \right] dx < \mu \int_{\mathbb{R}^2} \phi_{l_\varepsilon, (u_\mu^{(1)}, v_\mu^{(1)})}^t \left[\left(u_\mu^{(1)} \right)^2 + \left(v_\mu^{(1)} \right)^2 \right] dx$$

for $\varepsilon > 0$ sufficiently small. Then the symmetric case still holds in Theorem 1.5. Thus, system (S_{l_ε}) admits a vectorial radial solution $(\tilde{u}_{l_\varepsilon}, \tilde{v}_{l_\varepsilon}) \in \mathbf{H}_r$ satisfying

$$\delta_{l_\varepsilon} = J_{l_\varepsilon}(\tilde{u}_{l_\varepsilon}, \tilde{v}_{l_\varepsilon}) < 0 \text{ for } \varepsilon > 0 \text{ sufficiently small.}$$

Furthermore, by Theorem 1.9, system (S_{l_ε}) admits a vectorial ground state solution $(u_{l_\varepsilon}, v_{l_\varepsilon}) \in \mathbf{H}$ such that

$$J_{l_\varepsilon}(u_{l_\varepsilon}, v_{l_\varepsilon}) = \beta_{l_\varepsilon} < \delta_{l_\varepsilon} = J_{l_\varepsilon}(\tilde{u}_{l_\varepsilon}, \tilde{v}_{l_\varepsilon}) < 0 \text{ for } \varepsilon > 0 \text{ sufficiently small.}$$

Similar to the argument of Theorem 1.11, system (S_{l_ε}) possesses a nontrivial radial solution $(\bar{u}_{l_\varepsilon}, \bar{v}_{l_\varepsilon}) \in \mathbf{H}_r$ such that

$$J_{l_\varepsilon}(\bar{u}_{l_\varepsilon}, \bar{v}_{l_\varepsilon}) > 0 \text{ for } \varepsilon > 0 \text{ sufficiently small,}$$

which implies that

$$\beta_{l_\varepsilon} = J_{l_\varepsilon}(u_{l_\varepsilon}, v_{l_\varepsilon}) < \delta_{l_\varepsilon} = J_{l_\varepsilon}(\tilde{u}_{l_\varepsilon}, \tilde{v}_{l_\varepsilon}) < 0 < J_{l_\varepsilon}(\bar{u}_{l_\varepsilon}, \bar{v}_{l_\varepsilon}) \text{ for } \varepsilon > 0 \text{ sufficiently small.}$$

The proof is complete. $\square$

7 Appendix.

In this section, we present the details of the **proof of Proposition 2.11** in Section 2 above.

Proof Step 1. Since $\{(u_n, v_n)\}$ is bounded in $\mathbf{H}$, we may assume that, up to a subsequence,

$$\begin{aligned} (u_n, v_n) &\rightharpoonup (u_0, v_0) \text{ in } \mathbf{H}; \\ (u_n, v_n) &\rightarrow (u_0, v_0) \text{ in } L_{loc}^r(\mathbb{R}^2) \times L_{loc}^r(\mathbb{R}^2) \text{ for } 1 \leq r < 2_s^*; \\ (u_n, v_n) &\rightarrow (u_0, v_0) \text{ a.e. in } \mathbb{R}^2. \end{aligned} \quad (38)$$

Now let us prove that $J'_l(u_0, v_0) = 0$. Noting that $C_0^\infty(\mathbb{R}^2)$ is dense in $H^s(\mathbb{R}^2)$, it is sufficient to prove that $\langle J'_l(u_0, v_0), (\varphi, \psi) \rangle = 0$ for all $\varphi, \psi \in C_0^\infty(\mathbb{R}^2)$. According to the expression of $J'_l(u, v)$, we just have to prove that

$$\begin{aligned} o(1) &= \langle J'_l(u_n, v_n) - J'_l(u_0, v_0), (\varphi, \psi) \rangle \\ &= \langle (u_n - u_0, v_n - v_0), (\varphi, \psi) \rangle_{H^s} \\ &\quad + \int_{\mathbb{R}^2} l(x) \left(\phi_{l, (u_n, v_n)}^t u_n - \phi_{l, (u_0, v_0)}^t u_0 \right) \varphi dx \\ &\quad + \int_{\mathbb{R}^2} l(x) \left(\phi_{l, (u_n, v_n)}^t v_n - \phi_{l, (u_0, v_0)}^t v_0 \right) \psi dx \end{aligned}$$

$$\begin{aligned}
 & - \int_{\mathbb{R}^2} (|u_n|^{2p} u_n - |u_0|^{2p} u_0) \varphi \, dx \\
 & - \int_{\mathbb{R}^2} (|v_n|^{2p} v_n - |v_0|^{2p} v_0) \psi \, dx \\
 & - \int_{\mathbb{R}^2} (|v_n|^{p+1} |u_n|^{p-2} u_n - |v_0|^{p+1} |u_0|^{p-2} u_0) \varphi \, dx \\
 & - \int_{\mathbb{R}^2} (|u_n|^{p+1} |v_n|^{p-2} v_n - |u_0|^{p+1} |v_0|^{p-2} v_0) \psi \, dx
 \end{aligned} \tag{39}$$

for all $\varphi, \psi \in C_0^\infty(\mathbb{R}^2)$.

In fact, from (38), the condition (D_1) and the Sobolev inequality, there holds

$$\langle (u_n - u_0, v_n - v_0), (\varphi, \psi) \rangle_{H^s} = o_n(1) \tag{40}$$

for any $\varphi, \psi \in C_0^\infty(\mathbb{R}^2)$.

Again by (38), the condition (D_1) , the Sobolev inequality and a similar argument to that of Lemma 2.2(iv), there holds

$$\begin{aligned}
 & \int_{\mathbb{R}^2} l(x) \left(\phi_{l, (u_n, v_n)}^t u_n - \phi_{l, (u_0, v_0)}^t u_0 \right) \varphi \, dx \\
 & \leq l_{\max}^2 \int_{\mathbb{R}^2} (\phi_{u_n, v_n}^t u_n - \phi_{u_0, v_0}^t u_0) \varphi \, dx \\
 & = l_{\max}^2 \int_{\text{supp} \varphi} (\phi_{u_n, v_n}^t u_n - \phi_{u_n, v_n}^t u_0) \varphi \, dx \\
 & \quad + l_{\max}^2 \int_{\text{supp} \varphi} (\phi_{u_n, v_n}^t u_0 - \phi_{u_0, v_0}^t u_0) \varphi \, dx \\
 & \leq l_{\max}^2 \|\varphi\|_{H^s} \|\phi_{u_n, v_n}^t\|_{2_t^*} \left(\int_{\text{supp} \varphi} |u_n - u_0|^{\frac{4}{t+1}} \right)^{\frac{t+1}{4}} \\
 & \quad + l_{\max}^2 S^{-2} |u_0|_{\frac{4}{t+1}} \|\varphi\|_{H^s} \left(\int_{\text{supp} \varphi} \left| (-\Delta)^{\frac{t}{2}} (\phi_{u_n, v_n}^t - \phi_{u_0, v_0}^t) \right|^2 \right)^{\frac{1}{2}} \\
 & = o_n(1)
 \end{aligned} \tag{41}$$

for any $\varphi \in C_0^\infty(\mathbb{R}^2)$.

Analogously,

$$\int_{\mathbb{R}^2} l(x) \left(\phi_{l, (u_n, v_n)}^t v_n - \phi_{l, (u_0, v_0)}^t v_0 \right) \psi \, dx = o_n(1) \tag{42}$$

for any $\psi \in C_0^\infty(\mathbb{R}^2)$.

From [34, Lemma A.1] and (38), it is not difficult to obtain

$$\int_{\mathbb{R}^2} (|u_n|^{2p} u_n - |u_0|^{2p} u_0) \varphi \, dx = o_n(1) \tag{43}$$

and

$$\int_{\mathbb{R}^2} (|v_n|^{2p} v_n - |v_0|^{2p} v_0) \psi \, dx = o_n(1) \tag{44}$$

for all $\varphi, \psi \in C_0^\infty(\mathbb{R}^2)$.

Finally, to obtain (39), we just need to prove that

$$\int_{\mathbb{R}^2} (|v_n|^{p+1} |u_n|^{p-1} u_n - |v_0|^{p+1} |u_0|^{p-1} u_0) \varphi \, dx = o_n(1) \tag{45}$$

and

$$\int_{\mathbb{R}^2} (|u_n|^{p+1}|v_n|^{p-1}v_n - |u_0|^p|v_0|^{p-2}v_0) \psi dx = o_n(1) \quad (46)$$

for all $\varphi, \psi \in C_0^\infty(\mathbb{R}^2)$.

Next, we just prove that (45) holds, since (46) can be proved in a similar way.

Denote

$$\begin{aligned} I &= \int_{\mathbb{R}^2} (|v_n|^{p+1}|u_n|^{p-1}u_n - |v_0|^{p+1}|u_0|^{p-1}u_0) \varphi dx \\ &= \int_{\text{supp}\varphi} |v_n|^{p+1} (|u_n|^{p-1}u_n - |u_0|^{p-1}u_0) \varphi dx \\ &\quad + \int_{\text{supp}\varphi} (|v_n|^{p+1} - |v_0|^{p+1}) |u_0|^{p-1}u_0 \varphi dx \\ &= I_1 + I_2, \end{aligned}$$

where

$$I_1 = \int_{\text{supp}\varphi} |v_n|^{p+1} (|u_n|^{p-1}u_n - |u_0|^{p-1}u_0) \varphi dx$$

and

$$I_2 = \int_{\text{supp}\varphi} (|v_n|^{p+1} - |v_0|^{p+1}) |u_0|^{p-1}u_0 \varphi dx.$$

For the simplicity of calculation, we denote $f(t) = |t|^{p-1}t$ ($t \in \mathbb{R}$), then

$$|f(u_n)| = |u_n|^p \leq 1 + |u_n|^p = |u_n|^{\frac{2p+2}{(2p+2)/p}}$$

and

$$u_n \rightarrow u_0 \quad \text{in } L^{2p+2}(\text{supp}\varphi),$$

thus,

$$f(u_n) \rightarrow f(u_0) \quad \text{in } L^{\frac{2p+2}{p}}(\text{supp}\varphi) \quad (47)$$

by [34, Lemma A.1]. In view of the Hölder inequality, there holds

$$\begin{aligned} |I_1| &= \left| \int_{\text{supp}\varphi} |v_n|^{p+1} (|u_n|^{p-1}u_n - |u_0|^{p-1}u_0) \varphi dx \right| \\ &\leq \left(\int_{\text{supp}\varphi} |f(u_n) - f(u_0)|^{\frac{2p+2}{p}} \right)^{\frac{p}{2p+2}} \left(\int_{\text{supp}\varphi} |v_n \varphi|^{\frac{2(p+1)^2}{p+2}} \right)^{\frac{p+2}{2p+2}} \\ &\leq \left(\int_{\text{supp}\varphi} |f(u_n) - f(u_0)|^{\frac{2p+2}{p}} \right)^{\frac{p}{2p+2}} \left(\int_{\text{supp}\varphi} |v_n|^{2p+2} \right)^{\frac{1}{2}} \\ &\quad \times \left(\int_{\text{supp}\varphi} |\varphi|^{2p+2} \right)^{\frac{1}{2p+2}} = o_n(1) \end{aligned} \quad (48)$$

by (47).

Besides, we may deduce that

$$|v_n|^{p+1} \rightarrow |v_0|^{p+1} \quad \text{in } L^2(\text{supp}\varphi), \quad (49)$$

as $n \rightarrow \infty$. Indeed, the Hölder inequality and (38) imply that

$$\begin{aligned}
 & \left(\int_{\text{supp}\varphi} |v_n|^{p+1} - |v_0|^{p+1} \right)^2 \\
 & \leq C \int_{\text{supp}\varphi} |v_n - v_0|^2 \int_{\text{supp}\varphi} (|u_n|^p + |u_0|^p)^2 \\
 & \leq C \left(\int_{\text{supp}\varphi} |v_n - v_0|^{2p+2} \right)^{\frac{1}{p+1}} \left(\int_{\text{supp}\varphi} (|v_n|^p + |v_0|^p)^{\frac{2p+2}{p}} \right)^{\frac{p}{p+1}} \\
 & \leq C \left(\int_{\text{supp}\varphi} |v_n - v_0|^{2p+2} \right)^{\frac{1}{p+1}} \left(\int_{\text{supp}\varphi} (|v_n|^{2p+2} + |v_0|^{2p+2}) \right)^{\frac{p}{p+1}} \\
 & = o_n(1),
 \end{aligned}$$

which indicates that (49) holds.

Therefore, by (38), (49) and the Hölder inequality, we derive

$$\begin{aligned}
 |I_2| &= \left| \int_{\text{supp}\varphi} (|v_n|^{p+1} - |v_0|^{p+1}) |u_0|^{p-1} u_0 \varphi dx \right| \\
 &\leq \left(\int_{\text{supp}\varphi} |v_n|^{p+1} - |v_0|^{p+1} \right)^{\frac{1}{2}} \left(\int_{\text{supp}\varphi} |u_0^p \varphi|^2 \right)^{\frac{1}{2}} \\
 &\leq \left(\int_{\text{supp}\varphi} |v_n|^{p+1} - |v_0|^{p+1} \right)^{\frac{1}{2}} \left(\int_{\text{supp}\varphi} |u_0|^{2p+2} \right)^{\frac{p}{2p+2}} \\
 &\quad \times \left(\int_{\text{supp}\varphi} |\varphi|^{2p+2} \right)^{\frac{1}{2p+2}} = o_n(1).
 \end{aligned} \tag{50}$$

Then (48) and (50) imply that $I = o_n(1)$, that is, (45) holds. Thus, it follows from (40)-(46) that (39) holds, and then $J'_l(u_0, v_0) = 0$.

Step 2. Set $(u_n^{(1)}, v_n^{(1)}) := (u_n - u_0, v_n - v_0)$, then $(u_n^{(1)}, v_n^{(1)}) \rightarrow (0, 0)$ in $\mathbf{H}$. By Lemma 2.3, one has

$$\begin{aligned}
 & \int_{\mathbb{R}^2} l(x) \phi_{l, (u_n, v_n)}^t (u_n^2 + v_n^2) dx = \int_{\mathbb{R}^2} l(x) \phi_{l, (u_0, v_0)}^t (u_0^2 + v_0^2) dx \\
 & + \int_{\mathbb{R}^2} l(x) \phi_{l, (u_n^{(1)}, v_n^{(1)})}^t \left((u_n^{(1)})^2 + (v_n^{(1)})^2 \right) dx + o_n(1).
 \end{aligned} \tag{51}$$

Furthermore, it follows from Lemma 2.3 and Brezis–Lieb Lemma that

$$\| (u_n, v_n) \|_{\mathbf{H}}^2 = \| (u_0, v_0) \|_{\mathbf{H}}^2 + \| (u_n^{(1)}, v_n^{(1)}) \|_{\mathbf{H}}^2 + o_n(1), \tag{52}$$

$$\int_{\mathbb{R}^2} |u_n|^{2p+2} dx = \int_{\mathbb{R}^2} |u_0|^{2p+2} dx + \int_{\mathbb{R}^2} |u_n^{(1)}|^{2p+2} dx + o_n(1), \tag{53}$$

$$\int_{\mathbb{R}^2} |v_n|^{2p+2} dx = \int_{\mathbb{R}^2} |v_0|^{2p+2} dx + \int_{\mathbb{R}^2} |v_n^{(1)}|^{2p+2} dx + o_n(1) \tag{54}$$

and

$$\int_{\mathbb{R}^2} |u_n|^{p+1} |v_n|^{p+1} dx = \int_{\mathbb{R}^2} |u_0|^{p+1} |v_0|^{p+1} dx + \int_{\mathbb{R}^2} |u_n^{(1)}|^{p+1} |v_n^{(1)}|^{p+1} dx + o_n(1). \tag{55}$$

From (51)–(55), we derive

$$J_l(u_n^{(1)}, v_n^{(1)}) = J_l(u_n, v_n) - J_l(u_0, v_0) + o_n(1), \quad (56)$$

and

$$\langle J'_l(u_n^{(1)}, v_n^{(1)}), (u_n^{(1)}, v_n^{(1)}) \rangle = \langle J'_l(u_n, v_n), (u_n, v_n) \rangle - \langle J'_l(u_0, v_0), (u_0, v_0) \rangle + o_n(1) = o_n(1). \quad (57)$$

Now we set

$$\delta := \limsup_{n \rightarrow +\infty} \sup_{y \in \mathbb{R}^2} \int_{B_1(y)} (|u_n^{(1)}|^2 + |v_n^{(1)}|^2) dx \geq 0.$$

Case 1. $\delta = 0$. It follows from the vanishing lemma [30, Lemma II.4] that

$$\int_{\mathbb{R}^2} (|u_n|^r + |v_n|^r) dx \rightarrow 0 \text{ as } n \rightarrow \infty,$$

for all $2 < r < 2_s^*$.

Then

$$\int_{\mathbb{R}^2} F(u_n, v_n) dx \rightarrow 0, \quad (58)$$

and

$$\begin{aligned} & \int_{\mathbb{R}^2} l(x) \phi_{l, (u_n^{(1)}, v_n^{(1)})}^t \left[(u_n^{(1)})^2 + (v_n^{(1)})^2 \right] dx \\ & \leq l_{\max}^2 S^{-1} \|\phi_{l, (u_n^{(1)}, v_n^{(1)})}^t\|_{D_{l,2}} \left(|u_n^{(1)}|^{\frac{4}{1+t}} + |v_n^{(1)}|^{\frac{4}{1+t}} \right)^{\frac{t+1}{2}} \rightarrow 0 \end{aligned} \quad (59)$$

as $n \rightarrow \infty$.

Hence, by (57)–(59), we derive

$$\|(u_n^{(1)}, v_n^{(1)})\|_{\mathbf{H}}^2 = \int_{\mathbb{R}^2} F(u_n^{(1)}, v_n^{(1)}) dx - \int_{\mathbb{R}^2} l(x) \phi_{l, (u_n^{(1)}, v_n^{(1)})}^t \left[(u_n^{(1)})^2 + (v_n^{(1)})^2 \right] dx = o_n(1),$$

which implies that $(u_n^{(1)}, v_n^{(1)}) \rightarrow (0, 0)$ in $\mathbf{H}$, i.e. $(u_n, v_n) \rightarrow (u_0, v_0)$ as $n \rightarrow \infty$ in $\mathbf{H}$.

Case 2. $\delta > 0$. We may assume that there exists $\{y_n^{(1)}\} \subset \mathbb{R}^2$ such that

$$\int_{B_1(y_n^{(1)})} (|u_n^{(1)}|^2 + |v_n^{(1)}|^2) dx > \frac{\delta}{2}.$$

Define $(\tilde{u}_n^{(1)}, \tilde{v}_n^{(1)}) := (u_n^{(1)}(\cdot + y_n^{(1)}), v_n^{(1)}(\cdot + y_n^{(1)}))$. Then $(\tilde{u}_n^{(1)}, \tilde{v}_n^{(1)})$ is bounded in $\mathbf{H}$ and we may assume that $(\tilde{u}_n^{(1)}, \tilde{v}_n^{(1)}) \rightharpoonup (\omega^{(1)}, z^{(1)})$ in $\mathbf{H}$, $(\tilde{u}_n^{(1)}, \tilde{v}_n^{(1)}) \rightarrow (\omega^{(1)}, z^{(1)})$ in $L_{loc}^r(\mathbb{R}^2) \times L_{loc}^r(\mathbb{R}^2)$ for $1 \leq r < 2_s^*$, and $(\tilde{u}_n^{(1)}, \tilde{v}_n^{(1)}) \rightarrow (\omega^{(1)}, z^{(1)})$ a.e. in $\mathbb{R}^2$. Since $\int_{B_1(0)} (|\tilde{u}_n^{(1)}|^2 + |\tilde{v}_n^{(1)}|^2) dx > \frac{\delta}{2}$, we have $\int_{B_1(0)} (|\omega^{(1)}|^2 + |z^{(1)}|^2) dx > \frac{\delta}{2}$, which implies that $(\omega^{(1)}, z^{(1)}) \neq (0, 0)$. But, since $(u_n^{(1)}, v_n^{(1)}) \rightarrow (0, 0)$ in $\mathbf{H}$, then $\{y_n^{(1)}\}$ must be unbounded. Up to a subsequence, we may assume that $|y_n^{(1)}| \rightarrow +\infty$ as $n \rightarrow +\infty$.

Next, we will prove that $(J_{l_\infty})'(\omega^{(1)}, z^{(1)}) = 0$. Using a similar argument with that of (39), we can obtain that

$$\langle (J_{l_\infty})'(\tilde{u}_n^{(1)}, \tilde{v}_n^{(1)}), (\varphi, \psi) \rangle - \langle (J_{l_\infty})'(\omega^{(1)}, z^{(1)}), (\varphi, \psi) \rangle = o_n(1)$$

for any $\varphi, \psi \in C_0^\infty(\mathbb{R}^2)$. Thus, it suffices to prove that $\langle (J_{l_\infty})'(\tilde{u}_n^{(1)}, \tilde{v}_n^{(1)}), (\varphi, \psi) \rangle = o_n(1)$ for any $\varphi, \psi \in C_0^\infty(\mathbb{R}^2)$. Since $(u_n^{(1)}, v_n^{(1)}) \rightarrow (0, 0)$ in $\mathbf{H}$, similar to the argument of (39), we have

$$\langle J_l'(u_n^{(1)}, v_n^{(1)}), (\varphi(\cdot - y_n^{(1)}), \psi(\cdot - y_n^{(1)})) \rangle - \langle J_l'(0, 0), (\varphi(\cdot - y_n^{(1)}), \psi(\cdot - y_n^{(1)})) \rangle = o_n(1),$$

for any $\varphi, \psi \in C_0^\infty(\mathbb{R}^2)$. Thus, $\left\langle J_l'(u_n^{(1)}, v_n^{(1)}), (\varphi(\cdot - y_n^{(1)}), \psi(\cdot - y_n^{(1)})) \right\rangle = o_n(1)$, i.e.

$$\begin{aligned} & \int_{\mathbb{R}^2} (-\Delta)^{s/2} u_n^{(1)}(x) (-\Delta)^{s/2} \varphi(x - y_n^{(1)}) dx + \int_{\mathbb{R}^2} u_n^{(1)}(x) \varphi(x - y_n^{(1)}) dx \\ & + \int_{\mathbb{R}^2} (-\Delta)^{s/2} v_n^{(1)}(x) (-\Delta)^{s/2} \psi(x - y_n^{(1)}) dx + \int_{\mathbb{R}^2} v_n^{(1)}(x) \psi(x - y_n^{(1)}) dx \\ & + \int_{\mathbb{R}^2} l(x) \phi_{l, (u_n^{(1)}, v_n^{(1)})}^t u_n^{(1)} \varphi(x - y_n^{(1)}) dx + \int_{\mathbb{R}^2} l(x) \phi_{l, (u_n^{(1)}, v_n^{(1)})}^t v_n^{(1)} \psi(x - y_n^{(1)}) dx \\ & - \int_{\mathbb{R}^2} |u_n^{(1)}|^{2p} u_n^{(1)} \varphi(x - y_n^{(1)}) dx - \int_{\mathbb{R}^2} |v_n^{(1)}|^{2p} v_n^{(1)} \psi(x - y_n^{(1)}) dx \\ & - \int_{\mathbb{R}^2} |v_n^{(1)}|^{p+1} |u_n^{(1)}|^{p-2} u_n^{(1)} \varphi(x - y_n^{(1)}) dx \\ & - \int_{\mathbb{R}^2} |u_n^{(1)}|^{p+1} |v_n^{(1)}|^{p-2} v_n^{(1)} \psi(x - y_n^{(1)}) dx = o_n(1) \end{aligned} \quad (60)$$

for any $\varphi, \psi \in C_0^\infty(\mathbb{R}^2)$.

Besides, from the condition (D_1) and $|y_n^{(1)}| \rightarrow +\infty$ as $n \rightarrow +\infty$, we have

$$\begin{aligned} & \int_{\mathbb{R}^2} l(x) \phi_{l, (u_n^{(1)}, v_n^{(1)})}^t u_n^{(1)} \varphi(x - y_n^{(1)}) dx \\ & = \int_{\mathbb{R}^2} l(x + y_n^{(1)}) \phi_{l(x+y_n^{(1)}), (u_n^{(1)}(x+y_n^{(1)}), v_n^{(1)}(x+y_n^{(1)}))}^t u_n^{(1)}(x + y_n^{(1)}) \varphi(x) dx \\ & = \int_{\mathbb{R}^2} l_\infty \phi_{l_\infty, (\tilde{u}_n^{(1)}, \tilde{v}_n^{(1)})}^t \tilde{u}_n^{(1)} \varphi(x) dx \end{aligned} \quad (61)$$

and

$$\begin{aligned} & \int_{\mathbb{R}^2} l(x) \phi_{l, (u_n^{(1)}, v_n^{(1)})}^t v_n^{(1)} \psi(x - y_n^{(1)}) dx \\ & = \int_{\mathbb{R}^2} l(x + y_n^{(1)}) \phi_{l(x+y_n^{(1)}), (u_n^{(1)}(x+y_n^{(1)}), v_n^{(1)}(x+y_n^{(1)}))}^t v_n^{(1)}(x + y_n^{(1)}) \psi(x) dx \\ & = \int_{\mathbb{R}^2} l_\infty \phi_{l_\infty, (\tilde{u}_n^{(1)}, \tilde{v}_n^{(1)})}^t \tilde{v}_n^{(1)} \psi(x) dx \end{aligned} \quad (62)$$

for any $\varphi, \psi \in C_0^\infty(\mathbb{R}^2)$.

Then from (60)–(62), we obtain $\left\langle (J_{l_\infty})'(\tilde{u}_n^{(1)}, \tilde{v}_n^{(1)}), (\varphi, \psi) \right\rangle = o_n(1)$ for any $\varphi, \psi \in C_0^\infty(\mathbb{R}^2)$, i.e.,

$$(J_{l_\infty})'(\omega^{(1)}, z^{(1)}) = 0.$$

Now we claim that

$$\int_{\mathbb{R}^2} l(x) \phi_{l, (u_n^{(1)}, v_n^{(1)})}^t \left[\left(u_n^{(1)} \right)^2 + \left(v_n^{(1)} \right)^2 \right] dx = \int_{\mathbb{R}^2} l_\infty \phi_{l_\infty, (u_n^{(1)}, v_n^{(1)})}^t \left[\left(u_n^{(1)} \right)^2 + \left(v_n^{(1)} \right)^2 \right] dx. \quad (63)$$

Indeed, from the expression of $\phi_{l, (u, v)}^t$ and the Fubini theorem, we have

$$\begin{aligned} & \left| \int_{\mathbb{R}^2} l(x) \phi_{l, (u_n^{(1)}, v_n^{(1)})}^t \left[\left(u_n^{(1)} \right)^2 + \left(v_n^{(1)} \right)^2 \right] dx \right. \\ & \quad \left. - \int_{\mathbb{R}^2} l_\infty \phi_{l_\infty, (u_n^{(1)}, v_n^{(1)})}^t \left[\left(u_n^{(1)} \right)^2 + \left(v_n^{(1)} \right)^2 \right] dx \right| \\ &= \left| \int_{\mathbb{R}^2} l(x) \left(\phi_{l-l_\infty, (u_n^{(1)}, v_n^{(1)})}^t + \phi_{l_\infty, (u_n^{(1)}, v_n^{(1)})}^t \right) \left[\left(u_n^{(1)} \right)^2 + \left(v_n^{(1)} \right)^2 \right] \right. \\ & \quad \left. - \int_{\mathbb{R}^2} l_\infty \phi_{l_\infty, (u_n^{(1)}, v_n^{(1)})}^t \left[\left(u_n^{(1)} \right)^2 + \left(v_n^{(1)} \right)^2 \right] dx \right| \\ &\leq \underbrace{\int_{\mathbb{R}^2} |l(x) - l_\infty| \phi_{l-l_\infty, (u_n^{(1)}, v_n^{(1)})}^t \left[\left(u_n^{(1)} \right)^2 + \left(v_n^{(1)} \right)^2 \right] dx}_{I_1} \\ & \quad + 2 \underbrace{\int_{\mathbb{R}^2} |l(x) - l_\infty| \phi_{l_\infty, (u_n^{(1)}, v_n^{(1)})}^t \left[\left(u_n^{(1)} \right)^2 + \left(v_n^{(1)} \right)^2 \right] dx}_{I_2}. \end{aligned}$$

Since $l(x) \rightarrow l_\infty$ as $|x| \rightarrow \infty$, for any $\varepsilon > 0$, there exists $\bar{\rho}(\varepsilon) > 0$ such that for every $\rho > \bar{\rho}(\varepsilon)$,

$$|l(x) - l_\infty| < \varepsilon \quad \text{for all } |x| > \rho.$$

Since $(u_n^{(1)}, v_n^{(1)})$ is bounded in $\mathbf{H}$, we have $(u_n^{(1)}, v_n^{(1)})$ is bounded in $L^r(\mathbb{R}^2) \times L^r(\mathbb{R}^2)$ for $2 \leq r \leq 2_s^*$ by the Sobolev inequality. Note that $2 < \frac{4}{1+t} < 2_t^* \leq 2_s^*$, and thus $(u_n^{(1)}, v_n^{(1)})$ is bounded in $L^{\frac{4}{1+t}}(\mathbb{R}^2) \times L^{\frac{4}{1+t}}(\mathbb{R}^2)$. Then there exists $M_1 > 0$ such that

$$\begin{aligned} & \int_{\mathbb{R}^2 \setminus B_\rho(0)} |l(x) - l_\infty|^{\frac{2}{1+t}} \left(\left| u_n^{(1)} \right|^{\frac{4}{1+t}} + \left| v_n^{(1)} \right|^{\frac{4}{1+t}} \right) dx \\ & < \varepsilon \int_{\mathbb{R}^2 \setminus B_\rho(0)} \left(\left| u_n^{(1)} \right|^{\frac{4}{1+t}} + \left| v_n^{(1)} \right|^{\frac{4}{1+t}} \right) dx < M_1 \varepsilon. \end{aligned} \quad (64)$$

On the other hand, since $(u_n^{(1)}, v_n^{(1)}) \rightarrow (0, 0)$ in $H^s(B_\rho(0)) \times H^s(B_\rho(0))$ for any $\rho > 0$, we obtain

$$(u_n^{(1)}, v_n^{(1)}) \rightarrow (0, 0) \quad \text{in } L^{\frac{4}{1+t}}(B_\rho(0)) \times L^{\frac{4}{1+t}}(B_\rho(0)).$$

Therefore, for any $\varepsilon > 0$, we obtain

$$\int_{B_\rho(0)} |l(x) - l_\infty|^{\frac{2}{1+t}} \left(\left| u_n^{(1)} \right|^{\frac{4}{1+t}} + \left| v_n^{(1)} \right|^{\frac{4}{1+t}} \right) dx \leq M_1 \int_{B_\rho(0)} \left(\left| u_n^{(1)} \right|^{\frac{4}{1+t}} + \left| v_n^{(1)} \right|^{\frac{4}{1+t}} \right) dx < M_2 \varepsilon \quad (65)$$

for some $M_2 > 0$. Thus, the Sobolev inequality and (64)–(65) imply that

$$\begin{aligned} I_1 &\leq S^{-1} \|\phi_{l-l_\infty}^t(u_n^{(1)}, v_n^{(1)})\|_{D^{t,2}} \left[\int_{\mathbb{R}^2} |l(x) - l_\infty|^{\frac{2}{i+1}} \left(|u_n^{(1)}|^{\frac{4}{i+1}} + |v_n^{(1)}|^{\frac{4}{i+1}} \right) dx \right]^{\frac{i+1}{2}} \\ &\leq C \left[\int_{B_\rho(0)} |l(x) - l_\infty|^{\frac{2}{i+1}} \left(|u_n^{(1)}|^{\frac{4}{i+1}} + |v_n^{(1)}|^{\frac{4}{i+1}} \right) dx \right. \\ &\quad \left. + \int_{\mathbb{R}^2 \setminus B_\rho(0)} |l(x) - l_\infty|^{\frac{2}{i+1}} \left(|u_n^{(1)}|^{\frac{4}{i+1}} + |v_n^{(1)}|^{\frac{4}{i+1}} \right) dx \right]^{\frac{i+1}{2}} \\ &< C (M_1 \varepsilon + M_2 \varepsilon)^{\frac{i+1}{2}}, \end{aligned}$$

which implies that $I_1 = o_n(1)$. Similarly, $I_2 = o_n(1)$, and the claim is proved. Therefore, by (51)–(56) and (63), there holds

$$J_l(u_n, v_n) = J_l(u_0, v_0) + J_{l_\infty}(u_n^{(1)}, v_n^{(1)}) + o_n(1).$$

Step 3. Set

$$(u_n^{(2)}(\cdot), v_n^{(2)}(\cdot)) := (u_n^{(1)}(\cdot) - \omega^{(1)}(\cdot - y_n^{(1)}), v_n^{(1)}(\cdot) - z^{(1)}(\cdot - y_n^{(1)})).$$

Then $(u_n^{(2)}(\cdot), v_n^{(2)}(\cdot)) \rightarrow (0, 0)$ as $n \rightarrow \infty$ in $\mathbf{H}$.

From Lemma 2.3 and the Brezis–Lieb Lemma, there hold

$$\|(u_n^{(2)}, v_n^{(2)})\|_{\mathbf{H}}^2 = \|(u_n, v_n)\|_{\mathbf{H}}^2 - \|(u_0, v_0)\|_{\mathbf{H}}^2 - \left\| \left(\omega^{(1)}(\cdot - y_n^{(1)}), z^{(1)}(\cdot - y_n^{(1)}) \right) \right\|_{\mathbf{H}}^2 + o_n(1), \quad (66)$$

$$\int_{\mathbb{R}^2} |u_n^{(2)}|^{2p+2} dx = \int_{\mathbb{R}^2} |u_n|^{2p+2} dx - \int_{\mathbb{R}^2} |u_0|^{2p+2} dx - \int_{\mathbb{R}^2} |\omega^{(1)}(\cdot - y_n^{(1)})|^{2p+2} dx + o_n(1), \quad (67)$$

$$\int_{\mathbb{R}^2} |v_n^{(2)}|^{2p+2} dx = \int_{\mathbb{R}^2} |v_n|^{2p+2} dx - \int_{\mathbb{R}^2} |v_0|^{2p+2} dx - \int_{\mathbb{R}^2} |z^{(1)}(\cdot - y_n^{(1)})|^{2p+2} dx + o_n(1), \quad (68)$$

$$\begin{aligned} &\int_{\mathbb{R}^2} l(x) \phi_{l, (u_n^{(2)}, v_n^{(2)})}^t \left[(u_n^{(2)})^2 + (v_n^{(2)})^2 \right] dx \\ &= \int_{\mathbb{R}^2} l(x) \phi_{l, (u_n, v_n)}^t [(u_n)^2 + v_n^2] dx - \int_{\mathbb{R}^2} l(x) \phi_{l, (u_0, v_0)}^t (u_0^2 + v_0^2) dx \\ &\quad - \int_{\mathbb{R}^2} l(x) \phi_{l, (\omega^{(1)}(\cdot - y_n^{(1)}), z^{(1)}(\cdot - y_n^{(1)}))}^t \left[\left(\omega^{(1)}(\cdot - y_n^{(1)}) \right)^2 + \left(z^{(1)}(\cdot - y_n^{(1)}) \right)^2 \right] dx + o_n(1), \end{aligned} \quad (69)$$

$$\begin{aligned} &\int_{\mathbb{R}^2} l(x) \phi_{l, (u_n^{(2)}, v_n^{(2)})}^t u_n^{(2)} \varphi dx = \int_{\mathbb{R}^2} l(x) \phi_{l, (u_n, v_n)}^t u_n \varphi dx - \int_{\mathbb{R}^2} l(x) \phi_{l, (u_0, v_0)}^t u_0 \varphi dx \\ &\quad - \int_{\mathbb{R}^2} l(x) \phi_{l, (\omega^{(1)}(\cdot - y_n^{(1)}), z^{(1)}(\cdot - y_n^{(1)}))}^t \omega^{(1)}(\cdot - y_n^{(1)}) \varphi dx + o_n(1), \end{aligned} \quad (70)$$

$$\int_{\mathbb{R}^2} l(x) \phi_{l, (u_n^{(2)}, v_n^{(2)})}^t v_n^{(2)} \psi dx = \int_{\mathbb{R}^2} l(x) \phi_{l, (u_n, v_n)}^t v_n \psi dx - \int_{\mathbb{R}^2} l(x) \phi_{l, (u_0, v_0)}^t v_0 \psi dx$$

$$- \int_{\mathbb{R}^2} l(x) \phi_{l, (z^{(1)}(\cdot - y_n^{(1)}), z^{(1)}(\cdot - y_n^{(1)}))}^t \omega^{(1)}(\cdot - y_n^{(1)}) \psi dx + o_n(1), \quad (71)$$

and

$$\begin{aligned} \int_{\mathbb{R}^2} |u_n^{(2)}|^{p+1} |v_n^{(2)}|^{p+1} dx &= \int_{\mathbb{R}^2} |u_n|^{p+1} |v_n|^{p+1} dx - \int_{\mathbb{R}^2} |u_0|^{p+1} |v_0|^{p+1} dx \\ &- \int_{\mathbb{R}^2} |\omega^{(1)}(\cdot - y_n^{(1)})|^{p+1} |z^{(1)}(\cdot - y_n^{(1)})|^{p+1} dx + o_n(1) \end{aligned} \quad (72)$$

for any $\varphi, \psi \in H^s(\mathbb{R}^2)$.

In view of (66)–(72), we obtain

$$J_l(u_n^{(2)}, v_n^{(2)}) = J_l(u_n, v_n) - J_l(u_0, v_0) - J_{l_\infty}(\omega^{(1)}, z^{(1)}) + o_n(1),$$

$$J_{l_\infty}(u_n^{(2)}, v_n^{(2)}) = J_{l_\infty}(u_n^{(1)}, v_n^{(1)}) - J_{l_\infty}(\omega^{(1)}, z^{(1)}) + o_n(1),$$

and

$$\begin{aligned} \langle J_l'(u_n^{(2)}, v_n^{(2)}), (\varphi, \psi) \rangle &= \langle J_l'(u_n, v_n), (\varphi, \psi) \rangle - \langle J_l'(u_0, v_0), (\varphi, \psi) \rangle \\ &- \langle J_{l_\infty}'(\omega^{(1)}, z^{(1)}), (\varphi, \psi) \rangle + o_n(1) = o_n(1). \end{aligned}$$

Using similar arguments to those of Step 2, let

$$\delta_1 := \sup_{y \in \mathbb{R}^2} \int_{B_1(y)} \left[\left(u_n^{(2)} \right)^2 + \left(v_n^{(2)} \right)^2 \right] dx \geq 0.$$

If the vanishing occurs, then $\|(u_n^{(2)}, v_n^{(2)})\|_{\mathbf{H}} \rightarrow 0$, i.e.

$$\left\| \left(u_n - u_0 - \omega^{(1)}(\cdot - y_n^{(1)}), v_n - v_0 - z^{(1)}(\cdot - y_n^{(1)}) \right) \right\|_{\mathbf{H}} \rightarrow 0 \text{ as } n \rightarrow \infty,$$

then Proposition 2.11 holds with $k = 1$. If the vanishing does not occur, then there exist a sequence $\{y_n^{(2)}\} \subset \mathbb{R}^2$ and $(\omega^{(2)}, z^{(2)}) \in \mathbf{H}$ such that $(\tilde{u}_n^{(2)}, \tilde{v}_n^{(2)}) = (u_n^{(2)}(\cdot + y_n^{(2)}), v_n^{(2)}(\cdot + y_n^{(2)})) \rightharpoonup (\omega^{(2)}, z^{(2)})$ in $\mathbf{H}$. Furthermore, $(u_n^{(2)}, v_n^{(2)}) \rightarrow (0, 0)$ in $\mathbf{H}$ implies that $|y_n^{(2)}| \rightarrow +\infty$ and $|y_n^{(1)} - y_n^{(2)}| \rightarrow +\infty$ as $n \rightarrow +\infty$. By iterating this procedure, there exist sequences of points $\{y_n^{(k)}\} \subset \mathbb{R}^2$ such that $|y_n^{(k)}| \rightarrow +\infty$ and $|y_n^{(k)} - y_n^{(k')}| \rightarrow +\infty$ if $k \neq k'$ as $n \rightarrow +\infty$, and

$$\left(u_n^{(k)}(\cdot), v_n^{(k)}(\cdot) \right) = \left(u_n^{(k-1)}(\cdot) - \omega^{(k-1)}(\cdot - y_n^{(k-1)}), v_n^{(k-1)}(\cdot) - z^{(k-1)}(\cdot - y_n^{(k-1)}) \right)$$

with $k \geq 2$ such that

$$\left(u_n^{(k)}, v_n^{(k)} \right) \rightarrow (0, 0) \text{ in } \mathbf{H} \text{ and } (J_{l_\infty})'(\omega^{(k)}, z^{(k)}) = 0,$$

$$\begin{aligned} &\left\| \left(u_n - u_0 - \sum_{j=1}^{k-1} \omega^{(j)}(\cdot - y_n^{(j)}), v_n - v_0 - \sum_{j=1}^{k-1} z^{(j)}(\cdot - y_n^{(j)}) \right) \right\|_{\mathbf{H}}^2 \\ &= \left\| (u_n, v_n) \right\|_{\mathbf{H}}^2 - \left\| (u_0, v_0) \right\|_{\mathbf{H}}^2 - \sum_{j=1}^{k-1} \left\| \left(\omega^{(j)}(\cdot - y_n^{(j)}), z^{(j)}(\cdot - y_n^{(j)}) \right) \right\|_{\mathbf{H}}^2 + o_n(1), \end{aligned} \quad (73)$$

and

$$J_l(u_n, v_n) = J_l(u_0, v_0) + \sum_{j=1}^{k-1} J_{l_\infty}(\omega^{(j)}) + J_{l_\infty}(u_n^{(k)}, v_n^{(k)}) + o_n(1). \quad (74)$$

Since $\{(u_n, v_n)\}$ is bounded in $\mathbf{H}$, from (73) and (74), we know that the iteration must stop at some finite index $\ell + 1$. Hence, $(u_n^{(k)}, v_n^{(k)}) \rightarrow (0, 0)$ in $\mathbf{H}$. Again by (73) and (74), we can easily get the conclusions $(iv) - (v)$ hold. The proof is complete. $\square$

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Declarations

Conflict of interest The authors declare that there is no conflict of interest. We also declare that this manuscript has no associated data.

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