



Regular article

A pair of nontrivial bounded solutions for parametric nonlinear elliptic equations with unbalanced growth

Nikolaos S. Papageorgiou^{a,b}, Vicențiu D. Rădulescu^{c,d,e,f}, Zdeněk Šmarda^d^a Center of Applied Mathematics, Yulin Normal University, Yulin 537000, People's Republic of China^b National Technical University, Department of Mathematics, Zografou Campus, Athens 15780, Greece^c AGH University of Kraków, Faculty of Applied Mathematics, al. Mickiewicza 30, Kraków 30-059 Poland^d Brno University of Technology, Faculty of Electrical Engineering and Communication, Technická 3058/10, Brno 61600, Czech Republic^e School of Mathematics and Statistics, Northeast Petroleum University, Daqing 163318, People's Republic of China^f Scientific Research Center, Baku Engineering University, AZ0102 Baku, Azerbaijan

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ABSTRACT

We consider a parametric Dirichlet problem driven by a nonautonomous (p, q) -differential operator with unbalanced growth. In the reaction we have the competing effects of a parametric $(p-1)$ -superlinear term and of a $(p-1)$ -linear perturbation. Using variational tools, we show that for all small values of the positive parameter, the problem has at least two nontrivial bounded solutions.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2 -boundary $\partial\Omega$. In this paper, we study the following parametric nonlinear Dirichlet problem

$$\begin{cases} -\Delta_p^a u(z) - \Delta_q u(z) = \lambda |u(z)|^{r-2} u(z) + f(z, u(z)) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \quad \lambda > 0, \quad 1 < q < p < r. \end{cases} \quad (P_\lambda)$$

Let $C^{0,1}(\bar{\Omega})$ be the space of Lipschitz continuous functions on $\bar{\Omega}$ and $a \in C^{0,1}(\bar{\Omega}) \setminus \{0\}$, $a(z) \geq 0$ for all $z \in \bar{\Omega}$. For $s \in (1, \infty)$ we denote by Δ_s^a the weighted s -Laplace differential operator defined by $\Delta_s^a u = \operatorname{div}(a(z)|Du|^{s-2} Du)$. We assume that $\min_{\bar{\Omega}} a = 0$ and this has profound implications on the structure of the problem. If $\theta(z, t) = a(z)t^p + t^q$ for all $z \in \Omega$ and all $t \geq 0$, then $\theta(z, \cdot)$ has unbalanced growth, namely we have

$$t^q \leq \theta(z, t) \leq c_0(1 + t^p) \text{ for all } z \in \Omega, \text{ all } t \geq 0, \text{ for some } c_0 > 0.$$

This means that there is no global (up to the boundary) regularity theory for problems driven by such an operator. This in turn implies that we miss many powerful analytic tools which a global regularity theory provides (equivalence of Sobolev and Hölder minimizers, nonlinear versions of the Hopf maximum principle, strong comparison principles and more). This makes unbalanced

* Corresponding author at: AGH University of Kraków, Faculty of Applied Mathematics, al. Mickiewicza 30, Kraków 30-059 Poland.

E-mail addresses: npapg@math.ntua.gr (N.S. Papageorgiou), radulescu@inf.ucv.ro, radulescu@agh.edu.pl (V.D. Rădulescu), smarda@vut.cz (Z. Šmarda).

growth problems more difficult to treat. In the reaction (right-hand side) of problem (P_λ) , we have the combined effects of a parametric superlinear term of power-type $u \mapsto \lambda|u|^{r-2}u$ ($q < p < r$) with $\lambda > 0$ being the parameter and a Carathéodory perturbation $f(z, u)$. We assume that $f(z, \cdot)$ is $(p-1)$ -linear as $x \rightarrow \pm\infty$ and partially interacts with the spectrum of the Dirichlet operator $-\Delta_p^a$ (nonuniform nonresonance). Problem (P_λ) is a variant of the classical “concave–convex” problem, where the concave term is replaced by a $(p-1)$ -linear perturbation (not necessarily of power-type). Unbalanced growth concave–convex problems were studied by Liu–Papageorgiou [1]. Here, for problem (P_λ) , we show that if $\lambda > 0$ is small, then problem (P_λ) has at least two nontrivial bounded solutions.

2. Mathematical background and hypotheses

We say that $a \in L^1_{loc}(\Omega)$ is a weight if $a(z) > 0$ for a.a. $z \in \Omega$. For $p \in (1, \infty)$, we denote by $\mathcal{A}_p$ the class of p -Muckenhoupt weights, defined by

$$\mathcal{A}_p = \left\{ a \in L^1_{loc}(\Omega) : \sup_B \left(\frac{1}{|B|_N} \int_B a dz \right) \left(\frac{1}{|B|_N} \int_B a^{-1/(p-1)} dz \right)^{p-1} < \infty \right\},$$

where the supremum is taken over all balls and $|\cdot|_N$ denotes the Lebesgue measure in $\mathbb{R}^N$. It follows that $\mathcal{A}_p$ consists of those weights a for which the Hardy–Littlewood maximal operator is bounded on $L^p(\Omega)$.

Our hypotheses on $a(\cdot)$ and the exponents p, q, r are the following: H_0 : $a \in C^{0,1}(\bar{\Omega}) \cap \mathcal{A}_p$, $1 < q < p < r < q^* = \frac{Nq}{N-q}$ and $\frac{p}{q} < 1 + \frac{1}{N}$.

Remark. The last inequality means that p, q cannot be far apart, it leads to local regularity results (see [2]) and to a weak maximum principle (see [3]).

We denote by $L^0(\Omega)$ the space of measurable functions $u : \Omega \rightarrow \mathbb{R}$. As usual, we identify two such functions which differ on a Lebesgue-null set. We consider the following density functions

$$\theta(z, t) = a(z)t^p + t^q \quad \text{and} \quad \theta_0(z, t) = a(z)t^p \quad \text{for all } z \in \Omega, \text{ all } t \geq 0.$$

In what follows, $\eta = \theta$ or $\eta = \theta_0$. The generalized Lebesgue–Orlicz space $L^\eta(\Omega)$ is defined by

$$L^\eta(\Omega) = \left\{ u \in L^0(\Omega) : \rho_\eta(u) = \int_\Omega \eta(z, |u|) dz < \infty \right\}.$$

We equip $L^\eta(\Omega)$ with the so-called “Luxemburg norm” $\|\cdot\|$ defined by

$$\|u\|_\eta = \inf \left\{ \mu > 0 : \rho_\eta\left(\frac{u}{\mu}\right) \leq 1 \right\}.$$

With this norm, $L^\eta(\Omega)$ becomes a separable, reflexive (in fact, uniformly convex) Banach space. Using $L^\eta(\Omega)$, we can define the corresponding generalized Sobolev–Orlicz space $W^{1,\eta}(\Omega)$ by

$$W^{1,\eta}(\Omega) = \{u \in L^\eta(\Omega) : |Du| \in L^\eta(\Omega)\},$$

with Du being the weak gradient of u . We equip $W^{1,\eta}(\Omega)$ with the norm $\|\cdot\|_{1,\eta}$ defined by

$$\|u\|_{1,\eta} = \|u\|_\eta + \|Du\|_\eta \quad \text{for all } u \in W^{1,\eta}(\Omega),$$

where $\|Du\|_\eta = \||Du|\|_\eta$. Also, we set

$$W_0^{1,\eta}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{1,\eta}},$$

with $C_c^\infty(\Omega)$ being the space of C^∞ -functions with compact support in Ω . Both spaces $W^{1,\eta}(\Omega)$ and $W_0^{1,\eta}(\Omega)$ are separable and reflexive (in fact, uniformly convex). Moreover, on $W_0^{1,\eta}(\Omega)$ the Poincaré inequality holds, namely

$$\|u\|_\eta \leq \hat{c} \|Du\|_\eta \quad \text{for some } \hat{c} > 0, \text{ all } u \in W_0^{1,\eta}(\Omega).$$

For a more general result in this direction, see [4]. So, on $W_0^{1,\theta}(\Omega)$ we can consider the equivalent norm $\|\cdot\|$ defined by

$$\|u\| = \|Du\|_\theta \quad \text{for all } u \in W_0^{1,\theta}(\Omega).$$

For more on generalized Orlicz spaces, we refer to [5,6]. In what follows, $\hookrightarrow$ denotes continuous and dense embedding.

Proposition 1. *The following properties hold.*

- (a) $L^\theta(\Omega) \hookrightarrow L^s(\Omega)$, $W_0^{1,\theta}(\Omega) \hookrightarrow W_0^{1,s}(\Omega)$ for all $1 \leq s \leq q$.
- (b) $W_0^{1,\theta}(\Omega) \hookrightarrow L^s(\Omega)$ for all $1 \leq s \leq q^*$ and the embedding is compact if $1 \leq s < q^*$.
- (c) $W_0^{1,p}(\Omega) \hookrightarrow W_0^{1,\theta}(\Omega)$.
- (d) $W_0^{1,\theta_0}(\Omega) \hookrightarrow L^{\theta_0}(\Omega)$ compactly.

There is a close relation between $\rho_\theta(\cdot)$ and the norm $\|\cdot\|$.

Proposition 2. *The following properties hold.*

- (a) $\|u\| = \mu > 0 \iff \rho_\theta(\frac{Du}{\mu}) = 1$.
- (b) $\|u\| < 1$ (resp. $= 1, > 1$) $\iff \rho_\theta(Du) < 1$ (resp. $= 1, > 1$).
- (c) $\|u\| < 1 \Rightarrow \|u\|^p \leq \rho_\theta(Du) \leq \|u\|^q$.
- (d) $\|u\| > 1 \Rightarrow \|u\|^q \leq \rho_\theta(Du) \leq \|u\|^p$.
- (e) $\|u\| \rightarrow 0$ (resp. $\rightarrow +\infty$) $\iff \rho_\theta(Du) \rightarrow 0$ (resp. $\rightarrow +\infty$).

Given $u \in L^0(\Omega)$, we write $0 < u$ if for all $K \subseteq \Omega$ compact, we have $0 < c_K \leq u(z)$ for a.a. $z \in K$.

Consider the following eigenvalue problem.

$$-\Delta_p^a u(z) = \hat{\lambda}_1^a |u(z)|^{p-2} u(z) \text{ in } \Omega, \quad u|_{\partial\Omega} = 0.$$

From [7] we know that this problem has a smallest eigenvalue $\hat{\lambda}_1^a > 0$, which has the following variational characterization:

$$\hat{\lambda}_1^a = \inf \left\{ \frac{\rho_{\theta_0}(Du)}{\rho_{\theta_0}(u)} : u \in W_0^{1,\theta_0}(\Omega), u \neq 0 \right\}. \quad (1)$$

This eigenvalue is simple and isolated. Moreover, the infimum in (1) is realized on the one-dimensional eigenspace of $\hat{\lambda}_1^a$, the elements of which have fixed sign. We denote by $\hat{u}_1^a$ the positive eigenfunction with $\rho_{\theta_0}(\hat{u}_1^a) = 1$. We know that $\hat{u}_1^a \in W_0^{1,\theta_0}(\Omega) \cap L^\infty(\Omega)$, $0 < \hat{u}_1^a$.

Proposition 3. *If hypotheses H_0 hold, $\beta \in L^\infty(\Omega)$, $\beta(z) \leq \hat{\lambda}_1^a$ for a.a. $z \in \Omega$ and $\beta \not\equiv \hat{\lambda}_1^a$, then there exists $c^* > 0$ such that*

$$c^* \rho_{\theta_0}(Du) \leq \rho_{\theta_0}(Du) - \int_{\Omega} \beta(z) a(z) |u|^p dz \text{ for all } u \in W_0^{1,\theta_0}(\Omega).$$

Proof. We argue indirectly. So, suppose that the assertion of the proposition is not true. We can find $\{u_n\}_{n \in \mathbb{N}} \subseteq W_0^{1,\theta_0}(\Omega)$ such that

$$\rho_{\theta_0}(Du_n) - \int_{\Omega} \beta(z) a(z) |u_n|^p dz < \frac{1}{n} \cdot \rho_{\theta_0}(Du_n) \text{ for all } n \in \mathbb{N}. \quad (2)$$

Let $y_n = \frac{u_n}{\|u_n\|_{1,\theta_0}}$. Then $\|y_n\|_{1,\theta_0} = 1$ for all $n \in \mathbb{N}$ and so we may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,\theta_0}(\Omega), \quad y_n \rightarrow y \text{ in } L^{\theta_0}(\Omega) \text{ as } n \rightarrow \infty \text{ (see Proposition 1)}. \quad (3)$$

From (2), we have

$$\rho_{\theta_0}(Dy_n) - \int_{\Omega} \beta(z) a(z) |y_n|^p dz < \frac{1}{n} \quad (\text{since } \rho_{\theta_0}(Du_n / \|u_n\|_{1,\theta_0}) = 1). \quad (4)$$

From (3), we have $\rho_{\theta_0}(y_n - y) \rightarrow 0$ and so

$$\int_{\Omega} \beta(z) a(z) |y_n|^p dz \rightarrow \int_{\Omega} \beta(z) a(z) |y|^p dz. \quad (5)$$

The functional $\rho_{\theta_0}(\cdot)$ is continuous, convex, thus sequentially weakly lower semicontinuous. Therefore, if in (4) we pass to the limit as $n \rightarrow \infty$ and we use (3) and (5), we obtain

$$\begin{aligned} \rho_{\theta_0}(Dy) &\leq \int_{\Omega} \beta(z) a(z) |y|^p dz \leq \hat{\lambda}_1^a \rho_{\theta_0}(y) \\ \Rightarrow y &= 0 \text{ or } y = \pm \hat{u}_1^a \text{ (see (1)).} \end{aligned} \quad (6)$$

If $y = 0$, then from (4) we have

$$\rho_{\theta_0}(Dy_n) \rightarrow 0 \text{ in } W_0^{1,\theta_0}(\Omega),$$

a contradiction since $\|y_n\|_{1,\theta_0} = 1$ for all $n \in \mathbb{N}$.

If $y = \pm \hat{u}_1^a$, then $0 < |y|$ and so from (6), we have

$$\rho_{\theta_0}(Dy) < \hat{\lambda}_1^a \rho_{\theta_0}(y),$$

contradicting (1). This proves the Proposition. $\square$

The hypotheses on the reaction $f(z, x)$ are the following: H_1 : $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function such that

- (i) for every $\rho > 0$, there exists $\hat{\xi}_\rho \in L^\infty(\Omega)$ such that $|f(z, x)| \leq \hat{\xi}_\rho(z)$ for a.a. $z \in \Omega$, all $|x| \leq \rho$;
- (ii) there exist $\beta \in L^\infty(\Omega)$ and $\beta_0 > 0$ such that

$$\beta(z) \leq \hat{\lambda}_1^a \text{ for a.a. } z \in \Omega, \beta \not\equiv \hat{\lambda}_1^a,$$

$$-\beta_0 \leq \liminf_{x \rightarrow \pm\infty} \frac{f(z, x)}{a(z)|x|^{p-2}x} \leq \limsup_{x \rightarrow \pm\infty} \frac{f(z, x)}{a(z)|x|^{p-2}x} \leq \beta(z) \text{ uniformly for a.a. } z \in \Omega;$$

(iii) there exist $\tau \in [1, q)$ and $\delta > 0$ such that

$$c_1 |x|^\tau \leq f(z, x) \leq c_2 |x|^\tau \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \leq \delta, \text{ some } 0 < c_1 \leq c_2.$$

3. Pair of nontrivial solutions

In this section, we show that for $\lambda > 0$ small, problem (P_λ) has at least two nontrivial, bounded solutions.

For $\lambda > 0$, let $\phi_\lambda : W_0^{1,\theta}(\Omega) \rightarrow \mathbb{R}$ be the energy functional for problem (P_λ) , defined by

$$\phi_\lambda(u) = \frac{1}{p} \rho_{\theta_0}(Du) - \frac{1}{q} \|Du\|_q^q - \frac{\lambda}{r} \|u\|_r^r - \int_\Omega F(z, u) dz \quad \text{for all } u \in W_0^{1,\theta}(\Omega).$$

Proposition 4. *If hypotheses H_0, H_1 hold and $\lambda > 0$, then $\phi_\lambda(\cdot)$ satisfies the C-condition.*

Proof. Let $g_\lambda(z, x) = \lambda |x|^{r-2} x + f(z, x)$, $F(z, x) = \int_0^x f(z, s) ds$ and $G_\lambda(z, x) = \frac{\lambda}{r} |x|^r + F(z, x)$. Hypothesis $H_1(ii)$ implies that

$$\limsup_{x \rightarrow \pm\infty} \frac{pF(z, x)}{a(z)|x|^p} \leq \beta(z) \quad \text{uniformly for a.a. } z \in \Omega.$$

So, we can find $M > 1$ large such that

$$F(z, x) \leq \frac{1}{p} [\beta(z) + 1] a(z) |x|^p \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \geq M,$$

$$(-\beta_0 - 1) a(z) |x|^p \leq f(z, x) \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \geq M.$$

Since $r > p$, by taking $M > 1$ even bigger if necessary, for $\ell \in (p, r)$ we have

$$\begin{aligned} \frac{\lambda \ell}{r} |x|^\ell + \ell F(z, x) &\leq \frac{\lambda \ell}{r} |x|^\ell + \frac{\ell}{p} [\beta(z) + 1] a(z) |x|^p \\ &\leq \lambda |x|^\ell + (-\beta_0 - 1) a(z) |x|^p \\ &\leq \lambda |x|^\ell + f(z, x) \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \geq M, \\ &\Rightarrow 0 < \ell G_\lambda(z, x) \leq g_\lambda(z, x) \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \geq M. \end{aligned}$$

So, $g_\lambda(z, x)$ satisfies the Ambrosetti–Rabinowitz condition and this implies that $\phi_\lambda(\cdot)$ satisfies the C-condition (see [8, p. 59]). $\square$

Proposition 5. *If hypotheses H_0, H_1 hold, then there exists $\lambda^* > 0$ such that for every $\lambda \in (0, \lambda^*)$ we can find $\rho_\lambda > 0$ for which we have*

$$\phi_\lambda(u) \geq m_\lambda > 0 \quad \text{for all } u \in W_0^{1,\theta}(\Omega), \|u\| = \rho_\lambda.$$

Proof. On account of hypothesis H_1 , given $\varepsilon > 0$, we can find $c_\varepsilon > 0$ such that

$$F(z, x) \leq \frac{1}{p} [\beta(z) + \varepsilon] a(z) |x|^p + c_\varepsilon |x|^\tau \quad \text{for a.a. } z \in \Omega, \text{ for all } x \in \mathbb{R}. \quad (7)$$

Let $u \in W_0^{1,\theta}(\Omega)$ with $\|u\| \leq 1$. Then

$$\begin{aligned} \phi_\lambda(u) &\geq \frac{1}{p} \left[\rho_{\theta_0}(Du) - \int_\Omega \beta(z) a(z) |u|^p dz - \varepsilon \rho_{\theta_0}(u) \right] + \frac{1}{q} \|Du\|_q^q \\ &\quad - c_3 [\lambda \|u\|^r + \|u\|^\tau] \quad \text{for some } c_3 = c_3(\varepsilon) > 0 \text{ (see (7))} \\ &\geq \frac{1}{p} \left[c^* - \frac{\varepsilon}{\hat{\lambda}_1^q} \right] \rho_{\theta_0}(Du) + \frac{1}{q} \|Du\|_q^q - c_3 [\lambda \|u\|^r + \|u\|^\tau] \quad \text{(see Proposition 3 and (1)).} \end{aligned}$$

Choosing $\varepsilon \in (0, \hat{\lambda}_1^q c^*)$, we obtain

$$\begin{aligned} \phi_\lambda(u) &\geq c_4 \rho_\theta(Du) - c_3 [\lambda \|u\|^r + \|u\|^\tau] \quad \text{for some } c_4 = c_4(\varepsilon) > 0 \\ &\geq c_4 \|u\|^p - c_3 [\lambda \|u\|^r + \|u\|^\tau] \quad \text{(since } \|u\| \leq 1, \text{ see Proposition 2)} \\ &= [c_4 - c_1 (\lambda \|u\|^{r-p} + \|u\|^{\tau-p})] \|u\|^p. \end{aligned} \quad (8)$$

Set $k_\lambda(t) = \lambda t^{r-p} + t^{\tau-p}$, $t > 0$. Since $1 \leq \tau < q < p < r$, we see that

$$k_\lambda(t) \rightarrow +\infty \quad \text{as } t \rightarrow 0^+ \text{ and as } t \rightarrow +\infty.$$

Therefore, we can find $t_0 > 0$ such that

$$k_\lambda(t_0) = \inf_{t>0} k_\lambda(t) \Rightarrow k'_\lambda(t_0) = 0.$$

It follows that

$$\lambda(r-p)t_0^{r-p-1} = (p-\tau)t_0^{\tau-p-1},$$

hence

$$t_0 = \left[\frac{p - \tau}{\lambda(r - p)} \right]^{1/(r-\tau)}.$$

We have

$$k_\lambda(t_0) = \lambda \left[\frac{p - \tau}{\lambda(r - p)} \right]^{(r-p)/(r-\tau)} + \left[\frac{\lambda(r - p)}{p - \tau} \right]^{(p-\tau)/(r-\tau)}.$$

Since $(r - p)/(r - \tau) < 1$, we see that $k_\lambda(t_0) \rightarrow 0^+$ as $\lambda \rightarrow 0^+$. So, we can find $\lambda^* > 0$ such that if $\lambda \in (0, \lambda^*)$ and $\rho_\lambda = t_0(\lambda)$, we have

$$0 < \rho_\lambda \leq 1 \text{ and } \phi_\lambda(u) \geq m_\lambda > 0 \text{ for all } u \in W_0^{1,\theta}(\Omega), \|u\| = \rho_\lambda \text{ (see (8)).}$$

The proof is now complete. $\square$

Now we are ready for the multiplicity result.

Theorem 1. *If hypotheses H_0 , H_1 hold, then for all $\lambda > 0$ small problem (P_λ) has at least two distinct solutions $u_0, \hat{u} \in W_0^{1,\theta}(\Omega) \cap L^\infty(\Omega)$.*

Proof. Let $\bar{B}_\lambda = \{u \in W_0^{1,\theta}(\Omega) : \|u\| \leq \rho_\lambda = t_0(\lambda)\}$. The reflexivity of $W_0^{1,\theta}(\Omega)$ and the Eberlein–Smulian theorem imply that $\bar{B}_\lambda$ is sequentially weakly compact. On the other hand, using Proposition 1, we can easily see that $\phi_\lambda(\cdot)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass–Tonelli theorem, we can find $u_0 \in W_0^{1,\theta}(\Omega)$ such that

$$\phi_\lambda(u_0) = \inf \{ \phi_\lambda(u) : u \in \bar{B}_\lambda \}. \quad (9)$$

Let $u \in W_0^{1,\theta}(\Omega) \cap L^\tau(\Omega)$. We can find $t \in (0, 1)$ small such that

$$tu(z) \in [0, \delta] \text{ for a.a. } z \in \Omega, \|tu\| \leq \rho_\lambda. \quad (10)$$

Then from (10) and hypothesis $H_1(iii)$, we have

$$\begin{aligned} \phi_\lambda(tu) &\leq \frac{t^q}{q} [\rho_{\theta_0}(Du) + \|Du\|_q^q] - t^\tau c_1 \|u\|_\tau^\tau \text{ (recall that } q < p \text{ and } t \in (0, 1)) \\ &= \frac{t^q}{q} \rho_\theta(Du) - t^\tau c_1 \|u\|_\tau^\tau. \end{aligned}$$

Since $\tau < q$, choosing $t \in (0, 1)$ even smaller if necessary, we have

$$\phi_\lambda(tu) < 0 \Rightarrow \phi_\lambda(u_0) < 0 = \phi_\lambda(0) \text{ (see (9)).}$$

From Proposition 5, we know that

$$0 < m_\lambda \leq \phi_\lambda(u) \text{ for all } u \in W_0^{1,\theta}(\Omega), \|u\| = \rho_\lambda, 0 < \lambda < \lambda^*.$$

Therefore for $\lambda \in (0, \lambda^*)$, we have

$$\begin{aligned} 0 < \|u_0\| < \rho_\lambda &\Rightarrow \langle \phi'_\lambda(u_0), h \rangle = 0 \\ &\Rightarrow \int_\Omega [a(z)|Du_0|^{p-2} + |Du_0|^{q-2}](Du_0, Dh)_{\mathbb{R}^N} dz \\ &= \int_\Omega [\lambda|u_0|^{r-2}u_0 + f(z, u_0)]h dz \text{ for all } h \in W_0^{1,\theta}(\Omega), \\ &\Rightarrow u_0 \text{ is a solution of } (P_\lambda) \text{ (} 0 < \lambda < \lambda^* \text{)}. \end{aligned}$$

Let $u \in C_0^1(\bar{\Omega})$ with $u > 0$ in Ω . Then since $r > p$, we have

$$\phi_\lambda(tu) \rightarrow -\infty \text{ as } t \rightarrow +\infty. \quad (11)$$

Then relation (11) and Propositions 4 and 5, permit the use of the mountain pass theorem. So, we can find $\hat{u} \in W_0^{1,\theta}(\Omega)$ such that

$$\begin{aligned} \phi'_\lambda(\hat{u}) &= 0, \phi_\lambda(u_0) < 0 = \phi_\lambda(0) < m_\lambda \leq \phi_\lambda(\hat{u}), \\ \Rightarrow \hat{u} &\text{ is a solution of } (P_\lambda) \text{ (} 0 < \lambda < \lambda^* \text{), } \hat{u} \neq 0, \hat{u} \neq u_0. \end{aligned}$$

Finally, we show the $L^\infty(\Omega)$ -regularity of the solutions of (P_λ) . So, let $u \in W_0^{1,\theta}(\Omega)$ be a nontrivial solution of (P_λ) ($0 < \lambda < \lambda^*$). For $k > 1$, we define

$$D_k = \{z \in \Omega : |u(z)| > k\}.$$

We know that

$$\int_\Omega [a(z)|Du|^{p-2} + |Du|^{q-2}](Du, Dh)_{\mathbb{R}^N} dz = \int_\Omega [\lambda|u|^{r-2}u + f(z, u)]h dz \text{ for all } h \in W_0^{1,\theta}(\Omega). \quad (12)$$

In (12) we use the test function $h = (|u| - k)^+ \in W_0^{1,\theta}(\Omega)$. Then

$$\begin{aligned} & \int_{\Omega} [a(z)|Du|^{p-2} + |Du|^{q-2}](Du, D(|u| - k)^+)_{\mathbb{R}^N} dz = \rho_{\theta}(D(|u| - k)^+) \\ & \leq \int_{\Omega} [\lambda|u|^{r-2}u(|u| - k)^+ + c_5((|u| - k)^+ + ((|u| - k)^+)^p)] dz \\ & \quad \text{for some } c_5 > 0 \text{ (see hypotheses } H_1(i), (ii)) \\ & \leq c_6|D_k|_N^{1/r'} \|(|u| - k)^+\| \quad \text{for some } c_6 > 0, \end{aligned}$$

using Hölder's inequality and choosing $k > 1$ large so that $\|(|u| - k)^+\| \leq 1$. It follows that

$$\|(|u| - k)^+\|^{p-1} \leq c_6|D_k|_N^{1/r'} \quad (\text{see Proposition 2}). \quad (13)$$

Now let $k < m$. Then

$$\begin{aligned} (m - k)^p |D_m|_N^{\frac{p}{r}} & \leq \left(\int_{D_m} (|u| - k)^r \right)^{\frac{p}{r}} \leq \left(\int_{D_k} (|u| - k)^r \right)^{\frac{p}{r}} \quad (\text{since } D_m \subseteq D_k) \\ & \leq c_7 \|(|u| - k)^+\|^p \quad \text{for some } c_7 > 0 \text{ (see Proposition 1)}, \\ & \Rightarrow |D_m|_N \leq \frac{c_8}{(m - k)^r} |D_k|_N^{\frac{r p'}{r p}} \quad \text{for some } c_8 > 0 \text{ (see (13))}. \end{aligned} \quad (14)$$

Note that $1 < \frac{r p'}{r' p} = \frac{r-1}{p-1}$. So, from (14) and Lemma B.1 in [9, p. 63], we infer that there exists $M > 1$ large such that

$$|D_M|_N = 0 \Rightarrow u \in L^{\infty}(\Omega).$$

We conclude that $u_0, \hat{u} \in W_0^{1,\theta}(\Omega) \cap L^{\infty}(\Omega)$. $\square$

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Data availability

No data was used for the research described in the article.

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