NORMALIZED SOLUTIONS FOR QUASILINEAR (p, q)-EQUATIONS

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ABSTRACT. In this paper, we study the following quasilinear (p, q)-equation

$$-\Delta_p u - u\Delta_q u^2 + \lambda |u|^{p-2}u = \mu |u|^{l-2}u + |u|^{m-2}u, \quad \text{in } \mathbb{R}^N,$$

with prescribed mass

$$\int_{\mathbb{R}^N} |u|^p = c^p,$$

where $c > 0$, $\mu \geq 0$, $2 \leq p < q < N$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$,

$$\Delta_q u^2 = 2^{q-1}(|u|^{q-2}u\operatorname{div}(|\nabla u|^{q-2}\nabla u) + (q-1)|u|^{q-3}u|\nabla u|^q),$$

λ is a Lagrange multiplier and $p < l < m < p^* := \frac{Np}{N-p}$. We first consider the case $\frac{pq}{N} + 2q < m < p^*$, $\mu = 0$ and we prove the existence of normalized solutions in the purely supercritical case by using the perturbation method. Then we obtain multiplicity of normalized solutions in the case

$$p < l < \frac{p^2}{N} + p, \quad \frac{pq}{N} + 2q < m < p^*, \quad \mu > 0,$$

namely when the two nonlinearities have a different character with respect to the L^p-critical exponent. This case presents substantial differences concerning the purely supercritical case.

KEYWORDS: Quasilinear (p, q)-equations; Normalized solutions; Perturbation method; Combined nonlinearities.
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1. INTRODUCTION

In this paper, we consider the following quasilinear (p, q)-equation with lack of compactness

$$-\Delta_p u - u\Delta_q u^2 + \lambda |u|^{p-2}u = \mu |u|^{l-2}u + |u|^{m-2}u, \quad \text{in } \mathbb{R}^N. \tag{1.1}$$

We are interested in the existence of solutions with prescribed mass

$$\int_{\mathbb{R}^N} |u|^p = c^p,$$

where $2 \leq p < q < N$. Here, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ and

$$\Delta_q u^2 = 2^{q-1}(|u|^{q-2}u\operatorname{div}(|\nabla u|^{q-2}\nabla u) + (q-1)|u|^{q-3}u|\nabla u|^q),$$

$\mu \geq 0$, $p < l < m < p^* := \frac{Np}{N-p}$, λ is a Lagrange multiplier, and $c > 0$ is a constant.

The features of problem (1.1) are the following:

(i) The presence of two differential operators with different growth, which generates a double phase associated energy.
(ii) The problem combines the effect generated by the combined nonlinearities.
(iii) The corresponding energy functional is a non-autonomous variational integral that satisfies nonstandard growth conditions of (p, q)-type, following the terminology introduced in the basic papers of Marcellini [27, 28, 29, 30].
(iv) Due to the unboundedness of the domain, the Palais-Smale sequences do not have the compactness property.

Since the content of the paper is closely concerned with unbalanced growth, we briefly introduce in what follows the related background and applications and we recall some pioneering contributions to these fields. Equation (1.1) is driven by a differential operator with unbalanced growth due to the presence of the (p,q)-Laplacian operator. This type of problem comes from a general reaction-diffusion system:

$$u_t = \operatorname{div}[A(\nabla u)\nabla u] + c(x,u), \quad \text{and} \quad A(\nabla u) = |\nabla u|^{p-2} + |\nabla u|^{q-2},$$

where the function u is a state variable and describes the density or concentration of multicomponent substances, $\operatorname{div}[A(\nabla u)\nabla u]$ corresponds to the diffusion with coefficient $A(\nabla u)$ and $c(x,u)$ is the reaction and relates to source and loss processes. Originally, the idea to treat such operators comes from Zhikov [39] who introduced such classes to provide models of strongly anisotropic materials, see also the monograph of Zhikov et al.[40]. We refer to the remarkable works initiated by Marcellini [27, 28], where the author investigated the regularity and existence of solutions of elliptic equations with unbalanced growth conditions. For further contributions to this field, we refer to Eleuteri, Marcellini and Mascolo [13, 14].

The (p,q)-equation is also motivated by numerous models arising in mathematical physics. For instance, we can refer to the following Born-Infeld equation [8] that appears in electromagnetism, electrostatics and electrodynamics as a model based on a modification of Maxwell's Lagrangian density:

$$-\operatorname{div}\left(\frac{\nabla u}{(1-2|\nabla u|^2)^{\frac{1}{2}}}\right) = h(u) \quad \text{in } \Omega.$$

Indeed, by the Taylor formula, we have

$$(1-x)^{-\frac{1}{2}} = 1 + \frac{x}{2} + \frac{3}{2\cdot 2^2}x^2 + \frac{5!!}{3!\cdot 2^3}x^3 + \cdots + \frac{(2n-3)!!}{(n-1)!\cdot 2^{n-1}}x^{n-1} + \cdots \quad \text{for } |x| < 1.$$

Here, $(2n-3)!! := 1\times 3\times 5\times\cdots\times(2n-3)$ and $(n-1)! := 1\times 2\times 3\times\cdots\times(n-1)$. Taking $x = 2|\nabla u|^2$ and adopting the first order approximation, we obtain the (p,q)-equation for $p=2$ and $q = 4$. Furthermore, the n-th order approximation problem is driven by the multi-phase differential operator

$$-\Delta u - \Delta_4 u - \frac{3}{2}\Delta_6 u - \cdots - \frac{(2n-3)!!}{(n-1)!}\Delta_{2n} u.$$

We also refer to the following fourth-order relativistic operator

$$u \mapsto \operatorname{div}\left(\frac{|\nabla u|^2}{(1-|\nabla u|^4)^{\frac{3}{4}}}\nabla u\right),$$

which describes large classes of phenomena arising in relativistic quantum mechanics. Again, by Taylor's formula, we have

$$x^2(1-x^4)^{-\frac{3}{4}} = x^2 + \frac{3x^6}{4} + \frac{21x^{10}}{32} + \cdots .$$

This shows that the fourth-order relativistic operator can be approximated by the following operator

$$u \mapsto \Delta_4 u + \frac{3}{4}\Delta_8 u.$$

For more details on the physical backgrounds and other applications, we refer to Bahrouni et al. [5] (for phenomena associated with transonic flows) and to Benci et al. [6] (for models arising in quantum physics).

In the past few decades, the (p,q)-equation has been the subject of extensive mathematical studies. Using various variational and topological arguments, many authors studied the existence and multiplicity results of nontrivial solutions, ground state solutions, nodal solutions and some qualitative properties of solutions, respectively. We refer to [15, 31, 35] for the case of bounded domains. In this classical setting, we recall the seminal papers by Ni et al. [32], Li et al. [21], del Pino et al. [11, 12] and Ambrosetti et al. [4]. The regularity results, existence and multiplicity of solutions to (p, q)-equation on the whole space can be found in [3, 17, 38].

In the present paper, motivated by the fact that physicists are often interested in normalized solutions, we look for solutions to (1.1) having a prescribed L^p-norm. More precisely, the existence of normalized solutions can be formulated as the following problem: provided $c > 0$, $\mu \geq 0$ and $p < l < m < p^*$, we aim to search for $(u, \lambda) \in H \times \mathbb{R}$ solving (1.1) together with the normalization condition

$$\int_{\mathbb{R}^N} |u|^p \, dx = c^p,$$

where

$$H := \left\{ u \in W^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^q |\nabla u|^q \, dx < +\infty \right\}.$$

Solutions can be obtained as critical points of the energy functional I_μ under the constraint

$$u \in \tilde{S}(c) := \left\{ u \in H : \int_{\mathbb{R}^N} |u|^p \, dx = c^p \right\},$$

where

$$I_\mu^0(u) := \frac{1}{p} \int_{\mathbb{R}^N} |\nabla u|^p \, dx + \frac{2^{q-1}}{q} \int_{\mathbb{R}^N} |u|^q |\nabla u|^q \, dx - \frac{\mu}{l} \int_{\mathbb{R}^N} |u|^l \, dx - \frac{1}{m} \int_{\mathbb{R}^N} |u|^m \, dx.$$

Compared with the classical (p, q)-equation, the search for solutions of (1.1) presents a major difficulty: the functional associated with the term $u\Delta_q u^2$

$$V(u) = \int_{\mathbb{R}^N} |u|^q |\nabla u|^q \, dx$$

is nondifferentiable in H. In order to overcome this difficulty, we take the perturbation method, which has been applied firstly to the unconstrained situation in [25, 26] and then to constrained situation in [20, 23]. That is, for $\alpha \in (0, 1]$, we denote

$$I_\mu^\alpha(u) := \frac{\alpha}{\theta} \int_{\mathbb{R}^N} |\nabla u|^\theta \, dx + I_\mu^0(u)$$

on the space $E := W^{1,p}(\mathbb{R}^N) \cap W^{1,\theta}(\mathbb{R}^N)$, where $\frac{2Nq}{N+q} < \theta < \min\left\{N, \frac{2Nq+2q}{N+q}\right\}$. Then we easily know that E is a reflexive Banach space and $I_\mu^\alpha \in C^1(E)$. We consider I_μ^α on the following constraint:

$$S(c) := \left\{ u \in E : \int_{\mathbb{R}^N} |u|^p \, dx = c^p \right\}.$$

Now we collect some works close to our equation (1.1). When $p = q = 2$, $\mu = 0$, equation (1.1) is reduced to the classical quasilinear Schrödinger equation

$$-\Delta u - u\Delta u^2 + \lambda u = |u|^{m-2}u \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

where $m > 2$ and $N > 1$. Colin, Jeanjean and Squassina [10] and Jeanjean and Luo [19] consider the minimization problem

$$\tilde{m}(c) := \inf_{\tilde{S}(c)} I_0^0(u)$$

with $2 < m \leq 4 + \frac{4}{N}$. By the Gagliardo–Nirenberg inequality, they obtained some properties about $\tilde{m}(c)$ by applying some minimization approaches. After that, Zeng and Zhang [37] considered the existence and asymptotic behavior of the minimizers to

$$\inf_{u \in \tilde{S}(c)} I_0^0(u) + \int_{\mathbb{R}^N} a(x) u^2 \, dx,$$

where $a(x)$ is an infinite potential well. The difference from the previous method is that Jeanjean, Luo and Wang [20] proved the existence of mountain-pass-type normalized solution of (1.2) by using the perturbation method. Then Li and Zou [23] first considered the existence of normalized solutions for equation (1.2) in case of $m > 4 + \frac{4}{N}$ through the similar method. Moreover, they applied the index theory to obtain the existence of infinitely many normalized solutions.

Inspired by the above literature, we first study the normalized solutions of the quasilinear (p,q)-equation (1.1). In addition to overcoming the lack of differentiability of the associated functional, we need to introduce the Pohozaev manifold method to study the geometry of the associated functional influenced by the quasilinear (p,q)-Laplacian operator and the combined nonlinearities. Thus we define the Pohozaev set

$$\mathcal{P}_\mu^\alpha(c) := \left\{ u \in S(c) : P_\mu^\alpha(u) = 0 \right\},$$

where

$$\begin{aligned} P_\mu^\alpha(u) :=& \frac{N\theta + p\theta - pN}{p\theta} \alpha \int_{\mathbb{R}^N} |\nabla u|^\theta \, dx + \int_{\mathbb{R}^N} |\nabla u|^p \, dx + \frac{2^{q-1}(2qN + pq - pN)}{pq} \int_{\mathbb{R}^N} |u|^q |\nabla u|^q \, dx \\ & - \frac{lN - pN}{pl} \mu \int_{\mathbb{R}^N} |u|^l \, dx - \frac{mN - pN}{pm} \int_{\mathbb{R}^N} |u|^m \, dx. \end{aligned}$$

It is well known that any critical point of $I_\mu^\alpha|_{S(c)}$ stays in $\mathcal{P}_\mu^\alpha$, as a consequence of the Pohozaev identity. We denote the L^p-norm preserved transform

$$u \in S(c) \mapsto s * u(x) = e^{\frac{N}{p}s} u(e^s x) \in S(c)$$

and it is natural to study the fiber maps

$$\begin{aligned} (\Psi_\mu^\alpha)_u(s) := I_\mu^\alpha(s * u) =& \frac{\alpha}{\theta} e^{\frac{N}{p}s\theta + \theta s - Ns} \int_{\mathbb{R}^N} |\nabla u|^\theta \, dx + \frac{1}{p} e^{ps} \int_{\mathbb{R}^N} |\nabla u|^p \, dx \\ & + \frac{2^{q-1}}{q} e^{\frac{N}{p}s \cdot 2q + qs - Ns} \int_{\mathbb{R}^N} |u|^q |\nabla u|^q \, dx - \frac{\mu}{l} \cdot e^{\frac{N}{p}s \cdot l - Ns} \int_{\mathbb{R}^N} |u|^l \, dx \\ & - \frac{1}{m} \cdot e^{\frac{N}{p}s \cdot m - Ns} \int_{\mathbb{R}^N} |u|^m \, dx. \end{aligned}$$

We shall find that critical points of $(\Psi_\mu^\alpha)_u$ allow to project a function on $\mathcal{P}_\mu^\alpha$ and the monotonicity and convexity properties of $(\Psi_\mu^\alpha)_u$ strongly affect the structure of $\mathcal{P}_\mu^\alpha$. In this direction, we consider the decomposition of $\mathcal{P}_\mu^\alpha$ into the disjoint union $\mathcal{P}_\mu^\alpha = (\mathcal{P}_\mu^\alpha)^+ \cup (\mathcal{P}_\mu^\alpha)^0 \cup (\mathcal{P}_\mu^\alpha)^-$, where

$$\begin{aligned} (\mathcal{P}_\mu^\alpha)^+ &:= \left\{ u \in \mathcal{P}_\mu^\alpha : (\Psi_\mu^\alpha)_u''(0) > 0 \right\} \\ (\mathcal{P}_\mu^\alpha)^0 &:= \left\{ u \in \mathcal{P}_\mu^\alpha : (\Psi_\mu^\alpha)_u''(0) = 0 \right\} \\ (\mathcal{P}_\mu^\alpha)^- &:= \left\{ u \in \mathcal{P}_\mu^\alpha : (\Psi_\mu^\alpha)_u''(0) < 0 \right\}. \end{aligned}$$

Throughout this paper, we introduce some relevant results about the Sobolev spaces. For $p \in (1, +\infty)$ and $N > p$, we define $D^{1,p}(\mathbb{R}^N)$ as the closure of $C_0^\infty(\mathbb{R}^N)$ with respect to $\|\nabla u\|_p := \left(\int_{\mathbb{R}^N} |\nabla u|^p \, dx \right)^{\frac{1}{p}}$. $\| \cdot \|_s$ denotes the usual norm of the space $L^s(\mathbb{R}^N)$, $1 \leq s \leq +\infty$. Let $W^{1,p}(\mathbb{R}^N)$ be the usual Sobolev space endowed with the standard norm $\|u\|_{W^{1,p}(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} |\nabla u|^p + |u|^p\, dx\right)^{\frac{1}{p}}$. $W_r^{1,p}(\mathbb{R}^N) := \left\{u \in W^{1,p}(\mathbb{R}^N) : u(x) = u(|x|)\right\}$. For equation (1.1), we introduce the working space E endowed with the norm

$$\|u\|_E := \|u\|_{W^{1,p}(\mathbb{R}^N)} + \|u\|_{W^{1,\theta}(\mathbb{R}^N)}$$

and

$$E_r := \{u \in E : u(x) = u(|x|)\}.$$

Next, we need to give the well-known Sobolev embedding theorem and Gagliardo–Nirenberg inequality.

Lemma 1.1. [1] *Let $N > p$. There exists a constant $S > 0$ such that, for any $u \in D^{1,p}(\mathbb{R}^N)$,*

$$\|u\|_{p^*}^p \leq S^{-1}\|\nabla u\|_p^p.$$

Moreover, $W^{1,p}(\mathbb{R}^N)$ is embedded continuously into $L^m(\mathbb{R}^N)$ for any $m \in [p, p^*]$ and compactly into $L^m_{loc}(\mathbb{R}^N)$ for any $m \in [1, p^*)$, where $p^* := \frac{Np}{N-p}$.

Lemma 1.2. [3] *The space E is embedded continuously into $L^m(\mathbb{R}^N)$ for $m \in [p, \theta^*]$ and compactly into $L^m_{loc}(\mathbb{R}^N)$ for $m \in [1, \theta^*)$.*

Lemma 1.3. [2, 33] *The following results hold:*

(i) *Let $m \in (p, p^*)$. There exists a sharp constant $C_{N,m} > 0$ such that*

$$\|u\|_m \leq C_{N,m}\|\nabla u\|_p^{\delta_m}\|u\|_p^{1-\delta_m}, \quad \forall u \in W^{1,p}(\mathbb{R}^N), \tag{1.3}$$

where $\delta_m := \frac{N}{p} - \frac{N}{m}$.

(ii) *Let $1 < q < N$ and $1 \leq p < m < q^*$. Then there exists a sharp constant $K_{N,m} > 0$ such that*

$$\|u\|_m \leq K_{N,m}\|\nabla u\|_q^{\gamma_m}\|u\|_p^{1-\gamma_m}, \quad \forall u \in \mathcal{W}^q \tag{1.4}$$

where $\gamma_m := \frac{Nq(m-p)}{m[Nq-p(N-q)]}$, $\mathcal{W}^q := \left\{u \in L^p(\mathbb{R}^N) : \nabla u \in L^q(\mathbb{R}^N)\right\}$.

Remark 1.4. *In particular, by* (1.4)*, we also have*

$$\int_{\mathbb{R}^N} |u|^{\frac{m}{2}}\, dx \leq K_{N,\frac{m}{2}}^{\frac{m}{2}} \left(\int_{\mathbb{R}^N} |\nabla u|^q\, dx\right)^{\frac{m\bar{\gamma}_m}{2q}} \cdot \left(\int_{\mathbb{R}^N} |u|^{\frac{p}{2}}\, dx\right)^{\frac{m(1-\bar{\gamma}_m)}{p}}, \quad \forall u \in \overline{\mathcal{W}}^q, \tag{1.5}$$

where $1 < q < N$, $2 \leq p < m < 2 \cdot q^*$, $\bar{\gamma}_m := \frac{Nq(m-p)}{m[Nq-\frac{p}{2}(N-q)]}$, $\overline{\mathcal{W}}^q := \left\{u \in L^{\frac{p}{2}}(\mathbb{R}^N) : \nabla u \in L^q(\mathbb{R}^N)\right\}$. Then by replacing u with u^2 in (1.5)*, we immediately obtains the following equality*

$$\int_{\mathbb{R}^N} |u|^m\, dx \leq \overline{K}_{N,m} \left(\int_{\mathbb{R}^N} |u|^q|\nabla u|^q\, dx\right)^{\frac{m\bar{\gamma}_m}{2q}} \cdot \left(\int_{\mathbb{R}^N} |u|^p\, dx\right)^{\frac{m(1-\bar{\gamma}_m)}{p}}, \quad \forall u \in \hat{\mathcal{W}}^q, \tag{1.6}$$

where $\overline{K}_{N,m} := K_{N,\frac{m}{2}}^{\frac{m}{2}} \cdot 2^{\frac{m\bar{\gamma}_m}{2}}$, $\hat{\mathcal{W}}^q := \left\{u \in L^p(\mathbb{R}^N) : u\nabla u \in L^q(\mathbb{R}^N)\right\}$ and $1 < q < N, 2 \leq p < m < 2 \cdot q^*$. Now combining the definition of $I^0_\mu(u)$, (1.3) *and* (1.6)*, we find that $m = \frac{pq}{N} + 2q$ is L^p-critical exponent.*

The main results read as follows.

Theorem 1.5. *Assume that $\frac{pq}{N} + 2q < m < p^*$ and $\mu = 0$. Then exists a radially symmetric solution $u \in W^{1,p}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \backslash \{0\}$ of* (1.1) *for some $\lambda > 0$.*

Remark 1.6. *In Theorem* 1.5*, due to the failure of the compact embedding $E \hookrightarrow L^\kappa(\mathbb{R}^N)$ for $p < \kappa < p^*$, we recover the compactness of bounded sequences by using the symmetric decreasing arrangement.*

Remark 1.7. *The idea is to look for critical points of I_μ^α for $\alpha > 0$ small by using minimax and deformation argument first. Then having these critical points for the perturbed functional I_μ^α, we study the convergence of these critical points as $\alpha > 0$ and obtain a certain convergence to critical points of the original functional I_μ^0.*

Theorem 1.8. *Assume that $p < l < \frac{p^2}{N} + p$, $\frac{pq}{N} + 2q < m < p^*$, $\mu > 0$,*

$$\frac{\mu}{l} C_{N,l}^l c^{l(1-\delta_l)+\frac{m(1-\delta_m)(p-l\delta_l)}{m\delta_m - p}} < \frac{m\delta_m - p}{p(m\delta_m - l\delta_l)} \cdot \left(\frac{(p-l\delta_l)m}{p(m\delta_m - l\delta_l)C_{N,m}^m} \right)^{\frac{p-l\delta_l}{m\delta_m - p}} \tag{1.7}$$

and

$$\left[\frac{\mu}{l} C_{N,l}^l \frac{\left(\frac{N}{p}\cdot l - N\right)\left(\frac{N}{p}m - \frac{N}{p}l\right)}{\left(\frac{N}{p}m - N - p\right)} \right]^{\frac{1}{p-l\delta_l}} \cdot c^{\frac{l(1-\delta_l)}{p-l\delta_l} - \frac{m\delta_m - m}{m\delta_m - p}} < \left[\frac{m}{C_{N,m}^m} \frac{\left(p + N - \frac{N}{p}l\right)}{\left(\frac{N}{p}m - N\right)\left(\frac{N}{p}m - \frac{N}{p}l\right)} \right]^{\frac{1}{m\delta_m - p}}. \tag{1.8}$$

Then the following results hold:

(i) *$I_\mu^0|_{\tilde{S}(c)}$ has a critical point u at negative level $\bar{m}^* < 0$.*
(ii) *$I_\mu^0|_{\tilde{S}(c)}$ has a second critical point v at positive level $\bar{\sigma}^* > \bar{m}^*$.*
(iii) *Both u and v are radially symmetric functions in $\mathbb{R}^N$ and solve (1.1) for suitable $\lambda_u, \lambda_v > 0$.*

Remark 1.9. *In Theorem 1.8, since we need to study the geometry of the perturbed functional and show that $(\mathcal{P}_\mu^\alpha) = \emptyset$, the mass c is limited.*

Remark 1.10. *Different from the results about the normalized solutions for the classical quasilinear Schrödinger equation, we first study the quasilinear (p,q)-equation with the combined nonlinearities. This case has different character with respect to the L^p-critical exponent and leads to the assocaited functional having a more complex geometric structure. Hence it is necessary for us to discuss the manifold in blocks and obtain solutions with different properties in different regions.*

Remark 1.11. *We observe that when $p = q = 2$ and $\mu = 0$, equation (1.1) is reduced to the classical quasilinear Schrödinger equation (1.2). In particular, we point out that the results in Theorems 1.5 and 1.8 still hold in case of $1 < p = q < N$. In addition, although we only study normalized solutions of equation (1.1) in the purely supercritical case and the mixed supercritical and subcritical case, the results about normalized solutions of equation (1.1) in the purely subcritical case can also be obtained by the method introduced in [10, 19], where the condition $p < m \leq p + \frac{p^2}{N}$ for equation (1.1) is much closer to the condition $2 < m \leq 4 + \frac{4}{N}$ for equation (1.2).*

Remark 1.12. *Note that the quasilinear operators can be degenerate when $p > 2$. Similar to the argument of Lemma 2.6 in [26], we can see that the solutions obtained in Theorems 1.5 and 1.8 are strictly positive. Moreover, the quasilinear expression $\Delta_q u^2$ can be replaced by $\Delta_q(|u|^r u)$ for some $r > 1$ as in the porous media equation. This also allows us to deal with the existence and multiplicity of normalized solutions with the help of the perturbation method in this paper. However, based on the minimax and deformation argument, we need to give the new working space and Gagliardo–Nirenberg inequality to analyze the geometry of the energy functional and construct the corresponding Pohozaev manifold to obtain the normalized solutions of the equation.*

The paper is organized as follows. In Section 2, we consider the perturbed functional and give some properties of the associated Pohozaev manifold in case of $\mu = 0$. In Section 4, we study the convergence of the critical points for the perturbed functional as $\alpha \to 0^+$ in case of $\mu = 0$

and prove Theorem 1.5. In Section 5, we discuss the compactness of Palais-Smale sequences and properties of the associated Pohozaev manifold in case of $\mu > 0$. In Section 6, we give some results about the convergence of the critical points for the perturbed functional as $\alpha \to 0^+$ in case of $\mu > 0$ and prove Theorem 1.8.

2. Perturbed functional in case of $\mu = 0$

In this section, we discuss the case of $\mu = 0$ by taking the perturbation method. First of all, we give some properties of $\mathcal{P}_0^\alpha(c)$.

Lemma 2.1. *Assume that $0 < \alpha \le 1$ and $m > \frac{pq}{N} + 2q$. Then $\mathcal{P}_0^\alpha(c)$ is a C^1-submanifold of codimension 1 in $S(c)$, and hence a C^1-submanifold of codimension 2 in E.*

Proof. It follows from the definition of $\mathcal{P}_0^\alpha(c)$ that the set $\mathcal{P}_0^\alpha(c)$ is defined by the two equations $F(u) = 0$ and $P_0^\alpha(u) = 0$, where

$$F(u) := c^p - \|u\|_p^p.$$

We observe that $F(u) \in C^1(E)$. Now we verify that $d(P_0^\alpha, F) : E \mapsto \mathbb{R}^2$ is surjective. Otherwise, $dP_0^\alpha(u)$ and $dF(u)$ are linearly dependent, i.e., there exists $\xi \in \mathbb{R}$ such that for any $\psi \in E$

$$\begin{aligned}
&\frac{\theta N + p\theta - pN}{p}\alpha \int_{\mathbb{R}^N} |\nabla u|^{\theta-2}\nabla u \nabla \psi \, dx + p \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \nabla \psi \, dx \\
&+ \frac{2^{q-1}(2qN + pq - pN)}{p} \int_{\mathbb{R}^N} |\nabla u|^q |u|^{q-2} u\psi + |u|^q |\nabla u|^{q-2}\nabla u \nabla \psi \, dx \\
&- \frac{mN - pN}{p} \int_{\mathbb{R}^N} |u|^{m-2} u\psi \, dx = p\xi \int_{\mathbb{R}^N} |u|^{p-2} u\psi \, dx.
\end{aligned}$$

In particular, taking $\psi = u$ and $\psi = x \cdot \nabla u$ respectively, we get

$$\begin{aligned}
&\frac{\theta N + p\theta - pN}{p}\alpha \|\nabla u\|_\theta^\theta + p\|\nabla u\|_p^p + \frac{2^q(2qN + pq - pN)}{p}\|u\nabla u\|_q^q \\
&= \frac{mN - pN}{p}\|u\|_m^m + p\xi\|u\|_p^p
\end{aligned} \tag{2.1}$$

and

$$\begin{aligned}
&\frac{\theta N + p\theta - pN}{p\theta} \cdot \alpha(N - \theta)\|\nabla u\|_\theta^\theta + (N - p)\|\nabla u\|_p^p \\
&+ \frac{2^q(2qN + pq - pN)}{pq} \cdot (N - q)\|u\nabla u\|_q^q = \frac{mN - pN}{pm} \cdot N\|u\|_m^m + N\xi\|u\|_p^p.
\end{aligned} \tag{2.2}$$

Combining (2.1) and (2.2), we deduce that

$$\begin{aligned}
&(\theta N + p\theta - pN)\alpha \left(\frac{N}{p} - \frac{N - \theta}{\theta}\right) \|\nabla u\|_\theta^\theta + p^2\|\nabla u\|_p^p \\
&+ 2^q(2qN + pq - pN)\left(\frac{N}{p} - \frac{N - q}{q}\right)\|u\nabla u\|_q^q = (mN - pN)\left(\frac{N}{p} - \frac{N}{m}\right)\|u\|_m^m,
\end{aligned}$$

which along with $P_0^\alpha(u) = 0$ yields that

$$\begin{aligned}
&(\theta N + p\theta - pN)\alpha \left(\frac{N}{p} - \frac{N - \theta}{\theta} - \frac{mN - pN}{p\theta}\right) \|\nabla u\|_\theta^\theta + (p^2 - pN + mN)\|\nabla u\|_p^p \\
&+ 2^{q-1}(2qN + pq - pN)\left[2\left(\frac{N}{p} - \frac{N - q}{q}\right) - \frac{mN - pN}{pq}\right]\|u\nabla u\|_q^q = 0.
\end{aligned} \tag{2.3}$$

We note that $\frac{N}{p} - \frac{N-\theta}{\theta} - \frac{mN-pN}{p\theta} > 0$ since $\frac{2Nq}{N+q} < \theta < \frac{2Nq+2q}{N+q}$. In addition, when $m > \frac{pq}{N} + 2q$, $p^2 - pN + mN > 0$ and $2\left(\frac{N}{p} - \frac{N-q}{q} - \frac{mN-pN}{pq}\right) > 0$. Then in view of (2.3), we get $u = 0$, which contradicts the fact that $u \in S(c)$. The proof of Lemma 2.1 is completed. □

Lemma 2.2. *Assume that $m > \frac{pq}{N} + 2q$. For any $0 < \alpha \leq 1$ and any $u \in E \backslash \{0\}$, the following results hold:*

(i) *There exists a unique number $s_\alpha(u) \in \mathbb{R}$ such that $P_\mu(s_\alpha(u) * u) = 0$. $I_0^\alpha(s * u)$ is strictly increasing in $s \in (-\infty, s_\alpha(u))$ and is strictly decreasing in $s \in (s_\alpha(u), +\infty)$ and*

$$\lim_{s\to-\infty} I_0^\alpha(s*u) = 0^+, \quad \lim_{s\to+\infty} I_0^\alpha(s*u) = -\infty, \quad I_0^\alpha(s_\alpha(u)*u) > 0.$$

Hence $s_\alpha(u) < 0$ if and only if $P_0^\alpha(u) < 0$.

(ii) *The map $u \in E\backslash\{0\} \mapsto s_\alpha(u) \in \mathbb{R}$ is of class C^1.*

(iii) *$s_\alpha(u)$ is an even function with respect to $u \in E\backslash\{0\}$.*

Proof. (i) It follows from the definition of $P_0^\alpha(u)$ and the direct calculation that

$$\begin{aligned}
P_0^\alpha(s*u) = \frac{d}{ds} I_0^\alpha(s*u) &= \frac{N\theta + p\theta - pN}{p\theta} \cdot \alpha e^{\frac{Ns}{p}\cdot\theta + \theta s - Ns} \|\nabla u\|_\theta^\theta + e^{ps}\|\nabla u\|_p^p \\
&\quad + \frac{2^{q-1}(2qN + pq - pN)}{pq} \cdot e^{\frac{N}{p}s\cdot 2q + qs - Ns}\|u\nabla u\|_q^q - \frac{mN-pN}{pm} \cdot e^{\frac{N}{p}s\cdot m - Ns}\|u\|_m^m \\
&= e^{\frac{N}{p}s\cdot m - Ns}\left[\frac{N\theta + p\theta - pN}{p\theta} \cdot \alpha e^{\frac{Ns}{p}\cdot\theta + \theta s - \frac{N}{p}s\cdot m}\|\nabla u\|_\theta^\theta + e^{ps - \frac{N}{p}s\cdot m + Ns}\|\nabla u\|_p^p\right. \\
&\quad \left. + \frac{2^{q-1}(2qN + pq - pN)}{pq} \cdot e^{\frac{N}{p}s\cdot 2q + qs - \frac{N}{p}s\cdot m}\|u\nabla u\|_q^q - \frac{mN-pN}{pm}\|u\|_m^m\right].
\end{aligned} \tag{2.4}$$

We observe that $\frac{N}{p}\theta + \theta - \frac{N}{p}m < 0$, $p - \frac{N}{p}m + N < 0$ and $\frac{2qN}{p} + q - \frac{N}{p}m < 0$ since $\frac{2Nq}{N+q} < \theta < \frac{2Nq+2q}{N+q}$ and $m > \frac{pq}{N} + 2q$. Then in virtue of (2.4), we obtain that $P_0^\alpha(s*u) = 0$ has only one solution $s_\alpha(u) \in \mathbb{R}$. Moreover, $P_0^\alpha(s*u) > 0$ when $s < s_\alpha(u)$ and $P_0^\alpha(s*u) < 0$ when $s > s_\alpha(u)$. This means that $I_0^\alpha(s*u)$ is strictly increasing in $s \in (-\infty, s_\alpha(u))$ and is strictly decreasing in $s \in (s_\alpha(u), +\infty)$. In addition, Note that

$$\lim_{s\to-\infty} I_0^\alpha(s*u) = 0^+, \quad \lim_{s\to+\infty} I_0^\alpha(s*u) = -\infty,$$

which yields that

$$I_0^\alpha(s_\alpha(u)*u) = \max_{s\in\mathbb{R}} I_0^\alpha(s*u) > 0.$$

Hence $s_\alpha(u) < 0$ if and only if $P_0^\alpha(u) < 0$.

(ii) Suppose that $\Psi(s,u) = P_0^\alpha(s*u)$. Then from (i), we have $\Psi(s_\alpha(u), u) = 0$. On the other hand,

$$\begin{aligned}
\frac{\partial}{\partial s}\Psi(s,u) &= \left(\frac{N\theta + p\theta - pN}{p}\right)^2 \cdot \frac{\alpha}{\theta} e^{\frac{Ns}{p}\theta + \theta s - Ns}\|\nabla u\|_\theta^\theta + pe^{ps}\|\nabla u\|_p^p \\
&\quad + \frac{2^{q-1}}{q}\left(\frac{2qN + pq - pN}{p}\right)^2 \cdot e^{\frac{N}{p}s\cdot 2q + qs - Ns}\|u\nabla u\|_q^q \\
&\quad - \left(\frac{mN-pN}{p}\right)^2 \cdot \frac{1}{m} \cdot e^{\frac{N}{p}s\cdot m - Ns}\|u\|_m^m.
\end{aligned}$$

Then combining the fact that $P_0^\alpha(s_\alpha(u) * u) = 0$, it holds that

$$
\begin{aligned}
\frac{\partial}{\partial s}\Psi(s,u) &= \frac{N\theta + p\theta - pN}{p}\left(\frac{N\theta + p\theta - pN}{p} - \frac{mN - pN}{p}\right)\cdot \frac{\alpha}{\theta} e^{\frac{Ns}{p}\theta + \theta s - Ns}\|\nabla u\|_\theta^\theta \\
&\quad + \frac{2^{q-1}}{q}\cdot \frac{2qN + pq - pN}{p}\cdot\left(\cdot\frac{2qN + pq - pN}{p} - \frac{mN - pN}{p}\right) e^{\frac{2qNs}{p} + qs - Ns}\|u\nabla u\|_q^q \\
&\quad + \left(p - \frac{mN - pN}{p}\right) e^{ps}\|\nabla u\|_p^p \\
&< 0.
\end{aligned}
$$

Hence by the implicit function theorem in [9], the map $u \mapsto s_\alpha(u)$ is of class C^1.

(iii) From

$$P_0^\alpha(s_\alpha(u) * (-u)) = P_0^\alpha(-s_\alpha(u) * u) = P_0^\alpha(s_\alpha(u) * u) = 0,$$

one has $s_\alpha(-u) = s_\alpha(u)$ by the uniqueness. □

In what follows, we consider a minimization problem

$$m_0^\alpha(c) := \inf_{u \in \mathcal{P}_0^\alpha(c)} I_0^\alpha(u).$$

We easily find any critical point u of $I_0^\alpha|_{S(c)}$ is contained in $\mathcal{P}_0^\alpha(c)$ and if $m_0^\alpha(c)$ is achieved, then the minimizer is a ground state critical point of $I_0^\alpha|_{S(c)}$.

For any $u \in \mathcal{P}_0^\alpha(c)$ and $\alpha \in (0,1]$, we have

$$
\begin{aligned}
I_0^\alpha(u) &= I_0^\alpha(u) - \frac{p}{mN - pN} P_0^\alpha(u) \\
&= \left(\frac{\alpha}{\theta} - \frac{p}{mN - pN}\cdot\frac{N\theta + p\theta - pN}{p\theta}\alpha\right)\|\nabla u\|_\theta^\theta + \left(\frac{1}{p} - \frac{p}{mN - pN}\right)\|\nabla u\|_p^p \\
&\quad + \left(\frac{2^{q-1}}{q} - \frac{p}{mN - pN}\cdot\frac{2^{q-1}(2qN + pq - pN)}{pq}\right)\|u\nabla u\|_q^q \\
&\geq \left(\frac{2^{q-1}}{q} - \frac{p}{mN - pN}\cdot\frac{2^{q-1}(2qN + pq - pN)}{pq}\right)\|u\nabla u\|_q^q \\
&> 0.
\end{aligned}
$$

This means that $m_0^\alpha(c) \geq A(c) > 0$ for all $\alpha \in (0,1]$, where

$$A(c) := \left(\frac{2^{q-1}}{q} - \frac{p}{mN - pN}\cdot\frac{2^{q-1}(2qN + pq - pN)}{pq}\right)\inf_{0<\alpha\leq 1, u \in \mathcal{P}_0^\alpha(c)}\|u\nabla u\|_q^q.$$

Lemma 2.3. *There exists a small $r > 0$ independent of α such that for any $0 < \alpha \leq 1$,*

$$0 < \sup_{u \in B(r,c)} I_0^\alpha(u) < A(c) \quad \textit{and} \quad I_0^\alpha(u), P_0^\alpha(u) > 0 \quad \textit{for all } u \in B(r,c),$$

where

$$B(r,c) = \left\{u \in S(c) : \alpha\|\nabla u\|_\theta^\theta + \|\nabla u\|_p^p + \|u\nabla u\|_q^q \leq r\right\}.$$

Proof. By the definition of I_0^α, for $r > 0$ small enough and independent of α, it holds that

$$\sup_{u \in B(r,c)} I_0^\alpha(u) \leq \max\left\{\frac{1}{\theta}, \frac{1}{p}, \frac{2^{q-1}}{q}\right\} r < A(c).$$

In addition, by (1.6), for any $u \in \partial B(\rho, c)$ with $0 < \rho < r$ for a smaller $r > 0$, we get

$$\begin{aligned}\inf_{\partial B(\rho,c)} I_0^\alpha(u) &\geq \frac{\alpha}{\theta}\|\nabla u\|_\theta^\theta + \frac{1}{p}\|\nabla u\|_p^p + \frac{2^{q-1}}{q}\|u\nabla u\|_q^q - \frac{1}{m}\cdot \overline{K}_{N,m} c^{m(1-\bar{\gamma}_m)}\|u\nabla u\|_q^{\frac{m\bar{\gamma}_m}{2}} \\ &\geq C_1\rho > 0\end{aligned}$$

and

$$\begin{aligned}\inf_{\partial B(\rho,c)} P_0^\alpha(u) \geq\ & \frac{N\theta + p\theta - pN}{p\theta}\alpha\|\nabla u\|_\theta^\theta + \|\nabla u\|_p^p + \frac{2^{q-1}(2qN + pq - pN)}{pq}\|u\nabla u\|_q^q \\ & - \frac{mN - pN}{pm}\cdot \overline{K}_{N,m} c^{m(1-\bar{\gamma}_m)}\|u\nabla u\|_q^{\frac{m\bar{\gamma}_m}{2}} \\ \geq\ & C_2\rho > 0,\end{aligned}$$

where C_1, C_2 are positive constants independent of $\rho > 0$. To sum up, the proof of Lemma 2.3 is completed. □

Inspired by [18], in order to find a Palais-Smale sequence, we consider an auxiliary functional

$$\mathcal{I}_0^\alpha(s, u) := I_0^\alpha(s * u) : \mathbb{R} \times E \mapsto \mathbb{R}$$

and study $\mathcal{I}_0^\alpha$ on the radial space $\mathbb{R} \times S_r(c)$ with

$$S_r(c) := S(c) \cap E_r, \quad E_r = W_r^{1,\theta}(\mathbb{R}^N) \cap W_r^{1,2}(\mathbb{R}^N).$$

Obviously, $\mathcal{I}_0^\alpha$ is of class C^1. Moreover, it follows from the symmetric critical point principle in [34] that a Palais-Smale sequence for $\mathcal{I}_0^\alpha|_{\mathbb{R}\times S_r(c)}$ is a Palais-Smale sequence for $\mathcal{I}_0^\alpha|_{\mathbb{R}\times S(c)}$. Then we define the closed sublevel set by

$$(\mathbf{I}_0^\alpha)^\iota := \{u \in S(c) : I_0^\alpha(u) \leq \iota\}$$

and the minimax level

$$\sigma_0^\alpha(c) := \inf_{\gamma\in\Gamma_\alpha}\sup_{t\in[0,1]} \mathcal{I}_0^\alpha(\gamma(t)),$$

where the minimax class

$$\Gamma_\alpha := \left\{\gamma = (\beta_1, \beta_2) \in C([0,1], \mathbb{R} \times S_r(c)) : \gamma(0) \in \{0\} \times B(r, c), \gamma(1) \in \{0\} \times (\mathbf{I}_0^\alpha)^0\right\}.$$

Lemma 2.4. *For any $0 < \alpha \leq 1$, $m_0^\alpha(c) = \sigma_0^\alpha(c)$ and $\sigma_0^\alpha(c)$ is nondecreasing with respect to $\alpha \in (0, 1]$.*

Proof. On the one hand, for any $\gamma = (\beta_1, \beta_2) \in \Gamma_\alpha$, let us consider the function

$$g_\gamma(t) := P_0^\alpha(\beta_1(t) * \beta_2(t)).$$

From Lemma 2.3, we infer that $g_\gamma(0) = P_0^\alpha(\beta_2(0)) > 0$. Now we show that $g_\gamma(1) = P_0^\alpha(\beta_2(1)) < 0$. In fact, by $I_0^\alpha(\beta_2(1)) \leq 0$ and Lemma 2.2, we find that $s_\alpha(\beta_2(1)) < 0$, which yields that $P_0^\alpha(\beta_2(1)) < 0$. Then combining the continuous property of g_γ, we deduce that there exists $t_\gamma \in (0, 1)$ such that $g_\gamma(t_\gamma) = 0$, that is $\beta_1(t_\gamma) * \beta_2(t_\gamma) \in \mathcal{P}_0^\alpha(c)$. Thus

$$\max_{t\in[0,1]} \mathcal{I}_0^\alpha(\gamma(t)) \geq I_0^\alpha(\beta_1(t_\gamma) * \beta_2(t_\gamma)) \geq m_0^\alpha(c).$$

This means that $\sigma_0^\alpha(c) \geq m_0^\alpha(c)$.

On the other hand, if $u \in \mathcal{P}_0^\alpha(c) \cap E_r$, then

$$\gamma_u(t) := (0, ((1-t)s_1 + ts_2) * u) \in \gamma_\alpha,$$

where $s_1 \ll -1$ and $s_2 \gg 1$. Hence

$$I_0^\alpha(u) \geq \max_{t\in[0,1]} I_0^\alpha(((1-t)s_1 + ts_2) * u) \geq \sigma_0^\alpha(c),$$

which implies that

$$(m_0^\alpha)^r(c) := \inf_{u\in\mathcal{P}_0^\alpha(c)\cap E_r} I_0^\alpha(u) \geq \sigma_0^\alpha(c). \tag{2.5}$$

In addition, using the symmetric decreasing rearrangement in [24], we easily obtain $m_0^\alpha(c) \geq (m_0^\alpha)^r(c)$, which along with (2.5) yields that $m_0^\alpha(c) \geq (m_0^\alpha)^r(c)$. To sum up, $m_0^\alpha = \sigma_0^\alpha(c)$.

In the last, for any $0 < \alpha_1 < \alpha_2 \leq 1$, by $I_0^{\alpha_2}(u) \geq I_0^{\alpha_1}(u)$ and $\Gamma_{\alpha_2} \subset \Gamma_{\alpha_1}$, it holds that

$$\sigma_0^{\alpha_2}(c) = \inf_{\gamma\in\Gamma_{\alpha_2}} \sup_{t\in[0,1]} \mathcal{I}_0^{\alpha_2}(\gamma(t)) \geq \inf_{\gamma\in\Gamma_{\alpha_2}} \sup_{t\in[0,1]} \mathcal{I}_0^{\alpha_1}(\gamma(t)) \geq \inf_{\gamma\in\Gamma_{\alpha_1}} \sup_{t\in[0,1]} \mathcal{I}_0^{\alpha_1}(\gamma(t)) = \sigma_0^{\alpha_1}(c).$$

Thus $\sigma_0^\alpha(c)$ is nondecreasing with respect to $\alpha \in (0,1]$. □

In order to construct a Palais-Smale sequence of $\sigma_0^\alpha(c)$, we give the following well-known results.

Definition 2.1 (Definition 3.1,[16]). *Let $\mathcal{B}$ be a closed subset of $\mathcal{X}$. We say a class $\mathcal{F}$ of compact subsets of $\mathcal{X}$ is a homotopy stable family with boundary $\mathcal{B}$ provided:*

(i) *Every set in $\mathcal{F}$ contains $\mathcal{B}$.*
(ii) *For any set $\mathcal{A}$ in $\mathcal{F}$ and any $\eta \in C([0,1]\times\mathcal{X},\mathcal{X})$ satisfying $\eta(t,x) = x$ for all (t,x) in $(\{0\}\times\mathcal{X})\cup([0,1]\times\mathcal{B})$, we have $\eta(1,\mathcal{A}) \subset \mathcal{F}$.*

Theorem 2.5 (Theorem 5.2, [16]). *Let ϕ be a C^1-functional on a completely connected C^1-Finsler manifold $\mathcal{X}$ and consider a homotopy stable family $\mathcal{F}$ with an extended closed boundary $\mathcal{B}$. Set $\vartheta = \vartheta(\phi,\mathcal{F})$ and let $\overline{\mathcal{F}}$ be a closed subset of $\mathcal{X}$ satisfying*

$$\mathcal{A}\cap\overline{\mathcal{F}}\backslash\mathcal{B} \neq \emptyset \quad \textit{for all } \mathcal{A}\in\mathcal{F} \tag{2.6}$$

and

$$\sup\phi(\mathcal{B}) \leq \vartheta \leq \inf\phi(\overline{\mathcal{F}}). \tag{2.7}$$

Then for any sequence of sets $\mathcal{A}_n \subset \mathcal{F}$ such that $\lim_{n\to+\infty}\sup_{\mathcal{A}_n}\phi = \vartheta$, there exists a sequence $x_n \subset \mathcal{X}\backslash\mathcal{B}$ such that

(i) $\lim_{n\to+\infty}\phi(x_n) = \vartheta$.
(ii) $\lim_{n\to+\infty}\|d\phi(x_n)\| = 0$.
(iii) $\lim_{n\to+\infty} dist(x_n,\overline{F}) = 0$.
(iv) $\lim_{n\to+\infty}(x_n,\mathcal{A}_n) = 0$.

Lemma 2.6. *For any fixed $\alpha \in (0,1]$, there exists a sequence $u_n \in S_r(c)$ such that*

$$I_0^\alpha(u_n) \to \sigma_0^\alpha(c),\ (I_0^\alpha)'|_{S(c)}(u_n) \to 0,\ P_0^\alpha(u_n) \to 0 \quad \textit{and} \quad u_n^- \to 0 \textit{ a.e. in } \mathbb{R}^N.$$

Proof. It follows from Definition 2.1 that $\mathcal{F} = \{\mathcal{A} = \gamma([0,1]) : \gamma \in \Gamma_\alpha\}$ is a homotopy stable family of compact subsets of $\mathcal{X} = \mathbb{R}\times S_r(c)$ with boundary $\mathcal{B} = (\{0\}\times B(r,c))\cup(\{0\}\times\times(\mathbf{I}_0^\alpha)^0)$. Applying Theorem 2.5, we set $\overline{\mathcal{F}} = \{\mathcal{I}_0^\alpha \geq \sigma_0^\alpha(c)\}$ and (2.6), (2.7) with $\phi = \mathcal{I}_0^\alpha$, $\vartheta = \sigma_0^\alpha(c)$ are satisfied. Hence, taking a minimizing sequence $\{\gamma_n = (0,(\beta_2)_n)\} \subset \Gamma_\alpha$ with $(\beta_2)_n \geq 0$ a.e. in $\mathbb{R}^N$, there exists a Palais-Smale sequence $\{(s_n,w_n)\} \subset \mathbb{R}\times S_r(c)$ for $\mathcal{I}_0^\alpha|_{\mathbb{R}\times S_r(c)}$ at level $\sigma_0^\alpha(c)$, namely,

$$\partial_s\mathcal{I}_0^\alpha(s_n,w_n) \to 0 \quad \text{and} \quad \partial_u\mathcal{I}_0^\alpha(s_n,w_n) \to 0 \quad \text{as } n\to+\infty \tag{2.8}$$

with

$$|s_n| + \text{dist}_E(w_n,(\beta_2)_n([0,1])) \to 0 \quad \text{as } n\to+\infty. \tag{2.9}$$

Denote $u_n = s_n * w_n$. Then the first result in (2.8) implies that $P_0^\alpha(u_n) \to 0$ and the second result in (2.8) gives when $n \to +\infty$,

$$
\begin{aligned}
\|dI_0^\alpha|_{S(c)}(u_n)\| &= \sup_{\psi \in T_{u_n}S(c), \|\psi\|_E \le 1} |dI_0^\alpha(u_n)[\psi]| \\
&= \sup_{\psi \in T_{u_n}S(c), \|\psi\|_E \le 1} |dI_0^\alpha(s_n * w_n)[s_n * (-s_n)\psi]| \\
&= \sup_{\psi \in T_{u_n}S(c), \|\psi\|_E \le 1} |\partial_u \mathcal{I}_0^\alpha(s_n, w_n)[(-s_n) * \psi]| \\
&\le \|\partial_u \mathcal{I}_0^\alpha(s_n, w_n)\| \sup_{\psi \in T_{u_n}S(c), \|\psi\|_E \le 1} |(-s_n) * \psi| \\
&\le C\|\partial_u \mathcal{I}_0^\alpha(s_n, w_n)\| \to 0.
\end{aligned}
$$

In the last, (2.9) reads $u_n^- \to 0$ a.e. in $\mathbb{R}^N$. Therefore, the proof of Lemma 2.6 is completed. □

In the following, we shall show the compactness of the Palais-Smale sequence obtained in Lemma 2.6.

Lemma 2.7. *For any fixed $\alpha \in (0, 1]$, let $\{u_n\}$ be a sequence obtained in Lemma 2.6. Then there exists a $u_\alpha \in E \backslash \{0\}$ and a Lagrange multiplier $\lambda_\alpha \in \mathbb{R}$ such that up to a subsequence,*

$$
u_n \rightharpoonup u_\alpha \quad \text{in } E,
$$

$$
I_0^\alpha(u_\alpha) = \sigma_0^\alpha(c) \quad \text{and} \quad (I_0^\alpha)'(u_\alpha) + \lambda_\alpha |u_\alpha|^{p-2} u_\alpha = 0.
$$

Furthermore, if $\lambda_\alpha \neq 0$, then

$$
u_n \to u_\alpha \quad \text{in } E.
$$

Proof. Since $\{u_n\}$ is a sequence obtained in Lemma 2.6, combining Lemma 2.4 and the fact that $I_0^\alpha(u_n) \to \sigma_0^\alpha(c)$, $P_0^\alpha(u_n) \to 0$ as $n \to +\infty$, we easily get $\{u_n\}$ is bounded in E_r. Then using the Sobolev embedding theorem and interpolation, it holds that up to a subsequence, there exists a $u_\alpha \in E_r$ such that

$$
\begin{aligned}
&u_n \rightharpoonup u_\alpha \quad \text{in } E, \\
&u_n \to u_\alpha \quad \text{in } L^\kappa(\mathbb{R}^N) \text{ for all } \kappa \in (p, 2 \cdot q^*), \\
&u_n \to u_\alpha \ge 0 \quad \text{a.e. in } \mathbb{R}.
\end{aligned}
$$

Now we claim that $u_\alpha \neq 0$. Otherwise, when $n \to +\infty$, we obtain that

$$
\begin{aligned}
&\frac{N\theta + p\theta - pN}{p\theta} \alpha \|\nabla u_n\|_\theta^\theta + \|\nabla u_n\|_p^p + \frac{2^{q-1}(2qN + pq - pN)}{pq} \|u_n \nabla u_n\|_q^q \\
&= \frac{mN - pN}{pm} \|u_n\|_m^m + P_0^\alpha(u_n) + o_n(1) \to 0.
\end{aligned}
$$

This means that $I_0^\alpha(u_n) \to 0$ as $n \to +\infty$, which contradicts the fact $m_0^\alpha(c) = \sigma_0^\alpha(c) > 0$. Hence the claim is true. Next by Lemma 3 in [7] and $(I_0^\alpha)'|_{S(c)}(u_n) \to 0$, it holds that there exists a sequence $\{\lambda_n\} \in \mathbb{R}$ such that

$$
(I_0^\alpha)'(u_n) + \lambda_n |u_n|^{p-2} u_n \to 0 \quad \text{in } E^*. \tag{2.10}
$$

This implies that $\lambda_n = \frac{1}{c^p} \langle (I_0^\alpha)'(u_n), u_n \rangle + o_n(1)$ is bounded in $\mathbb{R}$. Then we assume that up to a subsequence, there exists $\lambda_\alpha \in \mathbb{R}$ such that $\lambda_n \to \lambda_\alpha$. Moreover, $(I_0^\alpha)'(u_\alpha) + \lambda_\alpha |u_\alpha|^{p-2} u_\alpha = 0$ and $P_0^\alpha(u_\alpha) = 0$. In addition, using the weak lower semicontinuous property, which is similar to Lemma 4.3 in [10], we have

$$
\begin{aligned}
\alpha \|\nabla u_n\|_\theta^\theta &\to \alpha \|\nabla u_\alpha\|_\theta^\theta, \\
\|\nabla u_n\|_p^p &\to \|\nabla u_\alpha\|_p^p
\end{aligned}
$$

and
$$\|u_n\nabla u_n\|_q^q \to \|u_\alpha \nabla u_\alpha\|_q^q.$$
That is, $I_0^\alpha(u_\alpha) = \lim_{n\to+\infty} I_0^\alpha(u_n) = \sigma_0^\alpha(c)$ and $\langle (I_0^\alpha)'(u_n), u_n\rangle \to \langle (I_0^\alpha)'(u_\alpha), u_\alpha\rangle$. This also means that $\lambda_\alpha\|u_n\|_p^p \to \lambda_\alpha\|u_\alpha\|_p^p$. So if $\lambda_\alpha \neq 0$, then $u_n \to u_\alpha$ in E. $\square$

Based on the above lemmas, we easily conclude the following theorem.

Theorem 2.8. *For any fixed $\alpha \in (0,1]$, there exists a $u_\alpha \in E_r\backslash\{0\}$ and a Lagrange multiplier $\lambda_\alpha \in \mathbb{R}$ such that*
$$(I_0^\alpha)'(u_\alpha) + \lambda_\alpha |u_\alpha|^{p-2}u_\alpha = 0, \quad I_0^\alpha(u_\alpha) = m_0^\alpha(c),$$
$$P_0^\alpha(u_\alpha) = 0, \quad 0 < \|u_\alpha\|_p^p \le c^p, \quad u_\alpha \ge 0.$$
Furthermore, if $\lambda_\alpha \neq 0$, then $\|u_\alpha\|_p^p = c^p$, i.e., $m_0^\alpha(c)$ is achieved and u_α is a ground state critical point of $I_0^\alpha|_{S(c)}$.

3. Convergence issues as $\alpha \to 0^+$ in case of $\mu = 0$

In this section, letting $\alpha \to 0^+$, we shall prove that the sequences of critical points of $I_0^\alpha|_{S(c)}$ obtained in Section 2 converge to critical points of $I_0^0|_{\tilde{S}(c)}$.

Lemma 3.1. *Assume that $\alpha_n \to 0^+$, $(I_0^{\alpha_n})'(u_{\alpha_n}) + \lambda_{\alpha_n}|u_{\alpha_n}|^{p-2}u_{\alpha_n} = 0$ with $\lambda_{\alpha_n} \ge 0$ and $I_0^{\alpha_n}(u_{\alpha_n}) \to \varrho \in (0,+\infty)$ for $u_{\alpha_n} \in S_r(c_n)$ with $0 < c_n \le c$. Then there exists a subsequence $u_{\alpha_n} \rightharpoonup u$ in $W^{1,p}(\mathbb{R}^N)$ with $u \neq 0$, $u \in W_r^{1,p}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ such that*
$$(I_0^0)'(u) + \lambda|u|^{p-2}u = 0, \quad I_0^0(u) = \varrho, \quad 0 < \|u\|_p^p \le c^p$$
and

(i) *If $u_{\alpha_n} \ge 0$ for every $n \in \mathbb{N}^+$, then $u \ge 0$.*
(ii) *If $\lambda \neq 0$, then $\lim_{n\to+\infty} c_n = \|u\|_p$.*

Proof. The idea of proof for Lemma 3.1 is inspired by [20, 23]. In the first, using the fact that $(I_0^{\alpha_n})'(u_{\alpha_n}) + \lambda_{\alpha_n}|u_{\alpha_n}|^{p-2}u_{\alpha_n} = 0$, we easily see that $P_0^{\alpha_n}(u_{\alpha_n}) = 0$ for every $n \in \mathbb{N}^+$. Moreover, it holds that
$$\sup_{n\ge1} \max\left\{\alpha_n\|\nabla u_{\alpha_n}\|_\theta^\theta,\ \|\nabla u_{\alpha_n}\|_p^p,\ \|u_{\alpha_n}\nabla u_{\alpha_n}\|_q^q\right\} < +\infty.$$
Therefore $\{u_{\alpha_n}\}$ is bounded in $W^{1,p}(\mathbb{R}^N)$. Now we claim that $\liminf_{n\to+\infty} c_n > 0$. Indeed, if $c_n \to 0$, then using (1.6), we have $\|u_n\|_m \to 0$, which along with $P_0^{\alpha_n}(u_{\alpha_n}) = 0$ yields that $I_0^{\alpha_n}(u_{\alpha_n}) \to 0$ which contradicts $\varrho > 0$. Hence the claim is true and $\lambda_{\alpha_n} = \frac{1}{c_n^p}(I_0^{\alpha_n})'(u_{\alpha_n})(u_{\alpha_n})$ is bounded in $\mathbb{R}$. Thus up to a subsequence, $\lambda_{\alpha_n} \to \lambda$ in $\mathbb{R}$, $u_{\alpha_n} \rightharpoonup u$ in $W^{1,p}(\mathbb{R}^N)$, $u_{\alpha_n} \to u$ in $L^\kappa(\mathbb{R}^N)$ for $\kappa \in (p, 2\cdot q^*)$ and $u_{\alpha_n} \to u$ a.e. on $\mathbb{R}^N$. In addition, if $u_{\alpha_n} \ge 0$ for every $n \in \mathbb{N}^+$, then $u \ge 0$. Furthermore, similar to the arguments in Lemma A.2 in [23], we also have $u_n\nabla u_n \to u\nabla u$ in $L^q_{\mathrm{loc}}(\mathbb{R}^N)$ and $\nabla u_{\alpha_n} \to \nabla u$ a.e. on $\mathbb{R}^N$. In what follows, we divide the remaining proof into three steps.

Step 1. We show that there exists a positive constant C such that $\|u_{\alpha_n}\|_\infty \le C$ and $\|u\|_\infty \le C$. Denote $R > 2$, $\nu > 0$ and
$$v_n = \begin{cases} R, & u_{\alpha_n} \ge R, \\ u_{\alpha_n}, & |u_{\alpha_n}| \le R, \\ -R, & u_{\alpha_n} \le -R. \end{cases}$$

Let $\psi = u_{\alpha_n}|v_{\alpha_n}|^{q\nu}$. Then $\psi \in E$. Then it follows from $(I_0^{\alpha_n})'(u_{\alpha_n}) + \lambda_{\alpha_n}|u_{\alpha_n}|^{p-2}u_{\alpha_n} = 0$ and $\lambda_{\alpha_n} \geq 0$ that

$$\begin{aligned}
&\int_{\mathbb{R}^N} |u_{\alpha_n}|^{m-2}u_{\alpha_n}\psi\, dx \\
&= \alpha_n \int_{\mathbb{R}^N} |\nabla u_{\alpha_n}|^{\theta-2}\nabla u_{\alpha_n} \cdot \nabla\psi\, dx + \int_{\mathbb{R}^N} |\nabla u_{\alpha_n}|^{p-2}\nabla u_{\alpha_n} \cdot \nabla\psi\, dx \\
&\quad + 2^{q-1} \int_{\mathbb{R}^N} |\nabla u_{\alpha_n}|^q |u_{\alpha_n}|^{q-2}u_{\alpha_n}\psi + |u_{\alpha_n}|^q |\nabla u_{\alpha_n}|^{q-2}\nabla u_{\alpha_n} \cdot \nabla\psi\, dx \\
&\quad + \lambda_{\alpha_n} \int_{\mathbb{R}^N} |u_{\alpha_n}|^{p-2}u_{\alpha_n}\psi\, dx \\
&\geq \alpha_n \int_{\mathbb{R}^N} |\nabla u_{\alpha_n}|^{\theta-2}\nabla u_{\alpha_n} \cdot \nabla\psi\, dx + \int_{\mathbb{R}^N} |\nabla u_{\alpha_n}|^{p-2}\nabla u_{\alpha_n} \cdot \nabla\psi\, dx \\
&\quad + 2^{q-1} \int_{\mathbb{R}^N} |u_{\alpha_n}|^q |\nabla u_{\alpha_n}|^{q-2}\nabla u_{\alpha_n} \cdot \nabla\psi\, dx \\
&= \alpha_n \int_{\mathbb{R}^N} |\nabla u_{\alpha_n}|^{\theta}|v_{\alpha_n}|^{q\nu}\, dx + \alpha_n \int_{\mathbb{R}^N} |\nabla u_{\alpha_n}|^{\theta-2} q\nu |v_{\alpha_n}|^{q\nu-2}u_{\alpha_n}v_{\alpha_n}\nabla u_{\alpha_n} \cdot \nabla v_{\alpha_n}\, dx \\
&\quad + \int_{\mathbb{R}^N} |\nabla u_{\alpha_n}|^{p}|v_{\alpha_n}|^{q\nu}\, dx + \int_{\mathbb{R}^N} |\nabla u_{\alpha_n}|^{p-2} q\nu |v_{\alpha_n}|^{q\nu-2}u_{\alpha_n}v_{\alpha_n}\nabla u_{\alpha_n} \cdot \nabla v_{\alpha_n}\, dx \\
&\quad + 2^{q-1} \int_{\mathbb{R}^N} |u_{\alpha_n}|^q |\nabla u_{\alpha_n}|^q |v_{\alpha_n}|^{q\nu} + |u_{\alpha_n}|^q |\nabla u_{\alpha_n}|^{q-2} q\nu |v_{\alpha_n}|^{q\nu-2}u_{\alpha_n}v_{\alpha_n}\nabla u_{\alpha_n} \cdot \nabla v_{\alpha_n}\, dx \\
&\geq 2^{q-1} \int_{\mathbb{R}^N} |u_{\alpha_n}|^q |\nabla u_{\alpha_n}|^q |v_{\alpha_n}|^{q\nu} + |u_{\alpha_n}|^q |\nabla u_{\alpha_n}|^{q-2} q\nu |v_{\alpha_n}|^{q\nu-2}u_{\alpha_n}v_{\alpha_n}\nabla u_{\alpha_n} \cdot \nabla v_{\alpha_n}\, dx \\
&= \frac{1}{2} \int_{\mathbb{R}^N} |v_{\alpha_n}|^{q\nu}(|\nabla u_{\alpha_n}^2|)^q\, dx + \frac{q2^{q-1}}{\nu^{q-1}} \int_{\mathbb{R}^N} |\nabla v_{\alpha_n}^{\nu}|^q |u_{\alpha_n}|^{2q}\, dx \\
&\geq \frac{1}{(2+\nu)^{q-1}} \int_{\mathbb{R}^N} |\nabla (u_{\alpha_n}^2 \cdot |v_{\alpha_n}|^{\nu})|^q\, dx \\
&\geq \frac{C}{(2+\nu)^{q-1}} \left(\int_{\mathbb{R}^N} |u_{\alpha_n}^2 \cdot |v_{\alpha_n}|^{\nu}|^{q^*}\, dx \right)^{\frac{q}{q^*}}.
\end{aligned}$$

On the other hand, by the interpolation inequality, one has

$$\begin{aligned}
\int_{\mathbb{R}^N} |u_{\alpha_n}|^{m-2}u_{\alpha_n}\psi\, dx &= \int_{\mathbb{R}^N} |u_{\alpha_n}|^m \cdot |v_{\alpha_n}|^{q\nu}\, dx \\
&\leq \left(\int_{\mathbb{R}^N} |u_{\alpha_n}|^{2\cdot q^*}\, dx \right)^{\frac{m-2q}{2\cdot q^*}} \cdot \left(\int_{\mathbb{R}^N} (|v_{\alpha_n}|^{\nu} u_{\alpha_n}^2)^{\frac{2q\cdot q^*}{2\cdot q^* - m + 2q}}\, dx \right)^{\frac{2\cdot q^* - m + 2q}{2\cdot q^*}} \\
&\leq C \cdot \left(\int_{\mathbb{R}^N} (|v_{\alpha_n}|^{\nu} u_{\alpha_n}^2)^{\frac{2q\cdot q^*}{2\cdot q^* - m + 2q}}\, dx \right)^{\frac{2\cdot q^* - m + 2q}{2\cdot q^*}},
\end{aligned}$$

where C is a positive constant. Combining the above inequalities, it holds that

$$\left(\int_{\mathbb{R}^N} |u_{\alpha_n}^2 \cdot |v_{\alpha_n}|^{\nu}|^{q^*}\, dx \right)^{\frac{q}{q^*}} \leq C(2+\nu)^{q-1} \cdot \left(\int_{\mathbb{R}^N} (|v_{\alpha_n}|^{\nu} u_{\alpha_n}^2)^{\frac{2q\cdot q^*}{2\cdot q^* - m + 2q}}\, dx \right)^{\frac{2\cdot q^* - m + 2q}{2\cdot q^*}}. \tag{3.1}$$

Let $(\nu_0 + 2)\zeta = 2 \cdot q^*$ and $\vartheta = \frac{q^*}{\zeta} > 1$ where $\zeta = \frac{2q\cdot q^*}{2\cdot q^* - m + 2q}$. Then taking $\nu = \nu_0$ in (3.1) and $R \to +\infty$, we deduce that

$$\|u_{\alpha_n}\|_{(2+\nu_0)\vartheta\cdot\zeta} \leq (C(2+\nu_0))^{\frac{1}{q(2+\nu_0)}} \|u_{\alpha_n}\|_{(2+\nu_0)\zeta}.$$

Denote $2+\nu_{i+1} = (2+\nu_i)\vartheta$ for $i \in \mathbb{N}$. Thus

$$\|u_{\alpha_n}\|_{(2+\nu_0)\vartheta\cdot\zeta} \leq \Pi_{k=0}^{i}(C(2+\nu_k))^{\frac{1}{q(2+\nu_k)}}\|u_{\alpha_n}\|_{(2+\nu_0)\zeta} \leq \tilde{C}\|u_{\alpha_n}\|_{(2+\nu_0)\zeta}, \tag{3.2}$$

where $\tilde{C}$ is a positive constant. Taking $i \to +\infty$ in (3.2), we have

$$\|u_{\alpha_n}\|_\infty \leq \hat{C} \quad \text{and} \quad \|u\|_\infty \leq \hat{C},$$

where $\hat{C}$ is a positive constant.

Step 2. We claim that $(I_0^0)'(u) + \lambda|u|^{p-2}u = 0$. Taking $\psi = \eta e^{-Mu_{\alpha_n}}$ with $\eta \in C_0^\infty(\mathbb{R}^N)$, $\eta \geq 0$, $M > 0$. we get

$$\begin{aligned}
0 &= ((I_0^{\alpha_n})'(u_{\alpha_n}) + \lambda_{\alpha_n}|u_{\alpha_n}|^{p-2}u_{\alpha_n})u_{\alpha_n} \\
&= \alpha_n \int_{\mathbb{R}^N} |\nabla u_{\alpha_n}|^{\theta-2}\nabla u_{\alpha_n}(\nabla\eta e^{-Mu_{\alpha_n}} - \eta M e^{-Mu_{\alpha_n}}\nabla u_{\alpha_n})\,dx \\
&\quad + \int_{\mathbb{R}^N} |\nabla u_{\alpha_n}|^{p-2}\nabla u_{\alpha_n}(\nabla\eta e^{-Mu_{\alpha_n}} - \eta M e^{-Mu_{\alpha_n}}\nabla u_{\alpha_n})\,dx \\
&\quad + 2^{q-1}\int_{\mathbb{R}^N} |u_{\alpha_n}|^q|\nabla u_{\alpha_n}|^{q-2}\nabla u_{\alpha_n}(\nabla\eta e^{-Mu_{\alpha_n}} - \eta M e^{-Mu_{\alpha_n}}\nabla u_{\alpha_n})\,dx \\
&\quad + 2^{q-1}\int_{\mathbb{R}^N} |\nabla u_{\alpha_n}|^q|u_{\alpha_n}|^{q-2}u_{\alpha_n}\eta e^{-Mu_{\alpha_n}}\,dx \\
&\quad + \lambda_{\alpha_n}\int_{\mathbb{R}^N} |u_{\alpha_n}|^{p-2}u_{\alpha_n}\eta e^{-Mu_{\alpha_n}}\,dx - \int_{\mathbb{R}^N} |u_{\alpha_n}|^{m-2}u_{\alpha_n}\eta e^{-Mu_{\alpha_n}}\,dx \\
&\leq \alpha_n \int_{\mathbb{R}^N} |\nabla u_{\alpha_n}|^{\theta-2}\nabla u_{\alpha_n}\nabla\eta e^{-Mu_{\alpha_n}}\,dx + \int_{\mathbb{R}^N} |\nabla u_{\alpha_n}|^{p-2}\nabla u_{\alpha_n}(\nabla\eta e^{-Mu_{\alpha_n}} - \eta M e^{-Mu_{\alpha_n}}\nabla u_{\alpha_n})\,dx \\
&\quad + 2^{q-1}\int_{\mathbb{R}^N} |u_{\alpha_n}|^q|\nabla u_{\alpha_n}|^{q-2}\nabla u_{\alpha_n}(\nabla\eta e^{-Mu_{\alpha_n}} - \eta M e^{-Mu_{\alpha_n}}\nabla u_{\alpha_n})\,dx \\
&\quad + 2^{q-1}\int_{\mathbb{R}^N} |\nabla u_{\alpha_n}|^q|u_{\alpha_n}|^{q-2}u_{\alpha_n}\eta e^{-Mu_{\alpha_n}}\,dx \\
&\quad + \lambda_{\alpha_n}\int_{\mathbb{R}^N} |u_{\alpha_n}|^{p-2}u_{\alpha_n}\eta e^{-Mu_{\alpha_n}}\,dx - \int_{\mathbb{R}^N} |u_{\alpha_n}|^{m-2}u_{\alpha_n}\eta e^{-Mu_{\alpha_n}}\,dx.
\end{aligned}$$

In particular, since $\alpha_n \to 0^+$ and $\|u_{\alpha_n}\|_\infty \leq C$ and u_{α_n} is bounded in E,

$$\alpha_n \int_{\mathbb{R}^N} |\nabla u_{\alpha_n}|^{\theta-2}\nabla u_{\alpha_n}\nabla\eta e^{-Mu_{\alpha_n}}\,dx \to 0.$$

Moreover, by the weak convergence of u_{α_n}, the Hölder inequality and the Lebesgue's dominated convergence theorem, we see that

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla u_{\alpha_n}|^{p-2}\nabla u_{\alpha_n}\nabla\eta e^{-Mu_{\alpha_n}} + 2^{q-1}|u_{\alpha_n}|^q|\nabla u_{\alpha_n}|^{q-2}\nabla u_{\alpha_n}\nabla\eta e^{-Mu_{\alpha_n}}\,dx \\
&\to \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u\nabla\eta e^{-Mu} + 2^{q-1}|u|^q|\nabla u|^{q-2}\nabla u\nabla\eta e^{-Mu}\,dx,
\end{aligned}$$

$$\lambda_{\alpha_n}\int_{\mathbb{R}^N} |u_{\alpha_n}|^{p-2}u_{\alpha_n}\eta e^{-Mu_{\alpha_n}}\,dx \to \lambda\int_{\mathbb{R}^N} |u|^{p-2}u\eta e^{-Mu}\,dx$$

and

$$\int_{\mathbb{R}^N} |u_{\alpha_n}|^{m-2}u_{\alpha_n}\eta e^{-Mu_{\alpha_n}}\,dx \to \int_{\mathbb{R}^N} |u|^{m-2}u\eta e^{-Mu}\,dx.$$

In addition, using the Fatou's lemma and taking M large enough, it holds that

$$\begin{aligned}
&\liminf_{n\to+\infty}\int_{\mathbb{R}^N} |\nabla u_{\alpha_n}|^p\cdot\eta M e^{-Nu_{\alpha_n}} + 2^{q-1}|u_{\alpha_n}|^q|\nabla u_{\alpha_n}|^q\cdot\eta M e^{-Mu_{\alpha_n}} - 2^{q-1}|\nabla u_{\alpha_n}|^q|u_{\alpha_n}|^{q-2}u_{\alpha_n}\eta e^{-Mu_{\alpha_n}}\,dx \\
&\geq \int_{\mathbb{R}^N} |\nabla u|^p\cdot\eta M e^{-Nu} + 2^{q-1}|u|^q|\nabla u|^q\cdot\eta M e^{-Mu} - 2^{q-1}|\nabla u|^q|u|^{q-2}u\eta e^{-Mu}\,dx.
\end{aligned}$$

Consequently, we have

$$
\begin{aligned}
0 \le & \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u(\nabla\eta e^{-Mu} - \eta M e^{-Mu}\nabla u)\,dx \\
& + 2^{q-1}\int_{\mathbb{R}^N} |u|^q|\nabla u|^{q-2}\nabla u(\nabla\eta e^{-Mu} - \eta M e^{-Mu}\nabla u)\,dx \\
& + 2^{q-1}\int_{\mathbb{R}^N} |\nabla u|^q|u|^{q-2}u\eta e^{-Mu}\,dx \\
& + \lambda\int_{\mathbb{R}^N} |u|^{p-2}u\eta e^{-Mu}\,dx - \int_{\mathbb{R}^N} |u|^{m-2}u\eta e^{-Mu}\,dx.
\end{aligned}
\tag{3.3}
$$

For any $\phi \in C_0^\infty(\mathbb{R}^N)$ with $\phi \ge 0$, we take a sequence of nonnegative functions $\eta_n \in C_0^\infty(\mathbb{R}^N)$ such that $\eta_n \to \phi e^{Mu}$ in $W^{1,p}(\mathbb{R}^N)$, $\eta_n \to \phi e^{Mu}$ a.e. in $\mathbb{R}^N$, and η_n is uniformly bounded in $L^\infty(\mathbb{R}^N)$. Thus by (3.3), we obtain that

$$
\begin{aligned}
0 \le & \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u\nabla\phi\,dx + 2^{q-1}\int_{\mathbb{R}^N} |u|^q|\nabla u|^{q-2}\nabla u\nabla\phi\,dx + 2^{q-1}\int_{\mathbb{R}^N} |\nabla u|^q|u|^{q-2}u\phi\,dx \\
& + \lambda\int_{\mathbb{R}^N} |u|^{p-2}u\phi\,dx - \int_{\mathbb{R}^N} |u|^{m-2}u\phi\,dx.
\end{aligned}
$$

Similarly, we can choose $\psi = \eta e^{Mu_{\alpha_n}}$ to get an opposite inequality. Note that $\phi = \phi^+ - \phi^-$ for any $\phi \in C_0^\infty(\mathbb{R}^N)$, we infer that $(I_0^0)'(u) + \lambda|u|^{p-2}u = 0$.

Step 3. We prove that if $\lambda \ne 0$, then $\|u\|_p = \lim_{n\to+\infty} c_n$. From the result that $(I_0^0)'(u) + \lambda|u|^{p-2}u = 0$, we immediately know that $P_0^0(u) = 0$. On the other hand, using the weak lower semicontinuous property, it holds that

$$
\begin{aligned}
&\alpha_n\|\nabla u_{\alpha_n}\|_\theta^\theta \to 0, \quad \|\nabla u_{\alpha_n}\|_p^p \to \|\nabla u\|_p^p \\
&\|u_{\alpha_n}\nabla u_{\alpha_n}\|_q^q \to \|u\nabla u\|_q^q, \quad \|u_{\alpha_n}\|_m^m \to \|u\|_m^m.
\end{aligned}
$$

This means that $I_0^0(u) = \lim_{n\to+\infty} I_0^{\alpha_n}(u_{\alpha_n}) = \varrho$ and $\langle (I_0^{\alpha_n})'(u_{\alpha_n}), u_{\alpha_n}\rangle \to \langle (I_0^0)'(u), u\rangle$. Then $\lambda_{\alpha_n}\|u_{\alpha_n}\|_p^p \to \lambda\|u\|_p^p$. Thus if $\lambda \ne 0$, then $\|u\|_p = \lim_{n\to+\infty} c_n$. The proof of Lemma 3.1 is completed. □

Proof of Theorem 1.5. From the fact that $m_0^\alpha(c) \ge A(c) > 0$ for all $\alpha \in (0,1]$ and Lemma 2.4, we find that

$$
m^*(c) := \lim_{\alpha\to 0^+} m_0^\alpha(c) \in (0,+\infty).
$$

Using Lemma 2.7, we obtain that

$$
\alpha_n \to 0^+, \quad (I_0^{\alpha_n})'(u_{\alpha_n}) + \lambda_{\alpha_n}|u_{\alpha_n}|^{p-2}u_{\alpha_n} = 0, \quad I_0^{\alpha_n}(u_{\alpha_n}) \to m^*(c) \tag{3.4}
$$

for $u_{\alpha_n} \in S_r(c_n)$ with $0 < c_n \le c$ and $u_{\alpha_n} \ge 0$. Moreover, $P_0^{\alpha_n}(u_{\alpha_n}) = 0$. In addition, testing the second result of (3.4) with u_{α_n}, we have

$$
0 = \alpha_n\|\nabla u_{\alpha_n}\|_\theta^\theta + \|\nabla u_{\alpha_n}\|_p^p + 2^q\|u_{\alpha_n}\nabla u_{\alpha_n}\|_q^q - \|u_{\alpha_n}\|_m^m + \lambda_{\alpha_n}\|u_{\alpha_n}\|_p^p.
$$

Then combining the condition that $\frac{pq}{N} + 2q < m < p^*$, we deduce that

$$
\begin{aligned}
\lambda_{\alpha_n}\cdot\frac{mN-pN}{pm}\|u_{\alpha_n}\|_p^p = & \left(\frac{N\theta + p\theta - PN}{p\theta} - \frac{mN-pN}{pm}\right)\alpha_n\|\nabla u_{\alpha_n}\|_\theta^\theta \\
& + \left(1 - \frac{mN-pN}{pm}\right)\|\nabla u_{\alpha_n}\|_p^p \\
& + \left(\frac{2^{q-1}(2qN+pq-pN)}{pq} - \frac{2^q(mN-pN)}{pm}\right)\|u_{\alpha_n}\nabla u_{\alpha_n}\|_q^q \\
> & \; 0.
\end{aligned}
$$

This means that $\lambda_{\alpha_n} > 0$. Next applying Lemma 3.1, there exists $u \neq 0$, $u \geq 0$ and $u \in W_r^{1,p}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $\lambda \in \mathbb{R}$ such that

$$(I_0^0)'(u) + \lambda |u|^{p-2}u = 0, \quad I_0^0(u) = m^*(c), \quad 0 < \|u\|_p^p \leq c^p.$$

Similarly, we also have $\lambda > 0$. Since $\lambda_{\alpha_n} \to \lambda$, we may suppose that $\lambda_{\alpha_n} \neq 0$ for n large enough. Hence $c_n = c$ and $\|u\|_p^p = c^p$. This means that u is a nontrivial nonnegative solution of (1.1). □

4. Perturbed functional in case of $\mu > 0$

In this section, we discuss the case of $\mu > 0$ by taking the perturbation method. First of all, we prove the convergence of special Palais-Smale sequences satisfying suitable additional conditions by applying the ideas introduced in [18].

Lemma 4.1. *Assume that $p < l < \frac{p^2}{N} + p$, $\frac{pq}{N} + 2q < m < p^*$. Let $\{u_n\} \subset S_r(c)$ be a Palais-Smale sequence for I_μ^α at level $\varrho \neq 0$ and $P_\mu^\alpha(u_n) \to 0$ as $n \to +\infty$. Then up to a subsequence, $u_n \to u$ strongly in E and there exists a Lagrange multiplier $\lambda > 0$.*

Proof. Since $P_\mu^\alpha(u_n) \to 0$ as $n \to +\infty$, we see that

$$\begin{aligned}&\frac{N\theta + p\theta - pN}{p\theta}\alpha\|\nabla u_n\|_\theta^\theta + \|\nabla u_n\|_p^p + \frac{2^{q-1}(2qN + pq - pN)}{pq}\|u_n \nabla u_n\|_q^q \\ &= \mu\frac{lN - pN}{pl}\|u_n\|_l^l + \frac{mN - pN}{pm}\|u_n\|_m^m + o_n(1).\end{aligned}$$

Combining the definition of $I_\mu^\alpha(u)$, we deduce that

$$\begin{aligned}I_\mu^\alpha(u_n) = &\left(1 - \frac{N\theta + p\theta - pN}{mN - pN}\right)\frac{\alpha}{\theta}\|\nabla u_n\|_\theta^\theta + \left(\frac{1}{p} - \frac{p}{mN - pN}\right)\|\nabla u_n\|_p^p \\ &+ \left(1 - \frac{2qN + pq - pN}{mN - pN}\right)\frac{2^{q-1}}{q}\|u_n \nabla u_n\|_q^q - \left(1 - \frac{lN - pN}{mN - pN}\right)\frac{\mu}{l}\|u_n\|_l^l + o_n(1),\end{aligned} \tag{4.1}$$

where the coefficients inside the brackets are positive. When $\mu > 0$, by (1.3), one has

$$\varrho + 1 \geq I_\mu^\alpha(u_n) \geq \left(\frac{1}{p} - \frac{p}{mN - pN}\right)\|\nabla u_n\|_p^p - \left(1 - \frac{lN - pN}{mN - pN}\right)\frac{\mu}{l} \cdot C_{N,l}^l \|\nabla u_n\|_p^{l\delta_l} \cdot c^{l(1-\delta_l)}.$$

Since $l\delta_l < p$, combining (4.1), we know that $\{u_n\}$ is bounded in E. Next, we observe that $E_r \hookrightarrow L^\kappa(\mathbb{R}^N)$ is compact for $\kappa \in (p, p^*)$. Then there exists $u \in E_r$ such that up to a subsequence, $u_n \rightharpoonup u$ weakly in E, $u_n \to u$ strongly in $L^\kappa(\mathbb{R}^N)$ for $\kappa \in (p, p^*)$ and a.e. in $\mathbb{R}^N$. Now, since $\{u_n\}$ is a bounded Palais-Smale sequence of $I_\mu^\alpha|_{S(c)}$, by the Lagrange multipliers rule, there exists $\lambda_n \in \mathbb{R}$ such that

$$\langle (I_\mu^\alpha)'(u_n), \psi\rangle + \lambda_n |u_n|^{p-2}u_n\psi = o_n(1) \tag{4.2}$$

for every $\psi \in E$. In particular, we take $\psi = u_n$. Then

$$\lambda_n c^p = -\alpha\|\nabla u_n\|_\theta^\theta - \|\nabla u_n\|_p^p - 2^q\|u_n \nabla u_n\|_q^q + \mu\|u_n\|_l^l + \|u_n\|_m^m + o_n(1),$$

which along with the boundedness of $\{u_n\}$ yields that $\{\lambda_n\}$ is bounded in $\mathbb{R}$. Thus up to a subsequence, there exists $\lambda \in \mathbb{R}$ such that $\lambda_n \to \lambda$ in $\mathbb{R}$. Moreover, recalling that $P_\mu^\alpha(u_n) \to 0$, we deduce that

$$\begin{aligned}\lambda_n c^p = &\left(\frac{N\theta + p\theta - pN}{p\theta} - 1\right)\alpha\|\nabla u_n\|_\theta^\theta + \left(\frac{2^{q-1}(2qN + pq - pN)}{pq} - 2^q\right)\|u_n \nabla u_n\|_q^q \\ &+ \mu\left(1 - \frac{lN - pN}{pl}\right)\|u_n\|_l^l + \left(1 - \frac{mN - pN}{pm}\right)\|u_n\|_m^m + o_n(1),\end{aligned}$$

where the coefficients inside the brackets are positive. Hence $\lambda \geq 0$, with equality only if $u \equiv 0$. In particular, if $u \equiv 0$, then

$$\mu\|u_n\|_l^l + \|u_n\|_m^m \to 0.$$

Combining $P_\mu^\alpha(u_n) \to 0$, we obtain that $I_\mu^\alpha(u_n) \to 0$, which contradicts the fact that $\varrho \neq 0$. That is, $\lambda_n \to \lambda > 0$. Finally, by the weak convergence and (4.2), it holds that

$$\langle (I_\mu^\alpha)'(u_n), \psi\rangle + \lambda|u|^{p-2}u\psi = 0$$

for every $\psi \in E$. Moreover, using the fact that $E_r \hookrightarrow L^\kappa(\mathbb{R}^N)$ is compact for $\kappa \in (p, p^*)$ and the weak lower semicontinuous property, similar to the proof of Lemma 2.2 in [22], we obtain that $u_n \to u$ strongly in E. $\square$

In order to obtain the existence of normalized solutions in case of a supercritical leading term with focusing subcritical perturbation, we consider the constrained functional $I_\mu^\alpha|_{S(c)}$. By (1.3), we have

$$I_\mu^\alpha(u) \geq \frac{1}{p}\|\nabla u\|_p^p - \frac{\mu}{l}\cdot C_{N,l}^l\|\nabla u\|_p^{l\delta_l}\cdot c^{l(1-\delta_l)} - \frac{1}{m}\cdot C_{N,m}^m\|\nabla u\|_p^{m\delta_m}\cdot c^{m(1-\delta_m)} \tag{4.3}$$

for every $u \in S(c)$. Hence, to analyze the geometry of the functional $I_\mu^\alpha|_{S(c)}$, it is useful to denote the function $h : \mathbb{R}^+ \mapsto \mathbb{R}$

$$h(t) := \frac{1}{p}\cdot t^p - \frac{\mu}{l}C_{N,l}^l c^{l(1-\delta_l)}t^{l\delta_l} - \frac{1}{m}C_{N,m}^m c^{m(1-\delta_m)}t^{m\delta_m}. \tag{4.4}$$

We observe $\mu > 0$ and $l\delta_l < p < m\delta_m$. So $h(0^+) = 0^-$ and $h(+\infty) = -\infty$. Furthermore, we have the following lemma.

Lemma 4.2. *Assume that the condition* (1.7) *holds. Then the function h has a local strict minimum at a negative level and a global strict maximum at a positive level. Moreover, there exist* $0 < R_1 < R_2$, *depending on* c, μ, *such that* $h(R_1) = 0 = h(R_2)$ *and* $h(t) > 0$ *when* $t \in (R_1, R_2)$.

Proof. Note that for $t > 0$, $h(t) > 0$ if and only if

$$\bar{h}(t) > \frac{\mu}{l}C_{N,l}^l c^{l(1-\delta_l)}, \quad \text{with } \bar{h}(t) := \frac{1}{p}t^{p-l\delta_l} - \frac{1}{m}C_{N,m}^m c^{m(1-\delta_m)}t^{m\delta_m - l\delta_l}.$$

It follows from the structure of $\bar{h}(t)$ that $\bar{h}(t)$ has a unique critical point on $(0, +\infty)$, which is a global maximum point at positive level, in

$$\bar{t} := \left(\frac{p - l\delta_l}{p}\cdot\frac{m}{(m\delta_m - l\delta_l)C_{N,m}^m}\right)^{\frac{1}{m\delta_m - p}}\cdot c^{-\frac{m(1-\delta_m)}{m\delta_m - p}},$$

the maximum level is

$$\bar{h}(\bar{t}) = \frac{m\delta_m - p}{p(m\delta_m - l\delta_l)}\cdot\left(\frac{(p - l\delta_l)m}{p(m\delta_m - l\delta_l)C_{N,m}^m}\right)^{\frac{p - l\delta_l}{m\delta_m - p}}\cdot c^{-\frac{m(1-\delta_m)(p - l\delta_l)}{m\delta_m - p}}.$$

Hence there exists an open interval (R_1, R_2) such that $h(t)$ is positive on this interval when $\bar{h}(\bar{t}) > \frac{\mu}{l}C_{N,l}^l c^{l(1-\delta_l)}$, that is, the condition (1.7) holds. Then we immediately obtain that $h(t)$ has a global maximum at positive level (R_1, R_2). In addition, by the fact that $h(0^+) = 0^-$, there exists a local minimum point at negative level in $(0, R_1)$. On the other hand, we observe that $h'(t) = 0$ if and only if

$$\tilde{h}(t) = \mu\delta_l C_{N,l}^l c^{l(1-\delta_l)}, \quad \text{with } \tilde{h}(t) := t^{p-l\delta_l} - \delta_m C_{N,m}^m c^{m(1-\delta_m)}t^{m\delta_m - l\delta_l}.$$

Obviously, $\tilde{h}(t)$ has only one critical point, which is a strict maximum. This means that the above equation has at most two solutions, which are the local minimum and the global maximum of h previously found. That is, $h(t)$ has no other critical points. $\square$

Lemma 4.3. *Assume that the condition* (1.8) *holds. Then* $(\mathcal{P}_\mu^\alpha)^0 = \emptyset$, *and* $\mathcal{P}_\mu^\alpha$ *is a* C^1*-submanifold of codimension* 2 *in* E.

Proof. We assume that there exists $u \in (\mathcal{P}_\mu^\alpha)^0$. Then using $P_\mu^\alpha(u) = 0$ and $(\Psi_\mu^\alpha)_u''(0) = 0$, we deduce that

$$\begin{aligned}&\frac{\alpha}{\theta}\left(\frac{N}{p}\theta + \theta - N\right)\|\nabla u\|_\theta^\theta + \|\nabla u\|_p^p + \frac{2^{q-1}}{q}\left(\frac{N}{p}\cdot 2q + q - N\right)\|u\nabla u\|_q^q \\ &= \frac{\mu}{l}\left(\frac{N}{p}\cdot l - N\right)\|u\|_l^l + \frac{1}{m}\left(\frac{N}{p}m - N\right)\|u\|_m^m\end{aligned}$$

and

$$\begin{aligned}&\frac{\alpha}{\theta}\left(\frac{N}{p}\theta + \theta - N\right)^2\|\nabla u\|_\theta^\theta + p\|\nabla u\|_p^p + \frac{2^{q-1}}{q}\left(\frac{N}{p}\cdot 2q + q - N\right)^2\|u\nabla u\|_q^q \\ &= \frac{\mu}{l}\left(\frac{N}{p}\cdot l - N\right)^2\|u\|_l^l + \frac{1}{m}\left(\frac{N}{p}m - N\right)^2\|u\|_m^m.\end{aligned}$$

Thus by (1.3), one has

$$\begin{aligned}\left(\frac{N}{p}m - N - p\right)\|\nabla u\|_p^p &\le \frac{\mu}{l}\left(\frac{N}{p}\cdot l - N\right)\left(\frac{N}{p}m - \frac{N}{p}l\right)\|u\|_l^l \\ &\le \frac{\mu}{l}\left(\frac{N}{p}\cdot l - N\right)\left(\frac{N}{p}m - \frac{N}{p}l\right)C_{N,l}^l\|\nabla u\|_p^{l\delta_l}\cdot c^{l(1-\delta_l)}\end{aligned}$$

and

$$\begin{aligned}\left(p + N - \frac{N}{p}l\right)\|\nabla u\|_p^p &\le \frac{1}{m}\left(\frac{N}{p}m - N\right)\left(\frac{N}{p}m - \frac{N}{p}l\right)\|u\|_m^m \\ &\le \frac{1}{m}\left(\frac{N}{p}m - N\right)\left(\frac{N}{p}m - \frac{N}{p}l\right)C_{N,m}^m\|\nabla u\|_p^{m\delta_m}\cdot c^{m(1-\delta_m)}.\end{aligned}$$

This implies that the lower and upper bounds $\|\nabla u\|_p$ are given by

$$\left[\frac{m}{C_{N,m}^m c^{m(1-\delta_m)}}\frac{\left(p + N - \frac{N}{p}l\right)}{\left(\frac{N}{p}m - N\right)\left(\frac{N}{p}m - \frac{N}{p}l\right)}\right]^{\frac{1}{m\delta_m - p}} \le \|\nabla u\|_p \le \left[\frac{\mu}{l}C_{N,l}^l c^{l(1-\delta_l)}\frac{\left(\frac{N}{p}\cdot l - N\right)\left(\frac{N}{p}m - \frac{N}{p}l\right)}{\left(\frac{N}{p}m - N - p\right)}\right]^{\frac{1}{p - l\delta_l}},$$

which contradicts the condition (1.8). That is, $(\mathcal{P}_\mu^\alpha)^0 = \emptyset$ holds. Next similar to the proof of Lemma 2.1, we easily show that $\mathcal{P}_\mu^\alpha$ is a C^1-submanifold of codimension 2 in E. Hence the proof of Lemma 4.3 is completed. $\square$

Based on Lemma 4.3, we see that the manifold $\mathcal{P}_\mu^\alpha$ is divided into two components $(\mathcal{P}_\mu^\alpha)^+$ and $(\mathcal{P}_\mu^\alpha)^-$.

Lemma 4.4. *For every* $u \in S(c)$, *the function* $(\Psi_\mu^\alpha)_u$ *has exactly two critical* $a_u, c_u \in \mathbb{R}$ *and two zeros* $b_u, d_u \in \mathbb{R}$, *with* $a_u < b_u < c_u < d_u$. *Furthermore, the following results hold:*

(i) $a_u * u \in (\mathcal{P}_\mu^\alpha)^+$, $c_u * u \in (\mathcal{P}_\mu^\alpha)^-$, *and if* $s * u \in \mathcal{P}_\mu^\alpha$, *then either* $s = a_u$ *or* $s = c_u$.

(ii) $\|\nabla(s * u)\|_p \le R_1$ *for every* $s \le b_u$, *and*

$$I_\mu^\alpha(a_u * u) = \min\left\{I_\mu^\alpha(s * u) : s \in \mathbb{R} \text{ and } \|\nabla(s * u)\|_p < R_1\right\} < 0$$

(iii)
$$I_\mu^\alpha(c_u * u) = \max\left\{I_\mu^\alpha(s*u) : s \in \mathbb{R}\right\} > 0,$$
and $(\Psi_\mu^\alpha)_u$ *is strictly decreasing and concave on* $(c_u, +\infty)$. *In particular, if* $c_u < 0$, *then* $P_\mu^\alpha(u) < 0$.

(iv) *The maps* $u \in S(c) \mapsto a_u \in \mathbb{R}$ *and* $u \in S(c) \mapsto c_u \in \mathbb{R}$ *are of class* C^1.

Proof. Let $u \in S(c)$. Note that $(\Psi_\mu^\alpha)'_u(s) = 0$ if and only if $s * u \in \mathcal{P}_\mu^\alpha$. Thus we first show that $(\Psi_\mu^\alpha)_u$ has at least two critical points. By (4.3) and (4.4), it holds that
$$(\Psi_\mu^\alpha)_u(s) = I_\mu^\alpha(s*u) \geq h(\|s*u\|_p) = h(e^s\|\nabla u\|_p).$$
This means that the C^2 function Ψ_μ^α is positive on $\left(\ln\left(\frac{R_1}{\|\nabla u\|_p}\right), \ln\left(\frac{R_2}{\|\nabla u\|_p}\right)\right)$. We easily check that $\Psi_\mu^\alpha(-\infty) = 0^-$, $\Psi_\mu^\alpha(+\infty) = -\infty$, which yields that $(\Psi_\mu^\alpha)_u$ has at least two critical points $a_u < c_u$, with a_u local minimum point on $\left(0, \ln\left(\frac{R_1}{\|\nabla u\|_p}\right)\right)$ at negative level, and c_u global maximum point at positive level. Next we claim that $(\Psi_\mu^\alpha)_u$ has no other critical points. Indeed, $(\Psi_\mu^\alpha)'_u(s) = 0$ is equivalent to $\phi(s) = \frac{1}{m}\left(\frac{N}{p}m - N\right)\|u\|_m^m$, with
$$\begin{aligned}\phi(s) :=& \frac{\alpha}{\theta}\left(\frac{N}{p}\theta + \theta s - N\right)e^{\frac{N}{p}s\theta + \theta - \frac{N}{p}sm}\|\nabla u\|_\theta^\theta + e^{ps+Ns-\frac{N}{p}sm}\|\nabla u\|_p^p \\ &+ \frac{2^{q-1}}{q}\left(\frac{N}{p}2q + q - N\right)e^{\frac{N}{q}s\cdot 2q + qs - \frac{N}{p}sm}\|u\nabla u\|_q^q - \frac{\mu}{l}\left(\frac{N}{p}l - N\right)e^{\frac{N}{p}sl - \frac{N}{p}sm}\|u\|_l^l.\end{aligned}$$
It follows from the structure of $\phi(s)$ that ϕ has a unique maximum point, and hence $(\Psi_\mu^\alpha)'_u(s) = 0$ has at most two solutions. That is, $(\Psi_\mu^\alpha)_u$ has exactly two critical points and no other critical points.

To sum up, it holds that $(\Psi_\mu^\alpha)_u$ has exactly two critical points: a_u is the local minimum on $\left(-\infty, \ln\left(\frac{R_1}{\|\nabla u\|_p}\right)\right)$ at negative level, and c_u is the global maximum at positive level. Furthermore, $a_u * u, c_u * u \in \mathcal{P}_\mu^\alpha$, and $s * u \in \mathcal{P}_\mu^\alpha$ implies that $s \in \{a_u.c_u\}$. Since $(\mathcal{P}_\mu^\alpha)^0 = \emptyset$, we easily see that $a_u * u \in (\mathcal{P}_\mu^\alpha)^+$ and $c_u * u \in (\mathcal{P}_\mu^\alpha)^-$. It follows from the monotonicity and the behavior of $(\Psi_\mu^\alpha)_u$ at infinity that $(\Psi_\mu^\alpha)_u$ has exactly two zeros b_u, d_u, with $a_u < b_u < c_u < d_u$. $(\Psi_\mu^\alpha)_u$ is concave on $[c_u, +\infty)$, and hence if $t_u < 0$, then $P_\mu^\alpha(u) = (\Psi_\mu^\alpha)'_u(0) = 0$.

In what follows, we apply the implicit function theorem on the C^1 function $\Phi(s,u) := (\Psi_\mu^\alpha)'_u(s)$. Then by $\Phi(a_u, u) = 0$, $\partial_s\Phi(a_u, u) = (\Psi_\mu^\alpha)''_u(a_u) < 0$ and $(\mathcal{P}_\mu^\alpha)^0 = \emptyset$, we deduce that $u \mapsto a_u$ is of class C^1. Similarly, we also find that $u \mapsto c_u$ is of class C^1. The proof of Lemma 4.4 is completed. □

Based on the results of Lemma 4.4, for $k > 0$, we denote
$$A_k := \left\{u \in S(c) : \|\nabla u\|_p < k\right\}, \quad \bar{m}_\mu^\alpha(c) := \inf_{u \in A_{R_1}} I_\mu^\alpha(u).$$
Then we easily find that $(\mathcal{P}_\mu^\alpha)^+$ is contained in A_{R_1} and $\sup_{u\in(\mathcal{P}_\mu^\alpha)^+} I_\mu^\alpha(u) \leq 0 \leq \inf_{u\in(\mathcal{P}_\mu^\alpha)^-} I_\mu^\alpha(u)$. Furthermore, we have the following results about $\bar{m}_\mu^\alpha(c)$.

Lemma 4.5. *It holds that* $\bar{m}_\mu^\alpha(c) \in (-\infty, 0)$,
$$\bar{m}_\mu^\alpha(c) = \inf_{u\in\mathcal{P}_\mu^\alpha} I_\mu^\alpha(u) = \inf_{u\in(\mathcal{P}_\mu^\alpha)^+} I_\mu^\alpha(u),$$
and
$$\bar{m}_\mu^\alpha(c) < \inf_{\overline{A_{R_1}}\setminus A_{R_1-\rho}} I_\mu^\alpha(u)$$

for $\rho > 0$ small enough.

Proof. For $u \in A_{R_1}$,

$$I_\mu^\alpha(u) \geq h(\|\nabla u\|_p) \geq \min_{t\in[0,R_1]} h(t) > -\infty,$$

which means that $\bar{m}_\mu^\alpha(c) > -\infty$. In addition, for any $u \in S(c)$, combining $\|\nabla(s * u)\|_p < R_1$ and $I_\mu^\alpha(s * u) < 0$ for $s \ll -1$, we have $\bar{m}_\mu^\alpha(c) < 0$. Next on one hand, since $(\mathcal{P}_\mu^\alpha)^+ \subset A_{R_1}$, $\bar{m}_\mu^\alpha(c) \leq \inf_{u\in(\mathcal{P}_\mu^\alpha)^+} I_\mu^\alpha(u)$. On the other hand, if $u \in A_{R_1}$, then $a_u * u \in (\mathcal{P}_\mu^\alpha)^+ \subset A_{R_1}$ and

$$I_\mu^\alpha(a_u * u) = \min\left\{ I_\mu^\alpha(s * u) : s \in \mathbb{R} \text{ and } \|\nabla(s * u)\|_p < R_1 \right\} \leq I_\mu^\alpha(u),$$

which implies that $\inf_{u\in(\mathcal{P}_\mu^\alpha)^+} I_\mu^\alpha(u) \leq \bar{m}_\mu^\alpha(c)$. Moreover, since $I_\mu^\alpha(u) > 0$ when $u \in (\mathcal{P}_\mu^\alpha)^-$, we obtain that $\inf_{u\in(\mathcal{P}_\mu^\alpha)^+} I_\mu^\alpha(u) = \inf_{u\in\mathcal{P}_\mu^\alpha} I_\mu^\alpha(u)$. Hence

$$\bar{m}_\mu^\alpha(c) = \inf_{u\in\mathcal{P}_\mu^\alpha} I_\mu^\alpha(u) = \inf_{u\in(\mathcal{P}_\mu^\alpha)^+} I_\mu^\alpha(u).$$

In the last, it follows from the continuity of h that there exists $\rho > 0$ such that $h(t) \geq \frac{\bar{m}_\mu^\alpha(c)}{2}$ when $t \in [R_1 - \rho, R_1]$. Thus combining $\inf_{u\in\mathcal{P}_\mu^\alpha} I_\mu^\alpha(u) \in (-\infty, 0)$, it holds that

$$I_\mu^\alpha(u) \geq h(\|\nabla u\|_p) \geq \frac{\bar{m}_\mu^\alpha(c)}{2} > \bar{m}_\mu^\alpha(c)$$

for each $u \in S(c)$ with $R_1 - \rho \leq \|\nabla u\|_p \leq R_1$. The proof of Lemma 4.5 is completed. ☐

To search for the second critical point for $I_\mu^\alpha|_{S(c)}$, we give the following lemma.

Lemma 4.6. *The following results hold:*

(i) *Assume that $I_\mu^\alpha(u) < \bar{m}_\mu^\alpha(c)$. Then the value c_u defined by Lemma 4.4 is negative.*

(ii) $\bar{\sigma}_\mu^\alpha(c) := \inf_{u\in(\mathcal{P}_\mu^\alpha)^-} I_\mu^\alpha(u) > 0$.

Proof. (i) It follows from Lemma 4.4 that Ψ_μ^α has exactly two critical points $a_u, c_u \in \mathbb{R}$ and two zeros $b_u, d_u \in \mathbb{R}$, with $a_u < b_u < c_u < d_u$. Obviously, if $d_u \leq 0$, then $c_u < 0$. Thus we suppose that $c_u > 0$. In particular, if $0 \in (b_u, d_u)$, then $I_\mu^\alpha(u) = (\Psi_\mu^\alpha)_u(0) > 0$, which is impossible by $I_\mu^\alpha(u) < \bar{m}_\mu^\alpha(c) < 0$. This means that $b_u > 0$. Then using Lemma 4.4(ii), one has

$$\begin{aligned}
\bar{m}_\mu^\alpha(c) > I_\mu^\alpha(u) = (\Psi_\mu^\alpha)_u(0) &\geq \inf_{s\in(-\infty,b_u]} \Psi_\mu^\alpha(s) \\
&\geq \inf\left\{ I_\mu^\alpha(s * u) : s \in \mathbb{R} \text{ and } \|\nabla(s * u)\|_p < R_1 \right\} = I_\mu^\alpha(a_u * u) \geq \bar{m}_\mu^\alpha(c),
\end{aligned}$$

which is a contradiction. Thus the value c_u is negative.

(ii) By Lemma 4.2, let $\bar{t}$ be the strict maximum of the function h at positive level. For each $u \in (\mathcal{P}_\mu^\alpha)^-$, there exists $\tau_u \in \mathbb{R}$ such that $\|\nabla(\tau_u * u)\| = \bar{t}$. On the other hand, it follows from Lemma 4.4 and $u \in (\mathcal{P}_\mu^\alpha)^-$ that the value 0 is the unique strict maximum of the function Ψ_μ^α. Hence

$$I_\mu^\alpha(u) = (\Psi_\mu^\alpha)_u(0) \geq (\Psi_\mu^\alpha)_u(\tau_u) = I_\mu^\alpha(\tau_u * u) \geq h(\|\nabla(\tau_u * u)\|_p) = h(\bar{t}) > 0,$$

which along with the arbitrary of $u \in (\mathcal{P}_\mu^\alpha)^-$ yields that

$$\bar{\sigma}_\mu^\alpha(c) = \inf_{u\in(\mathcal{P}_\mu^\alpha)^-} I_\mu^\alpha(u) > 0.$$

This completes the proof. ☐

Based on the above lemmas, we have the following theorem.

Theorem 4.7. *Under the assumptions* (1.7) *and* (1.8)*, the following results hold:*

(i) $I_\mu^\alpha|_{S(c)}$ *has a critical point* u_1 *at negative level* $\bar{m}_\mu^\alpha(c) < 0$ *which is an interior local minimizer of* I_μ^α *on the set* A_k *for a suitable* $k > 0$ *small enough.*

(ii) $I_\mu^\alpha|_{S(c)}$ *has a second critical point* u_2 *at level* $\bar{\sigma}_\mu^\alpha(c) > \bar{m}_\mu^\alpha(c)$.

(iii) u_1, u_2 *are radially symmetric functions in* $\mathbb{R}^N$ *and the Lagrange multipliers* $\lambda_1, \lambda_2 > 0$.

Proof. We first show the existence of a local minimizer. Let us consider a minimizing sequence $\{u_n\}$ for $I_\mu^\alpha|_{A_{R_1}}$. By the Schwarz rearrangement, we can further assume that $u_n \in E_r \cap S(c)$ is decreasing for every n. For each n, we take $a_{u_n} * u_n \in (\mathcal{P}_\mu^\alpha)^+$. Then by Lemma 4.4, $\|\nabla(a_{u_n} * u_n)\|_p < R_1$ and

$$I_\mu^\alpha(a_{u_n} * u_n) = \min\left\{I_\mu^\alpha(s * u_n) : s \in \mathbb{R} \text{ and } \|\nabla(s * u_n)\|_p < R_1\right\} \le I_\mu^\alpha(u_n),$$

we obtain a new minimizing sequence $\{v_n := a_{u_n} * u_n\}$ with $v_n \in E_r \cap S(c) \cap (\mathcal{P}_\mu^\alpha)^+$ radially decreasing for each n. In addition, it follows from Lemma 4.5 that $\|v_n\|_p < R_1 - \rho$ for each n. Hence by the Ekeland's variational principle, we find the existence of a new minimizing sequence $\{w_n\} \subset A_{R_1}$ for $\bar{m}_\mu^\alpha(c)$ with the property that $\|w_n - v_n\|_E \to 0$ as $n \to +\infty$. Moreover, $\{w_n\}$ satisfies all the assumptions of Lemma 4.1. So up to a subsequence, there exists w such that $w_n \to w$ strongly in E, w is an interior local minimizer for $I_\mu^\alpha|_{A_{R_1}}$ with some $\lambda > 0$ and $I_\mu^\alpha(w) = \bar{m}_\mu^\alpha(c)$.

Next we prove the existence of a second critical point for $I_\mu^\alpha|_{S(c)}$. Define

$$(\mathbf{I}_\mu^\alpha)^\iota := \left\{u \in S(c) : I_\mu^\alpha(u) \le \iota\right\}.$$

Inspired by [18], we consider the augmented functional

$$\mathcal{I}_\mu^\alpha(s, u) := I_\mu^\alpha(s * u) : \mathbb{R} \times E \mapsto \mathbb{R}$$

and study $\mathcal{I}_\mu^\alpha|_{\mathbb{R}\times S(c)}$. Obviously, $\mathcal{I}_\mu^\alpha$ is of class C^1. Theorem 1.28 in [36] indicates that a critical point for $\mathcal{I}_\mu^\alpha|_{\mathbb{R}\times(E_r\cap S(c))}$ is a critical point for $\mathcal{I}_\mu^\alpha|_{\mathbb{R}\times S(c)}$. Now we introduce the minimax class

$$\bar{\Gamma}_\alpha := \left\{\gamma = (\beta_1, \beta_2) \in C([0,1], \mathbb{R} \times (E_r \cap S(c))), \gamma(0) \in \{0\} \times (\mathcal{P}_\mu^\alpha)^+, \gamma(1) \in \{0\} \times (\mathbf{I}_\mu^\alpha)^{2\bar{m}_\mu^\alpha(c)}\right\}$$

with associated minimax level

$$\bar{\sigma}_\mu^\alpha(c) := \inf_{\gamma\in\bar{\Gamma}_\alpha} \max_{(s,u)\in\gamma([0,1])} \mathcal{I}_\mu^\alpha(s, u).$$

For any $u \in E_r \cap S(c)$, by Lemma 4.4, we know that there exists $s_0 \gg 1$ such that

$$\gamma_u : \tau \in [0,1] \mapsto (0, ((1-\tau)a_u + \tau s_0) * u) \in \mathbb{R} \times (E_r \cap S(c))$$

is a path in $\bar{\Gamma}_\alpha$. So $\Gamma_\alpha \ne \emptyset$ and $\bar{\sigma}_\mu^\alpha$ is a real number. We claim that for any $\gamma \in \bar{\Gamma}_\alpha$, there exists $\tau_\gamma \in (0,1)$ such that

$$\beta_1(\tau_\gamma) * \beta_2(\tau_\gamma) \in (\mathcal{P}_\mu^\alpha)^-. \tag{4.5}$$

Indeed, since $\gamma(0) = (\beta_1(0), \beta_2(0)) \in \{0\} \times (\mathcal{P}_\mu^\alpha)^+$, by Lemma 4.4, it holds that $c_{\beta_1(0)*\beta_2(0)} = c_{\beta_2(0)} > a_{\beta_2(0)} = 0$. On the other hand, since $I_\mu^\alpha(\beta_2(1)) = \mathcal{I}_\mu^\alpha(\gamma(1)) \le 2\bar{m}_\mu^\alpha(c) < 0$, by Lemma 4.6, it holds that $c_{\beta_1(1)*\beta_2(1)} = c_{\beta_2(1)} < 0$. In addition, we easily find that $c_{\beta_1(\tau)*\beta_2(\tau)}$ is continuous in τ. Thus for any $\gamma \in \bar{\Gamma}_\alpha$, there exists $\tau_\gamma \in (0,1)$ such that $c_{\beta_1(\tau_\gamma)*\beta_2(\tau_\gamma)} = 0$. That is $\beta_1(\tau_\gamma) * \beta_2(\tau_\gamma) \in (\mathcal{P}_\mu^\alpha)^-$. The claim is true.

In what follows, we show that $\bar{\sigma}_\mu^\alpha = \inf_{u\in(\mathcal{P}_\mu^\alpha)^-\cap E_r\cap S(c)} I_\mu^\alpha(u)$. In fact, for any $\gamma \in \bar{\Gamma}_\alpha$, by (4.5), one infers that

$$\max_{\gamma[0,1]} \mathcal{I}_\mu^\alpha \ge \mathcal{I}_\mu^\alpha(\gamma(\tau_\gamma)) = I_\mu^\alpha(\beta_1(\tau_\gamma) * \beta_2(\tau_\gamma)) \ge \inf_{u\in(\mathcal{P}_\mu^\alpha)^-\cap E_r\cap S(c)} I_\mu^\alpha(u).$$

This means that $\bar{\sigma}_\mu^\alpha \geq \inf_{u\in(\mathcal{P}_\mu^\alpha)^-\cap E_r\cap S(c)} I_\mu^\alpha(u)$. On the other hand, if $u \in (\mathcal{P}_\mu^\alpha)^- \cap E_r \cap S(c)$, then γ_u is a path in $\bar{\Gamma}_\alpha$ with

$$I_\mu^\alpha(u) = \mathcal{I}_\mu^\alpha(0,u) = \max_{\gamma_u([0,1])} \mathcal{I}_\mu^\alpha \geq \bar{\sigma}_\mu^\alpha.$$

This means that $\bar{\sigma}_\mu^\alpha \leq \inf_{u\in(\mathcal{P}_\mu^\alpha)^-\cap E_r\cap S(c)} I_\mu^\alpha(u)$. Then in virtue of Lemma 4.6, it holds that

$$\bar{\sigma}_\mu^\alpha = \inf_{(\mathcal{P}_\mu^\alpha)^-\cap E_r\cap S(c)} I_\mu^\alpha > 0 \geq \sup_{((\mathcal{P}_\mu^\alpha)^+\cup(\mathbf{I}_\mu^\alpha)^{2\bar{m}_\mu^\alpha(c)})\cap E_r\cap S(c)} I_\mu^\alpha = \sup_{(\{0\}\times(\mathcal{P}_\mu^\alpha)^+\cup\{0\}\times(\mathbf{I}_\mu^\alpha)^{2\bar{m}_\mu^\alpha(c)})\cap\mathbb{R}\times(E_r\cap S(c))} \mathcal{I}_\mu^\alpha.$$

In the last, we suppose that $\gamma_n(\tau) = ((\beta_1)_n(\tau), (\beta_2)_n(\tau))$ be any minimizing sequence for $\bar{\sigma}_\mu^\alpha(c)$ with the property that $(\beta_1)_n(\tau) = 0$ and $(\beta_2)_n(\tau) \geq 0$ a.e. in $\mathbb{R}^N$ for each $\tau \in [0,1]$. Similar to the discussion in Lemma 2.6, we obtain that there exists a Palais-Smale sequence $\{u_n\} \subset E_r \cap S(c)$ for $I_\mu^\alpha|_{E_r\cap S(c)}$ at level $\bar{\sigma}_\mu^\alpha(c)$ with $P_\mu^\alpha(u_n) \to 0$. Applying Lemma 4.1, there exists u such that $u_n \to u$ strongly in E with some $\lambda > 0$ and $I_\mu^\alpha(u) = \bar{\sigma}_\mu^\alpha(c) > 0$. The proof of Theorem 4.7 is completed. □

5. Convergence issues as $\alpha \to 0^+$ in case of $\mu > 0$

In this section, letting $\alpha \to 0^+$, we shall show that the sequence of critical points of $I_\mu^\alpha|_{S(c)}$ obtained in Section 4 convergence to critical points of $I_\mu^0|_{\tilde{S}(c)}$. Based on Lemma 4.1, similar to the proof of Lemma 3.1, it is not difficult for us to obtain the following lemma.

Lemma 5.1. *Assume that $\alpha \to 0^+$, $(I_\mu^{\alpha_n})'(u_{\alpha_n}) + \lambda_{\alpha_n}|u_{\alpha_n}|^{p-2}u_{\alpha_n} = 0$ with $\lambda_{\alpha_n} > 0$ and $I_\mu^{\alpha_n}(u_{\alpha_n}) \to \varrho \neq 0$ for $u_{\alpha_n} \in E_r \cap S(c)$. Then there exists a subsequence $u_{\alpha_n} \rightharpoonup u$ in $W^{1,p}(\mathbb{R}^N)$ with $u \not\equiv 0$, $u \in W_r^{1,p}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \cap S(c)$ and there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ such that*

$$(I_\mu^0)'(u) + \lambda|u|^{p-2}u = 0, \quad I_\mu^0(u) = \varrho.$$

Proof of Theorem 1.8. First of all, it follows from the proof of Lemmas 4.2 and 4.5 that

$$\bar{m}^*(c) := \lim_{\alpha\to 0^+} \bar{m}_\mu^\alpha(c) \in (-\infty, 0).$$

Then by Theorem 4.7, we take

$$\alpha_n \to 0^+, \quad (I_\mu^{\alpha_n})'(u_{\alpha_n}) + \lambda_{\alpha_n}|u_{\alpha_n}|^{p-2}u_{\alpha_n} = 0, \quad I_\mu^{\alpha_n}(u_{\alpha_n}) \to \bar{m}^*$$

for $u_{\alpha_n} \in E_r \cap S(c)$, $\lambda_{\alpha_n} > 0$. Combining Lemma 5.1, there exist $u \neq 0$, $u \in W^{1,p}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $\lambda > 0$ such that

$$(I_\mu^0)'(u) + \lambda|u|^{p-2}u = 0, \quad I_\mu^0(u) = \bar{m}^*(c) \quad, \|u\|_p^p = c^p.$$

That is, u is a nontrivial solution of (1.1). On the other hand, based on the proof of Lemma 4.6, we easily know that $\bar{\sigma}^*(c) := \lim_{\alpha\to 0^+} \bar{\sigma}_\mu^\alpha(c) > 0$. Then similar to the above argument, we obtain the existence of a second critical point for $I_\mu^0|_{\tilde{S}(c)}$. The proof of Theorem 1.8 is completed. □

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