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Results in Mathematics

Robin Problems Near Resonance at Any Nonprincipal Eigenvalue

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Abstract. We consider semilinear Robin problems near resonance with respect to a nonprincipal eigenvalue $\hat{\lambda}_m$. We distinguish two cases. In the first one the near resonance occurs from the right of $\hat{\lambda}_m$ and in the second from the left. For both cases, using variational tools, we produce two smooth solutions. We also provide conditions for these solutions to be nontrivial.

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1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following semilinear Robin problem

$$-\Delta u(z) = \hat{\lambda}_m u(z) + f(z, u(z)) \text{ in } \Omega, \quad \frac{\partial u}{\partial n} + \beta(z)u = 0 \text{ on } \partial\Omega. \qquad (1)$$

In this problem, $\{\hat{\lambda}_m\}_{m \geqslant 1}$ is the sequence of distinct eigenvalues of $-\Delta$ with Robin boundary condition. Depending on the asymptotic behavior of the quotient $\frac{f(z,x)}{x}$ as $x \to \pm\infty$ and in particular on how these limits relate to the eigenvalue $\hat{\lambda}_m$, we can have different existence and multiplicity results for

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problem (1). In the present work, we examine what happens when the reaction term $\hat{\lambda}_m x + f(z, x)$ asymptotically at $\pm\infty$ is near resonance with respect to the eigenvalue $\hat{\lambda}_m$ ($m \geqslant 2$) either from the right (that is, from above of $\hat{\lambda}_m$) or from the left (that is, from below of $\hat{\lambda}_m$).

This problem was first investigated for ordinary Dirichlet differential equations near resonance with respect to the principal eigenvalue, by Mawhin and Schmitt [9]. Their work was extended to semilinear Dirichlet partial differential equations by Chiappinelli et al. [2]. In both works, the approach is based on a combination of bifurcation theory and of degree theory. This method of proof requires stronger conditions on the perturbation $f(z, \cdot)$. A variational approach to the problem can be found in the works of Ma, Ramos and Sanchez [7] and Ramos and Sanchez [20]. Both papers deal with semilinear Dirichlet equations which are near resonance with respect to the principal eigenvalue. Their work was extended to equations driven by the Dirichlet p-Laplacian by Papageorgiou and Papalini [13]. Equations near resonance at higher eigenvalues, were investigated by Mawhin and Schmitt [10] and Lupo and Ramos [6] (for ordinary differential equations) and more recently by de Paiva and Massa [3] and Ke and Tang [5] (for semilinear elliptic equations). Both works deal with Dirichlet equations, the limits as $x \to \pm\infty$ exist and no interaction with the eigenvalue $\hat{\lambda}_m$ is allowed (strict nonresonance). Finally, we should mention also the works of Mugnai [12] and Rabinowitz et al. [19], which study Dirichlet elliptic equations near resonance at zero. To the best of our knowledge, no such analysis exists for Robin problems. We stress that for such problems, due to the failure of the Poincaré inequality on the Sobolev space $H^1(\Omega)$, the differential operator is not coercive.

First we deal with the situation in which the near resonance with respect to $\hat{\lambda}_m$ occurs from the right (that is, from values higher than $\hat{\lambda}_m$). We prove a multiplicity theorem producing two smooth solutions. In fact the first solution can be produced under conditions of resonance with respect to $\hat{\lambda}_m$. To obtain the second solution, we need to have conditions of nonuniform nonresonance with respect to $\hat{\lambda}_m$. Subsequently we deal with the case of near resonance from the left of $\hat{\lambda}_m$ (that is, from values smaller than $\hat{\lambda}_m$). Again we produce two smooth solutions.

In the next section, for easy reference, we recall the main mathematical tools which we will use in this work.

2. Mathematical Background

Let X be a Banach space and X^* its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X^*, X). Suppose that $\varphi \in C^1(X, \mathbb{R})$. We say that φ satisfies the "Cerami condition" (the "C-condition" for short), if the following property holds:

"Every sequence $\{u_n\}_{n\geqslant 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n\geqslant 1} \subseteq \mathbb{R}$ is bounded and

$$(1+||u_n||)\varphi'(u_n) \to 0 \text{ in } X^* \text{ as } n \to \infty,$$

admits a strongly convergent subsequence".

The Cerami condition is a compactness-type condition on the functional φ and it is weaker than the usual Palais–Smale condition. Nevertheless, if suffices to prove a deformation theorem from which one can derive the minimax theory of the critical values of φ (see, for example, Gasinski and Papageorgiou [4]). The next notion is central in this theory.

Definition 1. *Let Y be a Hausdorff topological space and let E_0, E and D be nonempty, closed subsets of Y such that $E_0 \subseteq E$. We say that the pair $\{E_0, E\}$ is linking with D in Y if:*

(a) $E_0 \cap D = \emptyset$;
(b) *for any $\gamma \in C(E, Y)$ such that $\gamma|_{E_0} = \mathrm{id}|_{E_0}$ we have $\gamma(E) \cap D \neq \emptyset$.*

Using this notion, we have the following general minimax theorem (see Gasinski and Papageorgiou [4, p. 644]).

Theorem 2. *Assume that X is a Banach space, E_0, E and D are nonempty, closed subsets of X such that $\{E_0, E\}$ is linking with D in X, $\varphi \in C^1(X, \mathbb{R})$ satisfies the C-condition,*

$$\sup_{E_0} \varphi < \inf_D \varphi$$

and $c = \inf_{\gamma\in\Gamma} \sup\limits_{u\in E} \varphi(\gamma(u))$, where $\Gamma = \{\gamma \in C(E, X) : \gamma|_{E_0} = id|_{E_0}\}$. Then $c \geqslant \inf_D \varphi$ and c is a critical value of φ (that is, there exists $u \in X$ such that $\varphi'(u) = 0$, $\varphi(u) = c$).

With suitable choices of the linking sets, from Theorem 2 we have as corollaries the main minimax theorems of the critical point theory. For future use, we state the so-called "saddle point theorem" due to Rabinowitz [18]. In what follows for $R > 0$, we define

$$B_R = \{u \in X : ||u|| < R\} \quad \text{and} \quad \partial B_R = \{u \in X,\ ||u|| = R\}.$$

Theorem 3. *Assume that $X = Y \oplus V$ with $\dim Y < +\infty$, $\varphi \in C^1(X, \mathbb{R})$, there exists $R > 0$ such that*

$$\max[\varphi(u) : u \in \partial B_R \cap Y] < \inf[\varphi(u) : u \in V] = m$$
$$\text{and}\ \ c = \inf_{\gamma\in\Gamma} \max_{u\in\bar{B}_R\cap Y} \varphi(\gamma(u))\ \text{where}$$
$$\Gamma = \{\gamma \in C(\bar{B}_R \cap Y, X) : \gamma|_{\partial B_R\cap Y} = \mathrm{id}|_{\partial B_R\cap Y}\}.$$

Then $c \geqslant m$ and c is a critical value of φ.

Remark 1. Theorem 3 follows from Theorem 2 by choosing

$$E_0 = \partial B_R \cap Y, E = \bar{B}_R \cap V \quad \text{and} \quad D = V$$

(see Gasinski and Papageorgiou [4, p. 649]).

Another abstract result which we will use, is the so-called "splitting spheres theorem" of Marino, Micheletti and Pistoia [8].

Theorem 4. *Assume that H is a Hilbert space, $H = Y \oplus V$ with $\dim Y < +\infty$, $\varphi \in C^1(X, \mathbb{R})$ satisfies the C-condition and there exist $r, \rho > 0$ such that*

$$\sup_{u\in\partial B_r\cap Y} \varphi(u) < a = \inf_{u\in\bar{B}_r\cap V} \varphi(u) \leqslant b = \sup_{u\in\bar{B}_r\cap Y} \varphi(u) < \inf_{u\in\partial B_\rho\cap V} \varphi(u).$$

Then there exists a critical point u_0 of φ such that $\varphi(u_0) \in [a, b]$.

In the study of problem (1) we will use the Sobolev space $H^1(\Omega)$, the Banach space $C^1(\overline{\Omega})$ and the boundary Lebesgue spaces $L^r(\partial\Omega), 1 \leqslant r \leqslant \infty$.

By $||\cdot||$ we denote the norm of the Sobolev space $H^1(\Omega)$ defined by

$$||u|| = \left[||u||_2^2 + ||Du||_2^2\right]^{1/2} \text{ for all } u \in H^1(\Omega).$$

On $\partial\Omega$ we employ the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma(\cdot)$. Using this measure we can define the Lebesgue spaces $L^r(\partial\Omega)$ $(1 \leqslant r \leqslant \infty)$ in the usual way. From the theory of Sobolev spaces, we know that there is a compact linear map $\gamma_0 : H^1(\Omega) \to L^2(\partial\Omega)$ known as the "trace map", such that

$$\gamma_0(u) = u|_{\partial\Omega} \quad \text{for all } u \in H^1(\Omega) \cap C(\overline{\Omega}).$$

In the sequel, for the sake of notational simplicity, we will drop the use of the trace map γ_0. The restrictions of Sobolev functions on $\partial\Omega$ are understood in the sense of traces.

We will use the spectrum of $-\Delta$ with the Robin boundary condition, So, we consider the following linear eigenvalue problem

$$-\Delta u(z) = \hat{\lambda} u(z) \quad \text{in } \Omega, \quad \frac{\partial u}{\partial n} + \beta(z) u = 0 \quad \text{on } \partial\Omega. \tag{2}$$

Here $\beta \in W^{1,\infty}(\partial\Omega)$ and $\beta(z) \geqslant 0$ for all $z \in \partial\Omega$. Using the spectral theorem for compact self-adjoint operators, we show that problem (2) has a sequence $\{\hat{\lambda}_n\}_{n\geqslant1} \subseteq \mathbb{R}_+$ of eigenvalues such that $\hat{\lambda}_n \to +\infty$. Also, there is a corresponding sequence $\{\hat{u}_n\}_{n\geqslant1} \subseteq H^1(\Omega)$ of eigenfunctions, which form an orthonormal basis of $H^1(\Omega)$ and an orthogonal basis of $L^2(\Omega)$. For every $n \in \mathbb{N}$, by $E(\hat{\lambda}_n)$ we denote the eigenspace corresponding to the eigenvalue $\hat{\lambda}_n$. The following facts are known for these eigenspaces and their corresponding eigenvalues (see Papageorgiou and Rădulescu [14,15]):

- $E(\lambda_n)$ is finite dimensional and $E(\lambda_n) \subseteq C^1(\overline{\Omega})$;
- The elements of $E(\hat{\lambda}_n)$ have the so-called "unique continuation property" (UCP for short), namely if $u \in E(\hat{\lambda}_n)$ and $u(\cdot)$ vanishes on a set of positive Lebesgue measure, then $u \equiv 0$;

- $\hat{\lambda}_1$ is simple, the elements of $E(\hat{\lambda}_1)$ do not change sign while the elements of $E(\hat{\lambda}_n)\backslash\{0\}$ ($n \geqslant 2$) are nodal (sign changing) and
$$H^1(\Omega) = \overline{\underset{\mathrm{n}\geqslant 1}{\oplus} E(\hat{\lambda}_n)}.$$
For every $n \in \mathbb{N}$, let
$$\bar{H}_n = \overset{n}{\underset{\mathrm{i=1}}{\oplus}} E(\hat{\lambda}_i) \text{ and } \hat{H}_n = \overline{\underset{\mathrm{i}\geqslant \mathrm{n}}{\oplus} E(\hat{\lambda}_i)}$$

and let $\vartheta : H^1(\Omega) \to \mathbb{R}$ be the C^1-functional defined by
$$\vartheta(u) = ||Du||_2^2 + \int_{\partial\Omega} \beta(z)u^2 d\sigma \quad \text{for all } u \in H^1(\Omega).$$

The eigenvalues $\{\hat{\lambda}_n\}_{n\geqslant 1}$ admit the following variational characterizations:
$$\hat{\lambda}_1 = \inf\left[\frac{\vartheta(u)}{||u||_2^2} : u \in H^1(\Omega), u \neq 0\right] \tag{3}$$
$$\begin{aligned}\hat{\lambda}_n &= \inf\left[\frac{\vartheta(u)}{||u||_2^2} : u \in \hat{H}_n, u \neq 0\right]\\ &= \sup\left[\frac{\vartheta(u)}{||u||_2^2} : u \in \bar{H}_n, u \neq 0\right],\ n \geqslant 2.\end{aligned} \tag{4}$$

In (3) the infimum is realized on the one dimensional eigenspace $E(\hat{\lambda}_1)$. In (4) both the infimum and the supremum are realized on the eigenspace $E(\hat{\lambda}_n)$.

In addition to (2), we will also consider the following linear eigenvalue problem:
$$-\Delta u(z) = \tilde{\lambda} m(z)u(z) \quad \text{in } \Omega, \quad \frac{\partial u}{\partial n} + \beta(z)u = 0 \quad \text{on } \partial\Omega. \tag{5}$$

In this problem, $m \in L^\infty(\Omega)$, $m(z) \geqslant 0$ for almost all $z \in \Omega$, $m \not\equiv 0$. Again we have a sequence of eigenvalues $\{\tilde{\lambda}_n(m)\}_{n\geq 1}$ such that $\tilde{\lambda}_n(m) \to +\infty$ as $n \to \infty$. In this case in the variational characterizations of the eigenvalues, the Rayleigh quotient has the form
$$\frac{\vartheta(u)}{\int_\Omega mu^2 dz} \quad \text{for all } u \in H^1(\Omega),\ u \neq 0.$$

The following strict monotonicity property of the function $m \mapsto \tilde{\lambda}_n(m)$ is an easy consequence of the UCP.

Lemma 5. *If $n \geqslant 2$, m, $\hat{m} \in L^\infty(\Omega)_+\backslash\{0\}$, $m(z) \leqslant \hat{m}(z)$ for almost all $z \in \Omega$, $m \not\equiv \hat{m}$, then $\tilde{\lambda}_n(\hat{m}) < \tilde{\lambda}_n(m)$.*

Another straightforward consequence of the UCP is the following lemma (see Papageorgiou and Rădulescu [16]).

Lemma 6. (a) *If* $\eta \in L^\infty(\Omega), \eta(z) \geqslant \hat{\lambda}_n$ $(n \in \mathbb{N})$ *for almost all* $z \in \Omega$, $\eta \not\equiv \hat{\lambda}_n$, *then there exists* $c_1 > 0$ *such that*

$$\vartheta(u) - \int_\Omega \eta(z) u^2 dz \leqslant -c_1 ||u||^2 \quad \textit{for all } u \in \bar{H}_n.$$

(b) *If* $\eta \in L^\infty(\Omega)$, $\eta(z) \leqslant \hat{\lambda}_n$ $(n \in \mathbb{N})$ *for almost all* $z \in \Omega$, $\eta \not\equiv \hat{\lambda}_n$, *there exists* $c_2 > 0$ *such that*

$$\vartheta(u) - \int_\Omega \eta(z) u^2 dz \geqslant c_2 ||u||^2 \quad \textit{for all } u \in \hat{H}_n.$$

For X a Banach space, $\varphi \in C^1(X, \mathbb{R})$ and $c \in \mathbb{R}$, we introduce the following sets:

$K_\varphi = \{u \in X : \varphi'(u) = 0\}$ (the critical set of φ),
$K_\varphi^c = \{u \in K_\varphi : \varphi(u) = c\}$ (the critical set of φ at the level $c \in \mathbb{R}$),
$\varphi^c = \{u \in X : \varphi(u) \leqslant c\}$ (the sublevel set of φ at $c \in \mathbb{R}$).

Let (Y_1, Y_2) be a topological pair such that $Y_2 \subseteq Y_1 \subseteq X$. For every $k \in \mathbb{N}_0$ by $H_k(Y_1, Y_2)$ we denote the kth relative singular homology group for the pair (Y_1, Y_2) with integer coefficients. Let $u_0 \in K_\varphi^c$ be isolated. Then the critical groups of φ at u_0 are defined by

$$C_k(\varphi, u_0) = H_k(\varphi^c \cap U, \varphi^c \cap U \backslash \{u_0\}) \quad \text{for all } k \in \mathbb{N}_0.$$

Here U is a neighborhood of u_0 such that $K_\varphi \cap \varphi^c \cap U = \{u_0\}$. The excision property of singular homology implies that this definition of critical groups is independent of the choice of the neighborhood U.

Finally we mention that by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$ and for every $u \in H^1(\Omega)$, $u^\pm = \max\{\pm u, 0\} \in H^1(\Omega)$. Moreover, $A \in \mathcal{L}(H^1(\Omega), H^1(\Omega)^*)$ is the linear operator defined by

$$\langle A(u), h \rangle = \int_\Omega (Du, Dh)_{\mathbb{R}^N} \, dz \quad \text{for all } u, h \in H^1(\Omega).$$

3. Near Resonance from the Right of $\hat{\lambda}_m$

In this section we examine what happens as we approach the eigenvalue $\hat{\lambda}_m$ from bigger values (from the right). So, we introduce the following conditions on the perturbation term $f(z, x)$. In what follows, we set $F(z, x) = \int_0^x f(z, s) ds$.

H_1 : $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

(i) for every $\rho > 0$, there exists $a_\rho \in L^\infty(\Omega)_+$ such that

$$|f(z, x)| \leqslant a_\rho(z) \text{ for almost all } z \in \Omega, \text{ all } |x| \leqslant \rho;$$

(ii) there exists a function $\eta \in L^\infty(\Omega)$ such that

$$0 \leqslant \eta(z) \leqslant \hat{\lambda}_{m+1} - \hat{\lambda}_m \text{ for almost all } z \in \Omega,\ \eta \not\equiv \hat{\lambda}_{m+1} - \hat{\lambda}_m\ (m \geqslant 2),$$
$$0 \leqslant \liminf_{x\to\pm\infty} \frac{f(z,x)}{x} \leqslant \limsup_{x\to\pm\infty} \frac{f(z,x)}{x}$$
$$\leqslant \eta(z) \text{ uniformly for almost all } z \in \Omega;$$

(iii) there exist $\tau \in (1,3)$ and functions $\eta_+,\ \eta_- \in L^\infty(\Omega)$ such that

$$\eta_+(z) = \liminf_{x\to+\infty} \frac{f(z,x)x - 2F(z,x)}{x^{\tau-1}} \text{ uniformly for almost all } z \in \Omega,$$
$$\eta_-(z) = \limsup_{x\to-\infty} \frac{f(z,x)x - 2F(z,x)}{|x|^{\tau-2}x} \text{ uniformly for almost all } z \in \Omega,$$

and for all $u \in E(\hat{\lambda}_m), u \neq 0$ we have

$$0 < \int_\Omega \eta_+(z)(u^+)^{\tau-1}dz - \int_\Omega \eta_-(z)(u^-)^{\tau-1}dz.$$

Remark 2. These hypotheses permit resonance at $\pm\infty$ with respect to $\hat{\lambda}_m$ (see hypothesis $H_1(ii)$). Hypothesis $H_1(iii)$ is a kind of generalized Landesman–Lazer condition and is more general than the condition used by Ke and Tang [5] where $\tau = 2$ (see hypothesis (F_-) in [5]). Consider a function $f(x)$ (for the sake of simplicity, we drop the z-dependence) such that

$$f(x) = \begin{cases} \eta x - \frac{1}{x^{1/2}} & \text{for all } x \geqslant 1 \\ \eta x + \frac{1}{|x|^{1/2}} & \text{for all } x \leqslant -1 \end{cases}$$

with $\eta \in [0, \hat{\lambda}_{m+1} - \hat{\lambda}_m)$ (in the interval $(-1,1)$, $f(\cdot)$ can be anything that preserves the continuity of $f(\cdot)$). This function satisfies hypotheses H_1 but does not fit in the framework of [3,5]. Another possibility is the function $f(x) = -|x|^{q-2}x$ with $1 < q < 2$. Finally we mention that we do not need any conditions on $f(z,\cdot)$ near zero.

For the rest of the paper, the following hypothesis concerning the boundary coefficient $\beta(\cdot)$ is in effect:

$H(\beta) : \beta \in W^{1,\infty}(\partial\Omega)$ and $\beta(z) \geqslant 0$ for all $z \in \partial\Omega$.

Let $\varphi : H^1(\Omega) \to \mathbb{R}$ be the energy (Euler) functional for problem (1) defined by

$$\varphi(u) = \frac{1}{2}\vartheta(u) - \frac{\hat{\lambda}_m}{2}||u||_2^2 - \int_\Omega F(z,u)dz \quad \text{for all } u \in H^1(\Omega).$$

Recall that $\vartheta(u) = ||Du||_2^2 + \int_{\partial\Omega} \beta(z)u^2 d\sigma$ for all $u \in H^1(\Omega)$. Evidently $\varphi \in C^1(H^1(\Omega))$.

Proposition 7. *If hypotheses $H(\beta)$, H_1 hold, then the functional φ satisfies the C-condition.*

Proof. Let $\{u_n\}_{n\geqslant 1} \subseteq H^1(\Omega)$ be a sequence such that

$$|\varphi(u_n)| \leqslant M_1 \text{ for some } M_1 > 0, \quad \text{all } n \in \mathbb{N}, \tag{6}$$

$$(1 + ||u_n||)\varphi'(u_n) \to 0 \quad \text{in } H^1(\Omega)^* \text{ as } n \to \infty. \tag{7}$$

From (7) we have

$$\begin{aligned} &|\langle \varphi'(u_n), h\rangle| \leqslant \frac{\epsilon_n ||h||}{1 + ||u_n||} \quad \text{for all } h \in H^1(\Omega) \text{ with } \epsilon_n \to 0^+, \\ &\Rightarrow \left| \langle A(u_n), h\rangle + \int_{\partial\Omega} \beta(z) u_n h d\sigma - \hat{\lambda}_m \int_\Omega u_n h dz - \int_\Omega f(z, u_n) h dz \right| \leqslant \frac{\epsilon_n ||h||}{1 + ||u_n||} \\ &\text{for all } n \in \mathbb{N}. \end{aligned} \tag{8}$$

Claim 1. *The sequence $\{u_n\}_{n\geqslant 1} \subseteq H^1(\Omega)$ is bounded.*

Arguing by contradiction, suppose that the Claim is not true. By passing to a subsequence if necessary, we may assume that $||u_n|| \to \infty$. Let $y_n = \frac{u_n}{||u_n||}$, $n \geqslant 1$. Then $||y_n|| = 1$ for all $n \in \mathbb{N}$ and so we may assume that

$$y_n \xrightarrow{w} y \text{ in } H^1(\Omega) \text{ and } y_n \to y \text{ in } L^2(\Omega) \text{ and in } L^2(\partial\Omega). \tag{9}$$

From (8) we have

$$\begin{aligned} &\left| \langle A(y_n), h\rangle + \int_{\partial\Omega} \beta(z) y_n h d\sigma - \hat{\lambda}_m \int_\Omega y_n h dz - \int_\Omega \frac{f(z, u_n)}{||u_n||} h dz \right| \leqslant \frac{\epsilon_n ||h||}{(1 + ||u_n||)||u_n||} \\ &\text{for all } n \in \mathbb{N}. \end{aligned} \tag{10}$$

Hypotheses $H_1(i), (ii)$ imply that

$$\begin{aligned} &|f(z, x)| \leqslant c_3(1 + |x|) \quad \text{for almost all } z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ some } c_3 > 0, \\ &\Rightarrow \left\{ \frac{f(\cdot, u_n(\cdot))}{||u_n||} \right\}_{n\geqslant 1} \subseteq L^2(\Omega) \text{ is bounded.} \end{aligned}$$

So, by passing to a subsequence if necessary and using hypothesis $H_1(ii)$, we have

$$\frac{f(\cdot, u_n(\cdot))}{||u_n||} \xrightarrow{w} \eta_0 y \text{ in } L^2(\Omega) \text{ with } 0 \leqslant \eta_0(z) \leqslant \eta(z) \quad \text{for almost all } z \in \Omega \tag{11}$$

(see Aizicovici, Papageorgiou and Staicu [1], proof of Proposition 14). In (10) we pass to the limit as $n \to \infty$ and use (9), (11). We obtain

$$\begin{aligned} &\langle A(y), h\rangle + \int_{\partial\Omega} \beta(z) y h d\sigma = \int_\Omega [\hat{\lambda}_m + \eta_0(z)] y h dz \quad \text{for all } h \in H^1(\Omega), \\ &\Rightarrow -\Delta y(z) = (\hat{\lambda}_m + \eta_0(z)) y(z) \quad \text{for almost all } z \in \Omega, \ \frac{\partial y}{\partial n} + \beta(z) y = 0 \text{ on } \partial\Omega \\ &\text{(see Papageorgiou and Rădulescu [14]).} \end{aligned} \tag{12}$$

First suppose that $\eta_0 \not\equiv 0$. Then

$$\hat{\lambda}_m \not\equiv \hat{\lambda}_m + \eta_0 \text{ and } \hat{\lambda}_m + \eta_0 \not\equiv \hat{\lambda}_{m+1} \text{ (see hypothesis } H_1(ii) \text{ and (11))}. \quad (13)$$

Using (13) and Lemma 5, we have

$$\tilde{\lambda}_m(\hat{\lambda}_m + \eta_0) < \tilde{\lambda}_m(\hat{\lambda}_m) = 1 \text{ and } 1 = \tilde{\lambda}_{m+1}(\hat{\lambda}_{m+1}) < \tilde{\lambda}_{m+1}(\hat{\lambda}_m + \eta_0). \quad (14)$$

From (12) and (14) it follows that $y = 0$.

On the other hand, if in (10) we choose $h = y_n - y \in H^1(\Omega)$, pass to the limit as $n \to \infty$ and use (9) and (11), then

$$\begin{aligned}
&\lim_{n\to\infty} \langle A(y_n), y_n - y\rangle = 0,\\
&\Rightarrow ||Dy_n||_2 \to ||Dy||_2,\\
&\Rightarrow Dy_n \to Dy \text{ in } L^2(\Omega, \mathbb{R}^N)\\
&\quad \text{(by the Kadec–Klee property, see Gasinski and Papageorgiou [4, p. 911])}\\
&\Rightarrow y_n \to y \text{ in } H^1(\Omega) \text{ (see (9)), hence } ||y|| = 1, \text{ a contradiction.}
\end{aligned}$$

Next suppose that $\eta_0(z) = 0$ for almost all $z \in \Omega$. From the previous argument we have that $y_n \to y$ in $H^1(\Omega)$ and so $||y|| = 1$. Also, from (12) we have $y \in E(\hat{\lambda}_m)\backslash\{0\}$. So, by the UCP it follows that $y(z) \neq 0$ for almost all $z \in \Omega$ and so

$$|u_n(z)| \to +\infty \text{ for almost all } z \in \Omega \text{ as } n \to \infty. \quad (15)$$

From (6) we have

$$||Du||_2^2 + \int_{\partial\Omega} \beta(z) u_n^2 d\sigma - \hat{\lambda}_m ||u_n||_2^2 - \int_\Omega 2F(z, u_n)dz \leqslant 2M_1 \quad \text{for all } n \in \mathbb{N}. \quad (16)$$

Also, in (8) we choose $h = u_n \in H^1(\Omega)$. Then

$$-||Du_n||_2^2 - \int_{\partial\Omega} \beta(z) u_n^2 d\sigma + \hat{\lambda}_m ||u_n||_2^2 + \int_\Omega f(z, u_n) u_n dz \leqslant \epsilon_n \quad \text{for all } n \in \mathbb{N}. \quad (17)$$

We add (16) and (17) and multiply with $\frac{1}{||u_n||^{\tau-1}}$. Then

$$\frac{1}{||u_n||^{\tau-1}} \int_\Omega [f(z, u_n)u_n - 2F(z, u_n)]dz \leqslant \frac{M_2}{||u_n||^{\tau-1}} \text{ for some } M_2 > 0, \text{ all } n \in \mathbb{N}. \quad (18)$$

Note that

$$\begin{aligned}
&\frac{1}{||u_n||^{\tau-1}} \int_\Omega [f(z, u_n)u_n - 2F(z, u_n)]dz\\
&\quad = \int_{\{u_n>0\}} \frac{f(z, u_n)u_n - 2F(z, u_n)}{u_n^{\tau-1}} y_n^{\tau-1} dz\\
&\qquad - \int_{\{u_n<0\}} \frac{f(z, u_n)u_n - 2F(z, u_n)}{|u_n|^{\tau-2}u_n} |y_n|^{\tau-1} dz.
\end{aligned}$$

Using (15), hypothesis $H_1(iii)$ and Fatou's lemma, we obtain

$$\liminf_{n\to\infty} \frac{1}{||u_n||^{\tau-1}} \int_\Omega [f(z,u_n)u_n - 2F(z,u_n)]dz$$
$$\geqslant \int_\Omega \eta_+(z)(y^+)^{\tau-1}dz - \int_{\partial\Omega} \eta_-(z)(y^-)^{\tau-1}dz,$$
$$\Rightarrow \int_\Omega \eta_+(z)(y^+)^{\tau-1}dz - \int_\Omega \eta_-(z)(y^-)^{\tau-1}dz \leqslant 0,\ y \in E(\hat{\lambda}_m)\backslash\{0\}$$
$$\text{(see (18) and recall that } ||u_n|| \to \infty) \tag{19}$$

Comparing (19) with hypothesis $H_1(iii)$, we reach a contradiction.
This proves the Claim.
Because of the Claim, we may assume that

$$u_n \xrightarrow{w} u \text{ in } H^1(\Omega) \text{ and } u_n \to u \text{ in } L^2(\Omega) \text{ and in } L^2(\partial\Omega). \tag{20}$$

In (8) we choose $h = u_n - u \in H^1(\Omega)$, pass to the limit as $n \to \infty$ and use (20). Then

$$\lim_{n\to\infty} \langle A(u_n), u_n - u\rangle = 0,$$
$$\Rightarrow u_n \to u \text{ in } H^1(\Omega) \text{ (as before via the Kadec-Klee property)},$$
$$\Rightarrow \varphi \text{ satisfies the C-condition.}$$

□

Recall that $\bar{H}_{m-1} = \oplus_{\mathrm{k}=1}^{m-1} E(\hat{\lambda}_k)$ and $\hat{H}_{m+1} = \overline{\underset{\mathrm{k}\geqslant \mathrm{m}+1}{\oplus} E(\hat{\lambda}_k)}$. We have the following orthogonal direct sum decomposition:

$$H^1(\Omega) = \bar{H}_{m-1} \oplus E(\hat{\lambda}_m) \oplus \hat{H}_{m+1}.$$

Proposition 8. *If hypotheses $H(\beta), H_1$ hold, then $\varphi|_{\bar{H}_{m-1}}$ is anticoercive (that is, $\varphi(u) \to -\infty$ as $||u|| \to \infty, u \in \bar{H}_{m-1}$).*

Proof. Hypotheses $H_1(i),(ii)$ imply that given $\epsilon > 0$, we can find $c_4 = c_4(\epsilon) > 0$ such that

$$F(z,x) \geqslant -\frac{\epsilon}{2}x^2 - c_4 \quad \text{for almost all } z \in \Omega,\ \text{all } x \in \mathbb{R}. \tag{21}$$

Let $u \in \bar{H}_{m-1}$. Then

$$\varphi(u) = \frac{1}{2}\vartheta(u) - \frac{\hat{\lambda}_m}{2}||u||_2^2 - \int_\Omega F(z,u)dz$$
$$\leqslant \frac{1}{2}\vartheta(u) - \frac{\hat{\lambda}_m - \epsilon}{2}||u||_2^2 + c_4|\Omega|_N.$$

Choosing $\epsilon \in (0, \hat{\lambda}_m - \hat{\lambda}_{m-1})$, we have

$$\varphi(u) \leqslant -\frac{c_0}{2}||u||_2^2 + c_4|\Omega|_N \text{ (see Lemma 6(a))},$$
$$\Rightarrow \varphi|_{\bar{H}_{m-1}} \text{ is anticoercive.}$$

The proof is complete. □

The above proposition implies that we can find $c_5 > 0$ such that

$$\sup[\varphi(u) : u \in \bar{H}_{m-1}] \leqslant c_5.$$

Proposition 9. *If hypotheses $H(\beta)$, H_1 hold, then we can find $\delta_0 > 0$ such that when $0 \leqslant \eta(z) \leqslant \delta_0$ for almost all $z \in \Omega$ and $0 < \gamma < \frac{1}{3-\tau}$ we have*

$$\varphi(u) > c_5 + 1 \textit{ for all } u \in E(\hat{\lambda}_m) \oplus \hat{H}_{m+1}, \ ||u|| = \frac{1}{\delta_0^\gamma}.$$

Proof. We argue by contradiction. So, suppose that the proposition is not true. Then for a sequence $\delta_n \to 0^+$ we can find $u_n \in E(\hat{\lambda}_m) \oplus \hat{H}_{m+1}$ ($n \in \mathbb{N}$) such that

$$\varphi(u_n) \leqslant c_5 + 1 \text{ and } 0 \leqslant \eta(z) \leqslant \delta_n \quad \text{for almost all } z \in \Omega, ||u_n|| = \frac{1}{\delta_n^\gamma} \quad \text{for all } n \in \mathbb{N}. \tag{22}$$

Let $y_n = \frac{u_n}{||u_n||}$, $n \in \mathbb{N}$. Then $||y_n|| = 1$ for all $n \in \mathbb{N}$ and so we may assume that

$$y_n \xrightarrow{w} y \text{ in } H^1(\Omega) \text{ and } y_n \to y \text{ in } L^2(\Omega) \text{ and in } L^2(\partial\Omega). \tag{23}$$

From (22) we have

$$\frac{1}{2}\vartheta(y_n) - \frac{\hat{\lambda}_m}{2}||y_n||_2^2 - \int_\Omega \frac{F(z, u_n)}{||u_n||^2} dz \leqslant \frac{c_5 + 1}{||u_n||^2} \quad \text{for all } n \in \mathbb{N}. \tag{24}$$

Recall that

$$\begin{aligned} &|F(z,x)| \leqslant c_6(1 + x^2) \text{ for almost all } z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ some } c_6 > 0 \\ &\quad \text{(see hypotheses } H_1(i), (ii)), \\ &\Rightarrow \left\{ \frac{F(\cdot, u_n(\cdot))}{||u_n||^2} \right\}_{n \geqslant 1} \subseteq L^1(\Omega) \text{ is uniformly integrable.} \end{aligned}$$

Then from the Dunford–Pettis theorem, we may assume that

$$\frac{F(\cdot, u_n(\cdot))}{||u_n||^2} \xrightarrow{w} g \quad \text{in } L^1(\Omega).$$

Moreover, since $\delta_n \to 0^+$, from hypothesis $H_1(ii)$ and (22), we see that $g = 0$. Hence, if in (24) we pass to the limit as $n \to \infty$, then

$$\begin{aligned} \vartheta(y) &\leqslant \hat{\lambda}_m ||y||_2^2 \text{ (see (23))}, \\ &\Rightarrow \vartheta(y) = \hat{\lambda}_m ||y||_2^2 \text{ (since } y \in E(\hat{\lambda}_m) \oplus \hat{H}_{m+1}, \text{ see (4))}, \\ &\Rightarrow y \in E(\hat{\lambda}_m). \end{aligned} \tag{25}$$

We have $y_n = y_n^0 + \hat{y}_n$ with $y_n^0 \in E(\hat{\lambda}_m)$, $\hat{y}_n \in \hat{H}_{m+1}$. Exploiting the orthogonality of the component spaces, from (24) we have

$$\frac{1}{2}\vartheta(\hat{y}_n) - \frac{\hat{\lambda}_m}{2}||\hat{y}_n||_2^2 - \int_\Omega \frac{F(z,u_n)}{||u_n||^2}dz \leqslant \frac{c_5+1}{||u_n||^2} \quad \text{for all } n \in \mathbb{N},$$

$$\Rightarrow \frac{c_1}{2}||\hat{y}_n||^2 - \int_\Omega \frac{F(z,u_n)}{||u_n||^2}dz \leqslant \frac{c_5+1}{||u_n||^2} \quad \text{for all } n \in \mathbb{N} \text{ (see Lemma 6(b))},$$

$$\Rightarrow \hat{y}_n \to 0 \text{ in } H^1(\Omega).$$

Recalling that $E(\hat{\lambda}_m)$ is finite dimensional, we finally have

$$y_n \to y = y^0 \in E(\hat{\lambda}_m) \text{ in } H^1(\Omega) \text{ (see (25))},$$
$$\Rightarrow ||y|| = 1 \text{ and so } y \in E(\hat{\lambda}_m)\backslash\{0\}.$$

The UCP implies that $y(z) \neq 0$ for almost all $z \in \Omega$ and so it follows that

$$|u_n(z)| \to +\infty \text{ for almost all } z \in \Omega.$$

Hypothesis $H_1(iii)$ implies that given $\epsilon > 0$, we can find $M_3 = M_3(\epsilon) > 0$ such that

$$f(z,x)x - 2F(z,x) \geqslant (\eta_+(z) - \epsilon)x^{\tau-1} \quad \text{for almost all } z \in \Omega, \text{ all } x \geqslant M_3, \tag{26}$$

$$f(z,x)x - 2F(z,x) \geqslant (\eta_-(z) + \epsilon)|x|^{\tau-2}x \quad \text{for almost all } z \in \Omega, \text{ all } x \leqslant -M_3. \tag{27}$$

For almost all $z \in \Omega$ and all $x \geqslant M_3$, we have

$$\begin{aligned}
\frac{d}{dx}\left(\frac{F(z,x)}{x^2}\right) &= \frac{f(z,x)x^2 - 2F(z,x)x}{x^4} \\
&= \frac{f(z,x)x - 2F(z,x)}{x^3} \\
&\geqslant \frac{\eta_+(z) - \epsilon}{x^{4-\tau}} \text{ (see (26))} \\
&= \frac{\eta_+(z) - \epsilon}{3-\tau}\ \frac{d}{dx}\left(-\frac{1}{x^{3-\tau}}\right), \\
&\Rightarrow \frac{F(z,x)}{x^2} - \frac{F(z,y)}{y^2} \geqslant \frac{\eta_+(z) - \epsilon}{3-\tau} \\
&\quad \left[-\frac{1}{x^{3-\tau}} + \frac{1}{y^{3-\tau}}\right] \text{ for almost all } z \in \Omega, \\
&\quad \text{all } x \geqslant y \geqslant M_3.
\end{aligned}$$

Letting $x \to +\infty$ and using hypothesis $H_1(ii)$, we obtain

$$\frac{1}{2}\eta(z) - \frac{F(z,x)}{y^2} \geqslant \frac{\eta_+(z) - \epsilon}{3-\tau}\ \frac{1}{y^{3-\tau}} \quad \text{for almost all } z \in \Omega, \text{ all } y \geqslant M_3,$$

$$\Rightarrow \frac{1}{2}\eta(z)y^2 - F(z,y) \geqslant \frac{\eta_+(z) - \epsilon}{3-\tau}y^{\tau-1} \quad \text{for almost all } z \in \Omega, \text{ all } y \geqslant M_3. \tag{28}$$

Hypothesis $H_1(i)$ implies that

$$\frac{1}{2}\eta(z)y^2 - F(z,y) \geqslant -c_7 \text{ for almost all } z \in \Omega, \text{ all } y \in [0, M_3]. \tag{29}$$

For almost all $z \in \Omega$ and all $x \leqslant -M_3$ we have

$$\begin{aligned}
\frac{d}{dx}\left(\frac{F(z,x)}{x^2}\right) &= \frac{f(z,x)x - 2F(z,x)}{|x|^2 x} \\
&\leqslant \frac{\eta_-(z)+\epsilon}{|x|^{4-\tau}} \text{ (see (27))}, \\
&= \frac{\eta_-(z)+\epsilon}{3-\tau}\ \frac{d}{dx}\left(-\frac{1}{|x|^{2-\tau}x}\right) \\
&\Rightarrow \frac{F(z,x)}{x^2} - \frac{F(z,y)}{y^2} \leqslant \frac{\eta_-(z)+\epsilon}{3-\tau}\left[-\frac{1}{|x|^{2-\tau}x} + \frac{1}{|y|^{2-\tau}y}\right] \\
&\quad \text{for almost all } z \in \Omega, \text{ all } y \leqslant x \leqslant -M_3.
\end{aligned}$$

Letting $y \to -\infty$ and using hypothesis $H_1(ii)$, we obtain

$$\begin{aligned}
&\frac{F(z,x)}{x^2} - \frac{1}{2}\eta(z) \leqslant \frac{\eta_-(z)+\epsilon}{3-\tau}\left(-\frac{1}{|x|^{2-\tau}x}\right) \text{ for almost all } z \in \Omega, \text{ all } x \leqslant -M_3, \\
&\Rightarrow \frac{1}{2}\eta(z)x^2 - F(z,x) \geqslant -\frac{\eta_+(z)+\epsilon}{3-\tau}|x|^{\tau-1} \text{ for almost all } z \in \Omega, \text{ all } x \leqslant -M_3.
\end{aligned} \tag{30}$$

In addition hypothesis $H_1(i)$ implies that

$$\frac{1}{2}\eta(z)x^2 - F(z,x) \geqslant -c_8 \text{ for almost all } z \in \Omega, \text{ all } x \in [-M_3, 0], \text{ some } c_8 > 0. \tag{31}$$

From (21) we have

$$\begin{aligned}
&\frac{1}{||u_n||^{\tau-1}}\left[\frac{1}{2}\vartheta(u_n) - \frac{\hat{\lambda}_m}{2}||u_n||_2^2 - \int_\Omega F(z,u_n)dz\right] \leqslant \frac{c_5+1}{||u_n||^{\tau-1}}, \\
&\Rightarrow \frac{1}{||u_n||^{\tau-1}}\left[-\int_{\{u_n \geqslant M_3\}} F(z,u_n)dz - \int_{\{0 \leqslant u_n < M_3\}} F(z,u_n)dz\right. \\
&\quad \left. - \int_{\{-M_3 < u_n < 0\}} F(z,u_n)dz - \int_{\{u_n \leqslant -M_3\}} F(z,u_n)dz\right] \leqslant \frac{c_5+1}{||u_n||^{\tau-1}} \\
&\quad (\text{since } u_n \in E(\hat{\lambda}_m) \oplus \hat{H}_{n+1}, \text{ see (4)}) \\
&\Rightarrow -\frac{1}{2||u_n||^{\tau-1}}\int_\Omega \eta(z)u_n^2 dz + \int_{\{u_n \geqslant M_3\}} \frac{\eta_+(z)-\epsilon}{3-\tau} y_n^{\tau-1} dz - \frac{c_7|\Omega|_N}{||u_n||^{\tau-1}} \\
&\quad - \int_{\{u_n \leqslant -M_3\}} \frac{\eta_-(z)+\epsilon}{3-\tau}|y_n|^{\tau-1} dz - \frac{c_8|\Omega|}{||u_n||^{\tau-1}}
\end{aligned}$$

$$\leqslant \frac{c_5+1}{||u_n||^{\tau-1}} \quad \text{for all } n \in \mathbb{N} \text{ (see (28), (29), (30), (31)).} \tag{32}$$

Note that

$$\begin{aligned}\frac{1}{2||u_n||^{\tau-1}}\int_\Omega \eta(z)u_n^2dz \leqslant \frac{\delta_n}{2||u_n||^{\tau-1}}||u_n||^2 &= \frac{1}{2}\delta_n||u_n||^{3-\tau}\\ &= \frac{1}{2}\delta_n\frac{1}{\delta_n^{(3-\tau)\gamma}} \text{ (see (22)).}\end{aligned}$$

Since $\gamma < \frac{1}{3-\tau}$ it follows that

$$\frac{1}{2||u_n||^{\tau-1}}\int_\Omega \eta(z)u_2^2dz \to 0.$$

So, if in (32) we pass to the limit as $n\to\infty$ and recall that $|u_n(z)|\to+\infty$ for almost all $z\in\Omega$, then we obtain

$$\frac{1}{3-\tau}\left[\int_\Omega(\eta_+(z)-\epsilon)(y^+)^{\tau-1}dz-\int_\Omega(\eta_-(z)+\epsilon)(y^-)^{\tau-1}dz\right]\leqslant 0.$$

Since $\epsilon>0$ is arbitrary, we let $\epsilon\to 0^+$ and have

$$\frac{1}{3-\tau}\left[\int_\Omega\eta_+(z)(y^+)^{\tau-1}dz-\int_\Omega\eta_-(z)(y^-)^{\tau-1}dz\right]\leqslant 0,\ y\in E(\hat{\lambda}_m)\backslash\{0\},$$

which contradicts hypothesis $H_1(iii)$. This proves the proposition. □

Proposition 10. *If hypotheses $H(\beta)$, H_1 hold and $\delta_0>0$ is as postulated by Proposition 9, then for every $\eta\in L^\infty(\Omega)$ satisfying $0\leqslant\eta(z)\leqslant\delta_0$ for almost all $z\in\Omega$, problem (1) admits a solution $u_0\in C^1(\overline{\Omega})$ such that $\varphi(u_0)\leqslant c_5$.*

Proof. Let $R_0=\frac{1}{\delta_0^\gamma}$. Invoking Proposition 8 we can find $r_0>0$ such that

$$\begin{aligned}&\sup\left[\varphi(u): u\in\bar{H}_{m-1}, ||u||=r_0\right]\\ &\quad< a=\inf\left[\varphi(u): u\in E(\hat{\lambda}_m)\oplus\hat{H}_{m+1}, ||u||\leqslant R_0\right]\\ &\quad\leqslant b=\sup\left[\varphi(u): u\in\bar{H}_{m-1}, ||u||\leqslant r_0\right] \text{ (since } a\leqslant\varphi(0)\leqslant b)\\ &\quad\leqslant\sup\left[\varphi(u): u\in\bar{H}_{m-1}\right]\\ &\quad\leqslant c_5<c_5+1\leqslant\inf\left[\varphi(u): u\in E(\hat{\lambda}_m)\oplus\hat{H}_{m+1}: ||u||=R_0\right] \text{ (see Proposition 9).}\end{aligned}$$

From Proposition 7 we know that φ satisfies the C-condition. So, we can apply Theorem 4 and find $u_0\in H^1(\Omega)$ such that

$$u_0\in K_\varphi \quad\text{and}\quad \varphi(u_0)\in[a,b]. \tag{33}$$

From (33) and Green's identity, we conclude that u_0 is a solution of (1) (see Papageorgiou and Rădulescu [14]) and the regularity theory implies that $u_0\in C^1(\overline{\Omega})$. □

By strengthening hypothesis $H_1(ii)$ we can produce a second solution $\hat{u} \neq u_0$.

The new conditions on the perturbation term $f(z,x)$, are the following:

H_2 : $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, hypotheses $H_2(i),(iii)$ are the same as the corresponding hypotheses $H_1(i),(iii)$ and

(ii) there exist functions $\hat{\eta}, \eta \in L^\infty(\Omega)$ such that

$$0 \leqslant \hat{\eta}(z) \leqslant \eta(z) \leqslant \hat{\lambda}_{m+1} - \hat{\lambda}_m \quad \text{for almost all } z \in \Omega,\ \hat{\eta} \not\equiv 0, \eta \not\equiv \hat{\lambda}_{m+1} - \hat{\lambda}_m,$$

$$\hat{\eta}(z) \leqslant \liminf_{x\to\pm\infty} \frac{f(z,x)}{x} \leqslant \limsup_{x\to\pm\infty} \frac{f(z,x)}{x} \leqslant \eta(z) \text{ uniformly for almost all } z \in \Omega.$$

Proposition 11. *If hypotheses $H(\beta)$, H_2 hold, then there exists $\delta_0 > 0$ such that if $\eta(z) \leqslant \delta_0$ for almost all $z \in \Omega$, problem (1) admits a second solution $\hat{u} \in C^1(\overline{\Omega})$, $\hat{u} \neq u_0$.*

Proof. By virtue of hypotheses $H_2(i),(ii)$, given $\epsilon > 0$, we can find $c_8 = c_8(\epsilon) > 0$ such that

$$\frac{1}{2}(\hat{\eta}(z) - \epsilon)x^2 - c_8 \leqslant F(z,x) \text{ for almost all } z \in \Omega,\ \text{all } x \in \mathbb{R}. \tag{34}$$

Let $u \in \bar{H}_{m-1} \oplus E(\hat{\lambda}_m)$. We have

$$\begin{aligned}\varphi(u) &= \frac{1}{2}\vartheta(u) - \frac{\hat{\lambda}_m}{2}||u||_2^2 - \int_\Omega F(z,u)dz \\ &\leqslant \frac{1}{2}\vartheta(u) - \frac{1}{2}\int_\Omega (\hat{\lambda}_m + \hat{\eta}(z))u^2dz + \frac{\epsilon}{2}||u||^2 + c_8|\Omega|_N.\end{aligned} \tag{35}$$

Note that

$$\hat{\lambda}_m \leqslant \hat{\lambda}_m + \hat{\eta}(z) \text{ for almost all } z \in \Omega \text{ and } \hat{\lambda}_m + \hat{\eta} \not\equiv \hat{\lambda}_m.$$

Since $u \in \bar{H}_{m-1} \oplus E(\hat{\lambda}_m) = \bar{H}_m$, invoking Lemma 6(a) we have

$$\vartheta(u) - \int_\Omega (\hat{\lambda}_m + \eta(z))u^2dz \leqslant -c_1||u||^2.$$

Using this in (35) we obtain

$$\varphi(u) \leqslant \frac{-c_1 + \epsilon}{2}||u||^2 + c_8|\Omega|_N.$$

Choosing $\epsilon \in (0, c_1)$, we infer that

$$\varphi|_{\bar{H}_{m-1}\oplus E(\hat{\lambda}_m) = \bar{H}_m} \text{ is anticoercive.} \tag{36}$$

On the other hand in a similar fashion since

$$\hat{\lambda}_m + \eta(z) \leqslant \hat{\lambda}_{m+1} \text{ for almost all } z \in \Omega \text{ and } \hat{\lambda}_m + \eta \not\equiv \hat{\lambda}_{m+1}$$

and using Lemma 6(b), we infer that

$$\varphi|_{\hat{H}_{m+1}} \text{ is coercive.} \tag{37}$$

On account of (36) and (37) and by choosing $\delta_0 > 0$ form Proposition 9 even smaller if necessary, we can find $\hat{R} > R_0 = \frac{1}{\delta_0^\gamma}$ such that

$$\varphi(u) \geqslant c_5 + 1 \text{ for all } u \in \hat{H}_{m+1} \text{ with } ||u|| \geqslant R_0, \tag{38}$$

$$\varphi(u) \leqslant c_5 \text{ for all } u \in \bar{H}_{m-1} \oplus E(\hat{\lambda}_m) \text{ with } ||u|| = \hat{R}. \tag{39}$$

We introduce the following sets

$$E_0 = \{u \in \bar{H}_{m-1} \oplus E(\hat{\lambda}_m) : ||u|| = \hat{R}\},\ E = \{u \in \bar{H}_{m-1} \oplus E(\hat{\lambda}_m) : ||u|| \leqslant \hat{R}\},$$
$$D = \{u \in \bar{H}_{m+1} : ||u|| \geqslant R_0\} \cup \{u \in E(\hat{\lambda}_m) \oplus \hat{H}_{m+1} : ||u|| = R_0\}.$$

From Lemma 4.6 of de Paiva and Massa [3], we know that $\{E_0, E\}$ is linking with D in $W^{1,p}(\Omega)$. Also, if $\eta(z) \leqslant \delta_0$ for almost all $z \in \Omega$, from Proposition 9 and (38), (39) we have

$$\sup_{E_0} \varphi \leqslant c_5 < c_5 + 1 \leqslant \inf_D \varphi.$$

Recall that φ satisfies the C–condition (see Proposition 7). So, we can apply Theorem 2 and find $\hat{u} \in H^1(\Omega)$ such that

$$\hat{u} \in K_\varphi \quad \text{and} \quad \varphi(\hat{u}) \geqslant c_5 + 1. \tag{40}$$

From (40) it follows that $\hat{u}$ solves problem (1) and $\hat{u} \in C^1(\overline{\Omega})$ (regularity theory). Moreover, from Proposition 10 we have

$$\varphi(u_0) \leqslant c_5 < c_5 + 1 \leqslant \varphi(\hat{u}),$$
$$\Rightarrow \hat{u} \neq u_0.$$

The proof is complete. $\square$

So, we can state the following multiplicity theorem for problems near resonance from the right of $\hat{\lambda}_m,\ m \geqslant 2$.

Theorem 12. *If hypotheses $H(\beta),\ H_2$ hold, then there exists $\delta_0 > 0$ such that if $\eta(z) \leqslant \delta_0$ for almost all $z \in \Omega$, problem (1) admits at least two distinct solutions $u_0,\ \hat{u} \in C^1(\overline{\Omega})$.*

4. Near Resonance from the Left of $\hat{\lambda}_m$

In this section we examine what happens as we approach $\hat{\lambda}_m$ from the left (that is from smaller values). In this case, we impose the following conditions on the perturbation term $f(z, x)$:

$H_3 : f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

(i) for every $\rho > 0$, there exists $a_\rho \in L^\infty(\Omega)_+$ such that

$$|f(z, x)| \leqslant a_\rho(z) \text{ for almost all } z \in \Omega,\ \text{all } |x| \leqslant \rho;$$

(ii) there exist functions $\hat{\eta}, \eta \in L^\infty(\Omega)$ and an integer $m \geqslant 2$ such that

$\hat{\lambda}_{m-1} - \hat{\lambda}_m \leqslant \hat{\eta}(z) \leqslant \eta(z) \leqslant 0$ for almost all $z \in \Omega$, $\hat{\eta} \not\equiv \hat{\lambda}_{m-1} - \hat{\lambda}_m$, $\eta \not\equiv 0$,
$\hat{\eta}(z) \leqslant \liminf\limits_{x \to \pm\infty} \dfrac{f(z,x)}{x} \leqslant \limsup\limits_{x \to \pm\infty} \dfrac{f(z,x)}{x} \leqslant \eta(z)$ uniformly for almost all $z \in \Omega$.

Remark 3. In this case we do not allow any resonance at $\pm\infty$ with respect to the eigenvalue $\hat{\lambda}_m$. Only nonuniform nonresonance (see hypothesis $H_3(ii)$ and compare with hypothesis $H_1(ii)$). On the other hand, hypotheses H_3 above are simpler that hypotheses H_2.

Proposition 13. *If hypotheses $H(\beta)$, H_3 hold, then $\varphi|_{\hat{H}_m}$ is coercive.*

Proof. Hypotheses $H_3(i), (ii)$ imply that given $\epsilon > 0$, we can find $c_9 = c_9(\epsilon) > 0$ such that

$$F(z,x) \leqslant \frac{1}{2}(\eta(z) + \epsilon)x^2 + c_9 \text{ for almost all } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{41}$$

For $u \in \hat{H}_m$ we have

$$\begin{aligned} \varphi(u) &= \frac{1}{2}\vartheta(u) - \frac{\hat{\lambda}_m}{2}||u||_2^2 - \int_\Omega F(z,u)dz \\ &\geqslant \frac{1}{2}\vartheta(u) - \frac{1}{2}\int_\Omega (\hat{\lambda}_m + \eta(z))u^2 dz - \frac{\epsilon}{2}||u||^2 - c_9|\Omega|_N \text{ (see (41))}. \end{aligned} \tag{42}$$

Note that

$$\hat{\lambda}_m + \eta(z) \leqslant \hat{\lambda}_m \text{ for almost all } z \in \Omega, \ \hat{\lambda}_m + \eta \not\equiv \hat{\lambda}_m.$$

So, from Lemma 6(b) and (42) we have

$$\varphi(u) \geqslant \frac{c_2 - \epsilon}{2}||u||^2 - c_9|\Omega|_N. \tag{43}$$

Choosing $\epsilon \in (0, c_2)$, from (43) we conclude that $\varphi|_{\hat{H}_m}$ is coercive. $\square$

Proposition 14. *If hypotheses $H(\beta)$, H_3 hold, then $\varphi|_{\bar{H}_{m-1}}$ is anticoercive.*

Proof. Hypotheses $H_3(i), (ii)$ imply that given $\epsilon > 0$, we can find $c_{10} = c_{10}(\epsilon) > 0$ such that

$$F(z,x) \geqslant \frac{1}{2}(\hat{\eta}(z) - \epsilon)x^2 - c_{10} \text{ for almost all } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{44}$$

Then for $u \in \bar{H}_{m-1}$ we have

$$\begin{aligned} \varphi(u) &= \frac{1}{2}\vartheta(u) - \frac{\hat{\lambda}_m}{2}||u||_2^2 - \int_\Omega F(z,u)dz \\ &\leqslant \frac{1}{2}\int_\Omega [\hat{\lambda}_{m-1} - \hat{\lambda}_m - \hat{\eta}(z)]u^2 dz + \frac{\epsilon}{2}||u||^2 + c_{10}|\Omega|_N \text{ (see (4) and (44))}. \end{aligned} \tag{45}$$

Note that

$$\hat{\lambda}_{m-1} \leqslant \hat{\lambda}_m + \hat{\eta}(z) \text{ for almost all } z \in \Omega,\ \hat{\lambda}_{m-1} \not\equiv \hat{\lambda}_m + \hat{\eta}.$$

So, from Lemma 6(a) and (45) we have

$$\varphi(u) \leqslant \frac{-c_1 + \epsilon}{2}||u||^2 + c_{10}|\Omega|_N. \tag{46}$$

Choosing $\epsilon \in (0, c_1)$ from (46) we infer that $\varphi|_{\bar{H}_{m-1}}$ is anticoercive. $\square$

Since hypothesis $H_3(ii)$ does not allow complete resonance with respect to $\hat{\lambda}_m$, but only nonuniform nonresonance, a simplified version of the proof of Proposition 7 leads to the following result.

Proposition 15. *If hypotheses $H(\beta)$, H_3 hold, then the energy functional φ satisfies the C-condition.*

Now we are ready to generate the first solution. Note that for this solution we do not require to be close to $\hat{\lambda}_m$ (no restriction on $\hat{\eta}$).

Proposition 16. *If hypotheses $H(\beta)$, H_3 hold, then problem* (1) *has a solution $u_0 \in C^1(\overline{\Omega})$.*

Proof. Proposition 13 implies that

$$m_0 = \inf[\varphi(u) : u \in \hat{H}_m] > -\infty. \tag{47}$$

On the other hand Proposition 14 implies that for $R > 0$ big we have

$$\sup[\varphi(u) : u \in \bar{H}_{m-1},\ ||u|| = R] < m_0. \tag{48}$$

Then Proposition 15 and (47) and (48) above, permit the use of Theorem 3 (the saddle point theorem). So, we can find $u_0 \in H^1(\Omega)$ such that

$$u_0 \in K_\varphi \text{ and } m_0 \leqslant \varphi(u_0).$$

Hence u_0 is a solution of (1) and $u_0 \in C^1(\overline{\Omega})$ (regularity theory). $\square$

Next, using again a variational argument based on the saddle point theorem (see Theorem 3), we will produce a second smooth solution $\hat{u}$ for (1). To do this, we need to impose a restriction on $\hat{\eta}$ (see hypothesis $H_3(ii)$; equation near resonance).

Proposition 17. *If hypotheses $H(\beta)$, H_3 hold, then we can find $\delta_0 > 0$ and $R_0 > 0$ such that when $-\delta_0 \leqslant \hat{\eta}(z)$ for almost all $z \in \Omega$ we have*

$$\varphi(u) < m_0 \quad \textit{for all } u \in \bar{H}_m,\ ||u|| = R_0 \ (\textit{see } (47)).$$

Proof. We argue by contradiction. Assuming that the proposition is not true for $\delta_n \to 0^+$ and $R_n \to +\infty$, we can find $u_n \in \bar{H}_m$ such that

$$m_0 \leqslant \varphi(u_n),\ ||u_n|| = R_n \quad \text{for all } n \in \mathbb{N}. \tag{49}$$

Let $y_n = \frac{u_n}{||u_n||}$, $n \in \mathbb{N}$. Then $||y_n|| = 1$ for all $n \in \mathbb{N}$ and exploiting the fact that $\bar{H}_m$ is finite dimensional, we have (at least for a subsequence) that

$$y_n \to y \quad \text{in } H^1(\Omega). \tag{50}$$

From (49) we have

$$\frac{m_0}{||u_n||^2} \leqslant \frac{1}{2}\vartheta(y_n) - \frac{\hat{\lambda}_m}{2}||y_n||_2^2 - \int_\Omega \frac{F(z,u_n)}{||u_n||^2}dz \quad \text{for all } n \in \mathbb{N}. \tag{51}$$

As before (see the proof of Proposition 9) we have that

$$\left\{\frac{F(\cdot,u_n(\cdot))}{||u_n||^2}\right\}_{n\geqslant 1} \subseteq L^1(\Omega) \text{ is uniformly integrable.}$$

So, by the Dunford–Pettis theorem, we may assume that

$$\frac{F(\cdot,u_n(\cdot))}{||u_n||^2} \xrightarrow{w} g \quad \text{in } L^1(\Omega). \tag{52}$$

Hypothesis $H_3(ii)$ implies that

$$g = \frac{1}{2}\eta_0 y^2 \text{ with } \hat{\eta}(z) \leqslant \eta_0(z) \leqslant \eta(z) \text{ for almost all } z \in \Omega \tag{53}$$

(see Aizicovici, Papageorgiou and Staicu [1], proof of Proposition 14). So, passing to the limit as $n \to \infty$ in (51) and using (50), (52), (53), we obtain

$$\vartheta(y) - \int_\Omega (\hat{\lambda}_m + \eta_0(z))y^2 dz \geqslant 0 \text{ (recall that } ||u_n|| \to \infty, \text{ see (49)).} \tag{54}$$

We have

$$\hat{\lambda}_m + \eta_0(z) \leqslant \hat{\lambda}_m \text{ for almost all } z \in \Omega,\ \hat{\lambda}_m + \eta_0 \not\equiv \hat{\lambda}_m.$$

From Lemma 6(a) and (54), we have

$$c_1||y||^2 \leqslant 0,$$
$$\Rightarrow y = 0.$$

But from (50) it follows that $||y|| = 1$, a contradiction. □

Using Theorem 3 (the saddle point theorem) we will produce another solution $\hat{u}$ of (1). A priori we cannot say that $\hat{u} \neq u_0$. This will be shown in the proof of Theorem 19. The new solution is valid only when we are near resonance.

Proposition 18. *If hypotheses $H(\beta)$, H_3 hold, then there exists $\delta_0 > 0$ such that when $-\delta_0 \leqslant \hat{\eta}(z)$, problem (1) has a solution $\hat{u} \in C^1(\overline{\Omega})$.*

Proof. Let $m_0 > -\infty$ be as in (47). Proposition 13 implies that $\varphi|_{\hat{H}_{m+1}}$ is coercive. So, we have

$$m_1 = \inf[\varphi(u) : u \in \hat{H}_{m+1}] \geqslant m_0 > -\infty. \tag{55}$$

Let $\delta_0, R_0 > 0$ as postulated by Proposition 17. We have

$$\varphi(u) < m_0 \text{ for all } u \in \bar{H}_m,\ ||u|| = R_0. \tag{56}$$

From Proposition 15 we know that φ satisfies the C-condition. This fact together with (55), (56) permit the use of Theorem 3 (the saddle point theorem). So, we can find $\hat{u} \in H^1(\Omega)$ such that

$$\hat{u} \in K_\varphi \text{ and } m_1 \leqslant \varphi(\hat{u}).$$

Then $\hat{u}$ solves (1) and $\hat{u} \in C^1(\overline{\Omega})$ (regularity theory). □

Next we show that $\hat{u} \neq u_0$ and we have the multiplicity theorem when we are near resonance from the left of $\hat{\lambda}_m$.

Theorem 19. *If hypotheses $H(\beta)$, H_3 hold, then there exists $\delta_0 > 0$ such that when $-\delta_0 \leqslant \hat{\eta}(z)$ for almost all $z \in \Omega$, problem* (1) *has at least two distinct smooth solutions*

$$u_0, \hat{u} \in C^1(\overline{\Omega}),\ u_0 \neq \hat{u}.$$

Proof. Propositions 16 and 18 produced two solutions $u_0,\ \hat{u} \in C^1(\overline{\Omega})$. We need to show that they are distinct.

Both solutions where produced via an application of the saddle point theorem (see Theorem 3). More precisely, we have

$$\varphi(u_0) = \inf_{\gamma \in \Gamma_1} \sup[\varphi(\gamma(u)) : u \in \bar{H}_{m-1},\ ||u|| = R], \tag{57}$$

with

$$\Gamma_1 = \left\{\gamma \in C(\bar{B}_R^{m-1}, H^1(\Omega)) : \gamma|_{\partial B_R^{m-1}} = \mathrm{id}|_{\partial B_R^{m-1}}\right\}$$

and $\bar{B}_R^{m-1} = \{u \in \bar{H}_{m-1} : ||u|| \leqslant R\}$, $\partial B_R^{m-1} = \{u \in \bar{H}_{m-1} : ||u|| = R\}$ (see the proof of Proposition 16).

Also, from the proof Proposition 18, we have

$$\varphi(\hat{u}) \geqslant m_1 \geqslant m_0 \text{ (see (56), (55))}. \tag{58}$$

We can always assume that $R > R_0$. Let $w_0 \in E(\hat{\lambda}_m)$ with $||w_0|| = 1$ and consider the map $\gamma : \bar{B}_R^{m-1} \to H^1(\Omega)$ defined by

$$\hat{\gamma}(u) = \begin{cases} u + \sqrt{R_0^2 - ||u||^2 w_0} & \text{if } u \in \bar{H}_{m-1},\ ||u|| \leqslant R_0 \\ u & \text{if } u \in \bar{H}_{m-1},\ R_0 < ||u|| \leqslant R. \end{cases}$$

Evidently $\hat{\gamma} \in \Gamma_1$ and so from Proposition 17 (see also (56)), we have

$$\begin{aligned} &\sup \varphi(\hat{\gamma}(u)) < m_0, \\ &\quad \Rightarrow \varphi(u_0) < m_0 \text{ (see (57))}, \\ &\quad \Rightarrow u_0 \neq \hat{u} \text{ (see (58))}. \end{aligned}$$

The proof is complete. □

If there is a set $\Omega_0 \subseteq \Omega$ with $|\Omega_0|_N > 0$ and $f(z,0) \neq 0$ for all $z \in \Omega_0$, then u_0, $\hat{u}$ are nontrivial. However, if $f(z,0) = 0$ for almost all $z \in \Omega$, then we cannot guarantee the nontriviality of the solutions. To achieve this, we need to impose a condition on $f(z,\cdot)$ near zero.

So, the new hypotheses on the perturbation term $f(z,x)$ are the following:

H_4 : $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that hypotheses $H_4(i),(ii)$ are the same as the corresponding hypotheses $H_3(i),(ii)$ and

(iii) there exist $\hat{\delta} > 0, \tau \in (1,2)$ and $c_{11} > 0$ such that

$$c_{11}x^2 \leqslant f(z,x)x \leqslant \tau F(z,x) \text{ for almost all } z \in \Omega, \text{ all } |x| \leqslant \hat{\delta}$$
$$0 < \operatorname{ess\,inf}_\Omega F(\cdot, \pm\hat{\delta}).$$

Theorem 20. *If hypotheses $H(\beta)$, H_4 hold, then there exists $\delta_0 > 0$ such that when $-\delta_0 \leqslant \hat{\eta}(z)$ for almost all $z \in \Omega$, problem (1) has at least two nontrivial smooth solutions*

$$u_0, \hat{u} \in C^1(\overline{\Omega}),\ u_0 \neq \hat{u}.$$

Proof. From Theorem 19 we already have two distinct smooth solutions

$$u_0, \hat{u} \in C^1(\overline{\Omega}),\ u_0 \neq \hat{u}.$$

From hypothesis $H_4(iii)$ and Proposition 8 of Papageorgiou and Rădulescu [15] (see also Proposition 6 of [17]) we have

$$C_k(\varphi, 0) = 0 \text{ for all } k \in \mathbb{N}_0. \tag{59}$$

On the other hand, we know that both $u_0, \hat{u}$ were produced via the saddle point theorem (see the proofs of Propositions 16 and 18). So, from Proposition 6.80, p. 168 of Motreanu, Motreanu and Papageorgiou [11], we have

$$C_{d_{m-1}}(\varphi, u_0) \neq 0 \text{ and } C_{d_m}(\varphi, \hat{u}) \neq 0 \tag{60}$$

with $d_{m-1} = \dim \bar{H}_{m-1},\ d_m = \dim \bar{H}_m$. Comparing (59) and (60) we conclude that

$$u_0 \neq 0,\ \hat{u} \neq 0.$$

The proof is now complete. □

The nontriviality of u_0, $\hat{u}$ can be established, if instead we impose the following conditions on $f(z,x)$.

H_5 : $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that hypotheses $H_5(i),(ii)$ are the same as the corresponding hypotheses $H_3(i),(ii)$ and

(iii) there exists a function $\xi \in L^\infty(\Omega)$ such that

$$\xi(z) \leqslant \hat{\lambda}_1 - \hat{\lambda}_m \text{ for almost all } z \in \Omega,\ \xi \not\equiv \hat{\lambda}_1 - \hat{\lambda}_m$$
$$\limsup_{x \to 0} \frac{2F(z,x)}{x^2} \leqslant \xi(z) \text{ uniformly for almost all } z \in \Omega.$$

We have the following result.

Theorem 21. *If hypotheses $H(\beta)$, H_5 hold, then there exists $\delta_0 > 0$ such that when $-\delta_0 \leqslant \hat{\eta}(z)$ for almost all $z \in \Omega$, problem* (1) *has at least two nontrivial smooth solutions*

$$u_0,\ \hat{u} \in C^1(\overline{\Omega}),\ u_0 \neq \hat{u}.$$

Proof. From Theorem 19 we already have two distinct smooth solutions

$$u_0,\ \hat{u} \in C^1(\overline{\Omega}),\ u_0 \neq \hat{u}.$$

Hypothesis $H_5(iii)$ implies that given $\epsilon > 0$, we can find $\delta > 0$ such that

$$F(z,x) \leqslant \frac{1}{2}(\xi(z)+\epsilon)x^2 \text{ for almost all } z \in \Omega,\ |x| \leqslant \delta. \tag{61}$$

Then for $u \in C^1(\overline{\Omega})$ with $||u||_{C^1(\overline{\Omega})} \leqslant \delta$, we have

$$\begin{aligned} \varphi(u) &= \frac{1}{2}\vartheta(u) - \frac{\hat{\lambda}_m}{2}||u||_2^2 - \int_\Omega F(z,u)dz \\ &\geqslant \frac{1}{2}\vartheta(u) - \frac{1}{2}\int_\Omega (\hat{\lambda}_m + \xi(z))u^2 - \frac{\epsilon}{2}||u||^2. \end{aligned} \tag{62}$$

Note that

$$\hat{\lambda}_m + \xi(z) \leqslant \hat{\lambda}_1 \text{ for almost all } z \in \Omega,\ \hat{\lambda}_m + \xi \not\equiv \hat{\lambda}_1.$$

So, using Lemma 6(b), we have

$$\vartheta(u) - \int_\Omega (\hat{\lambda}_m + \xi(z))u^2 dz \geqslant c_{12}||u||^2 \text{ for all } u \in H^1(\Omega), \text{ some } c_{12} > 0. \tag{63}$$

Using (63) in (62) we obtain

$$\varphi(u) \geqslant \frac{c_{12}-\epsilon}{2}||u||^2. \tag{64}$$

Choosing $\epsilon \in (0, c_{12})$, from (64) we infer that $u = 0$ is a local $C^1(\overline{\Omega})$-minimizer of φ. Proposition 3 of Papageorgiou and Rădulescu [14] implies that $u = 0$ is a local $H^1(\Omega)$-minimizer of φ. Hence

$$C_k(\varphi, u) = \delta_{k,0}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0. \tag{65}$$

Here $\delta_{k,0}$ is the Kronecker symbol defined by

$$\delta_{k,0} = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k \neq 0. \end{cases}$$

Comparing (65) with (60) and since $m \geqslant 2$, we conclude that $u_0,\ \hat{u} \in C^1(\overline{\Omega})$ are nontrivial smooth solutions of (1). □

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