Differential Inequalities Having a Liouville-Type Property

In *Problem* 06-005, THE PROPOSER (Vicenţiu Rădulescu, University of Craiova, Romania) introduced the differential inequality

$$(1) \qquad \left(x^2 f''(x) + pxf'(x) - q|f(x)|^{r-1}f(x)\right) \operatorname{sgn} f(x) \geq 0 \qquad \text{for all } x > 0,$$

where constants p, q, and r are real, q and r are nonnegative, and f is twice differentiable on $(0, \infty)$.

(a) Prove that if $q > 0$, then either f vanishes identically or there exists $A > 0$ such that both f and f' do not vanish in $[A, +\infty)$. Establish a corresponding result for $q = 0$.

(b) Suppose $q > 0$ and let f be a nontrivial solution of (1) that is positive and increasing in $[A, \infty)$. Prove that f is unbounded.

(c) Suppose $q = 0$ and let f be a nontrivial solution of (1) such that f is positive and increasing in $[A, \infty)$. Prove that if $p \leq 1$, then f is unbounded. Is the condition $p \leq 1$ necessary?

Composite Solution by FEN QIN[1] (Shippensburg University, Shippensburg, PA) *and* THE PROPOSER[2].

(a) Suppose $q > 0$ and f is a nontrivial solution of (1). Observe that $-f$ also satisfies (1). If $f(a) > 0$, $f'(a) = 0$, and $f''(a) \leq 0$, then the first term in

$$a^2 f''(a) + paf'(a) - q(f(a))^r$$

is nonpositive, the second is 0, and the third is negative, which is impossible because (1) is then violated. Hence f has neither a positive local maximum nor a negative local minimum. By Rolle's Theorem, f has at most one zero. Hence there exists $A > 0$ for which $f \neq 0$ in $[A, \infty)$. By replacing f by $-f$ as necessary, we may assume that $f > 0$ in $[A, \infty)$. Multiplying (1) by x^{p-2} gives

$$(2) \qquad (x^p f'(x))' = x^p f''(x) + px^{p-1}f'(x) \geq qx^{p-2}(f(x))^r > 0.$$

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Thus f' is increasing in $[A,\infty)$. By adjusting the value of A as necessary, neither f nor f' vanishes in $[A,\infty)$.

In case $q=0$ we claim that either f is constant or there exists $A>0$ such that neither f nor f' vanishes in $[A,\infty)$. Indeed, let f be a nonconstant solution of (1) and suppose $x_0>0$ is a local maximum of f and $f(x_0)>0$. We can assume that f is nonconstant in some neighborhood of x_0, that is, $f'\not\equiv 0$ in $I:=(x_0-\delta,x_0+\delta)\subset(0,+\infty)$. It follows from (1) that $(x^p f'(x))'\geq 0$ for all $x\in I$. Then $f'(x_0)=0$ shows that $f'\leq 0$ in $(x_0-\delta,x_0)$ and $f'\geq 0$ in $(x_0,x_0+\delta)$. Because x_0 is a local maximum of f, this forces $f(x)=f(x_0)$ for all $x\in I$, a contradiction. Let us now prove that we can choose $A>0$ large enough so that $f'\neq 0$ in $[A,\infty)$. For this purpose it is enough to show that if f is a nonconstant solution having constant sign in $[A,\infty)$, then f' vanishes at most one time in $[A,\infty)$. Indeed, assume that $f'(x_1)=f'(x_2)=0$ for some $A<x_1<x_2$. Because $f'\not\equiv 0$ and $x^2f''(x)+pxf'(x)\geq 0$ in $[x_1,x_2]$,

$$
\begin{aligned}
0<\int_{x_1}^{x_2} x^p (f'(x))^2\,dx &= x^p f(x) f'(x)\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} x^{p-2} f(x)\left(x^2 f''(x)+pxf'(x)\right)dx \\
&\leq x^p f(x) f'(x)\Big|_{x_1}^{x_2} = 0,
\end{aligned}
$$

a contradiction.

(b) Suppose $q>0$ and f is a solution of (1) that is positive and increasing in $[A,\infty)$. In view of (2), there exists $A>0$ such that $x^p f'$ is increasing in $[A,\infty)$. Hence $x^p f'(x)\geq A^p f'(A)$ for all $x\geq A$, which gives $f'(x)>A^p f'(A)x^{-p}>0$. If f is bounded in $[A,\infty)$, then $\lim_{x\to\infty} f(x)=C$ for some positive constant C. Integrating (2) and dividing both sides by x^p yields

$$
\begin{aligned}
f'(x) > \frac{x^p f'(x)-A^p f'(A)}{x^p} &\geq \frac{q}{x^p}\int_A^x s^{p-2}(f(s))^r\,ds \\
&\geq \frac{q(f(A))^r}{x^p}\int_A^x s^{p-2}\,ds \geq q(f(A))^r Q(x),
\end{aligned}
$$

where

$$Q(x) = \begin{cases} \dfrac{x^{-1} - A^{p-1}x^{-p}}{p-1} & \text{if} \quad p > 1, \\ \dfrac{\ln(x/A)}{x} & \text{if} \quad p = 1, \\ \dfrac{A^{p-1}x^{-p} - x^{-1}}{1-p} & \text{if} \quad p < 1. \end{cases}$$

Notice that $\int_A^\infty Q(x)dx = \infty$ whatever the value of p. Integrating $f'(x) > q(f(A))^r Q(x)$ shows that as $x \to \infty$,

$$C - f(A) \geq f(x) - f(A) = \int_A^x f'(s)\, ds > q(f(A))^r \int_A^x Q(s)\, ds \to \infty,$$

which is a contradiction.

(c) Suppose $q = 0$. Let f be an any solution of (1) that is positive and increasing in $[A, \infty)$. Hence

$$(x^p f'(x))' \geq 0 \qquad \text{for all } x \geq A.$$

It follows that

$$f(x) \geq \begin{cases} \dfrac{A^p f'(A)}{1-p} x^{1-p} - C & \text{if} \quad p < 1, \\ A f'(A) \log x - C & \text{if} \quad p = 1 \end{cases}$$

for all $x > A$, where $C > 0$. Because $f'(A) > 0$, it follows that f is unbounded for any $p \leq 1$. This property does not remain true if $p > 1$. Indeed, the function $f(x) = 1 - (x+1)^{1-p}$ is a positive, increasing, and *bounded* (since $p > 1$) solution of the differential inequality.