STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LV**, Number 4, December 2010

REMARKS ON A LIMITING CASE IN THE TREATMENT OF NONLINEAR PROBLEMS WITH MOUNTAIN PASS GEOMETRY

VICENŢIU RĂDULESCU

Abstract. We study a class of nonlinear elliptic problem with linear growth and Dirichlet boundary condition. By means of the mountain pass theorem, we establish the existence of a positive solution. The proof of the Palais-Smale condition differs with respect to the standard case that corresponds to nonlinearities with a superlinear behaviour.

1. **Introduction**

The mountain pass theorem of A. Ambrosetti and P. Rabinowitz [2] is a result of great importance in the determination of critical points to energy functionals that occur in the theory of partial differential equations. The original version of A. Ambrosetti and P. Rabinowitz corresponds to the case of mountains of positive altitude. Their proof relies on some deep deformation techniques developed by R. Palais and S. Smale [11, 12], who put the main ideas of the Morse theory into the framework of differential topology on infinite dimensional manifolds. H. Brezis and L. Nirenberg provided in [4] a simpler proof which combines two major tools: Ekeland's variational principle and the pseudogradient lemma. Ekeland's variational principle is the nonlinear version of the Bishop–Phelps theorem and it may be also viewed as a generalization of Fermat's theorem. As pointed out by H. Brezis and F. Browder [3], the mountain pass theorem "extends ideas already present in Poincaré and Birkhoff". An important contribution is due to P. Pucci and J. Serrin [14, 15, 16], who studied he case of mountains of zero altitude.

In its simplest form, the mountain pass theorem considers functions $J : X \to \mathbb{R}$ of class C^1, where X is a real Banach space. It is assumed that J satisfies the following geometric conditions:

(H1) *there exist two numbers $R > 0$ and $c_0 \in \mathbb{R}$ such that $J(u) \geq c_0$ for every*

Received by the editors: 20.09.2010.

2000 *Mathematics Subject Classification.* 35J60, 58E05.

Key words and phrases. Mountain pass geometry, Palais-Smale compactness condition, critical point, linear growth.

$u \in S_R := \{v \in X;\ \|v\| = R\}$;
(H2) $J(0) < c_0$ *and* $J(e) < c_0$ *for some* $e \in X$ *with* $\|e\| > R$.
With an additional compactness condition of Palais-Smale type it then follows that the function J has a critical point $u_0 \in X \setminus \{0, e\}$, with corresponding critical value $c \geq c_0$. In essence, this critical value occurs because 0 and e are low points on either side of the "mountain" S_R, so that between 0 and e there must be a lowest critical point, that is, a mountain pass. Condition (H2) signifies that the mountain should have positive altitude. P. Pucci and J. Serrin [14, 15] proved that the mountain pass theorem continues to hold for a mountain of zero altitude, provided it also has nonzero thickness. In addition, if $c = c_0$, then the "pass" itself occurs precisely on the mountain. Roughly speaking, P. Pucci and J. Serrin showed that the mountain pass theorem still remains true if (H1) is strengthened a little, to the form
(H1)$'$ *there exist real numbers* c_0, R, r *such that* $0 < r < R$ *and* $J(u) \geq c_0$ *for every* $u \in A := \{v \in X;\ r < \|v\| < R\}$,
while hypothesis (H2) is weakened and replaced with
(H2)$'$ $J(0) \leq c_0$ *and* $J(e) \leq c_0$ *for some* $e \in X$ *with* $\|e\| > R$.

As stated above, the Palais-Smale compactness condition is also crucial for the mountain pass theorem. Let X be a real Banach space and $J \in C^1(X, \mathbb{R})$. We recall that J satisfies the Palais-Smale condition if any sequence $(u_n)_n$ in X such that

$$(J(u_n))_n \text{ is bounded and } J'(u_n) \to 0,$$

admits a convergent subsequence. As pointed out by M. Struwe [19, p. 169], "recent advances in the calculus of variations have shown that the Palais-Smale condition holds for problems in a broad range of energies. Moreover, the failure of the Palais-Smale condition at certain levels reflects highly interesting phenomena related to internal symmetries of the systems under study, which geometrically can be described as *separation of spheres*, or mathematically as *singularities*, respectively as *change in topology*. Speaking in physical terms, we might observe *phase transitions* or *particle creation* at the energy levels where the Palais-Smale condition fails".

The mountain pass theorem has the following simple geometric interpretation. Consider two valleys A and B such that A is surrounded by a mountain ridge that separates it from B. To go from A to B, we must cross the mountain chain. If we want to climb as little as possible, we would have to consider the maximal elevation of each path. The path with the minimal one of these elevations will cross a mountain pass.

We refer to the books by A. Ambrosetti and A. Malchiodi [1], M. Ghergu and V. Rădulescu [7], Y. Jabri [8], A. Kristály, V. Rădulescu, and Cs. Varga [9], J. Mawhin and M. Willem [10], P. Rabinowitz [17], M. Schechter [18], M. Struwe [19], M. Willem [20], and W. Zou [21] for relevant applications of the mountain pass theory. We also refer to the recent paper by P. Pucci and V. Rădulescu [13] for a history of this result and several applications, including in the nonsmooth setting.

2. **Main Result**

The main application of the mountain pass theorem provided by A. Ambrosetti and P. Rabinowitz [2] concerns the Emden–Fowler equation (see R. Emden [5] and R. H. Fowler [6])

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary. The standard example here is given by $f(u) = |u|^{p-1}u$, where $1 < p < (N+2)/(N-2)$ for $N \geq 3$ ($1 < p < \infty$ if $N = 1$ or $N = 2$). More generally, the nonlinear term in problem (2.1) can be assumed to satisfy the following assumptions:
(1) $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function such that

$$|f(u)| \leq C\,(1 + u^p) \qquad \text{for all } u \geq 0,$$

for some $C > 0$, where $1 < p < (N+2)/(N-2)$;
(2) $f(0) = f'(0) = 0$;
(3) there exists $\mu > 2$ such that

$$0 < \mu F(u) \leq u f(u) \qquad \text{for all } u \text{ large enough},$$

where $F(u) := \int_0^u f(t)\,dt$.

The purpose of the present paper is to study problem (2.1) in the case of f has a linear behaviour. This means that f fulfills the same growth condition as in hypothesis (1) above, but provided that $p = 1$. Simple examples show that we cannot expect that a solution exists in all cases. Indeed, if we consider the simplest case corresponding to $f(u) = \lambda u$, where λ is a positive parameter, then problem (2.1) has a solution if and only if $\lambda = \lambda_1$. Here, λ_1 stands for the first eigenvalue of $(-\Delta)$ in $H_0^1(\Omega)$. Even if we drop the restriction that the solution is positive, a nontrivial solution of (2.1) does not exist if λ is not an eigenvalue of the Laplace operator. These simple remarks show that the lowest eigenvalue λ_1 of $(-\Delta)$ must play a central role in the existence of a solution to problem (2.1), provided that the nonlinear term f has a linear growth.

Our main result is stated in what follows. We point out that the nonlinear term is not assumed to satisfy the above Ambrosetti-Rabinowitz technical assumption (3).

Theorem 2.1. *Assume $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function such that $f(0) = 0$,*

$$f'(0) < \lambda_1 \tag{2.2}$$

and

$$\lambda_1 < \lim_{u\to\infty} \frac{f(u)}{u} < \infty\,. \tag{2.3}$$

Then problem (2.1) has at least a solution.

Proof. Since we are looking for positive solutions, it is natural to consider the continuous function

$$f_0(u) := \begin{cases} f(u) & \text{if } u \geq 0 \\ 0 & \text{if } u < 0. \end{cases}$$

By our hypotheses, there is some $C > 0$ such that for all $u \in \mathbb{R}$,

$$|f_0(u)| \leq C(1 + |u|). \tag{2.4}$$

Assumption (2.3) implies that $C > \lambda_1$. Relation (2.3) also shows that

$$f_0(u) \geq C_1 u - C_2 \qquad \text{for all } u \geq 0,$$

for some C_1, $C_2 > 0$ with $C_1 > \lambda_1$.

Set $F_0(u) := \int_0^u f(t)\,dt$. Therefore

$$|F_0(u)| \leq C\left(u + \frac{u^2}{2}\right)$$

and

$$F_0(u) \geq C_1\,\frac{u^2}{2} - C_3 \qquad \text{for all } u \geq 0,$$

where $C_3 > 0$.

We associate to (2.1) the energy functional $J : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$J(u) := \int_\Omega |\nabla u|^2\,dx - \int_\Omega F_0(u)\,dx \qquad \text{for all } u \in H_0^1(\Omega).$$

It follows from (2.4) and the Sobolev embedding theorem that J is well defined on $H_0^1(\Omega)$. Moreover, J is of class C^1 and for all $v \in H_0^1(\Omega)$,

$$J'(u)(v) = \int_\Omega [\nabla u \nabla v - f_0(u)v]\,dx.$$

We first argue that the geometric assumptions of the mountain pass theorem are fulfilled. Fix $1 < p < (N+2)/(N-2)$. In order to check condition (H1) we observe that hypothesis (2.3) implies that there are some $0 < \alpha < \lambda_1$ and $C_4 > 0$ such that for all $u \in H_0^1(\Omega)$

$$|f_0(u)| \leq \alpha\, |u| + C_4 |u|^p .$$

Thus, $F_0(u) \leq \alpha u^2/2 + C_5\, |u|^{p+1}$, where $C_5 > 0$. It follows that

$$J(u) \geq \frac{1}{2}\int_\Omega |\nabla u|^2\, dx - \frac{\alpha}{2}\int_\Omega u^2\, dx - C_5 \int_\Omega |u|^{p+1}\, dx.$$

By Poincaré's inequality we have

$$\int_\Omega u^2\, dx \leq \frac{1}{\lambda_1}\int_\Omega |\nabla u|^2\, dx \qquad \text{for all } u \in H_0^1(\Omega).$$

We deduce that

$$J(u) \geq \frac{\lambda_1 - \alpha}{2\lambda_1}\, \|u\|^2_{H_0^1(\Omega)} - C_5 \int_\Omega |u|^{p+1}\, dx,$$

for all $u \in H_0^1(\Omega)$. Using Sobolev embeddings we conclude that $J(u) > 0$ for all u with $\|u\| = R$, provided that $R > 0$ is small enough. This shows that condition (H1) is fulfilled.

Now, we prove that assumption (H2) in the mountain pass theorem holds true. Let $e_1 > 0$ be the first eigenfunction of $(-\Delta)$ in $H_0^1(\Omega)$, hence $\int_\Omega |\nabla e_1|^2\, dx = \lambda_1 \int_\Omega e_1^2\, dx$. Thus, for all $t > 0$,

$$J(te_1) \leq \frac{t^2\lambda_1^2}{2}\int_\Omega e_1^2\, dx - \frac{t^2 C_1^2}{2}\int_\Omega e_1^2\, dx + C_3|\Omega| < 0,$$

provided that t is sufficiently large. This follows from the fact that $C_1 > \lambda_1$, which is a direct consequence of our assumption (2.3).

To complete the proof, it remains to show that the energy functional J satisfies the Palais-Smale condition. For this purpose, let (u_n) be a sequence in $H_0^1(\Omega)$ such that

$$\sup_n |J(u_n)| < \infty \tag{2.5}$$

and

$$\lim_{n\to\infty} \|J'(u_n)\|_{H^{-1}(\Omega)} = 0. \tag{2.6}$$

We now claim that

$$(u_n) \text{ is bounded in } H_0^1(\Omega). \tag{2.7}$$

We first observe that relation (2.6) implies

$$-\Delta u_n = f_0(u_n) + \psi_n\, , \tag{2.8}$$

where $\|\psi_n\|_{H^{-1}(\Omega)} \to 0$. Taking into account the linear growth of f_0 and using (2.8), we deduce that our claim (2.7) follows after proving that (u_n) is bounded in

$L^2(\Omega)$. Arguing by contradiction, we assume that $\|u_n\|_{L^2(\Omega)} \to \infty$. It follows that $v_n := u_n/\|u_n\|_{L^2(\Omega)}$ satisfies

$$-\Delta v_n = \frac{f_0(u_n)}{\|u_n\|_{L^2(\Omega)}} + \frac{\psi_n}{\|u_n\|_{L^2(\Omega)}}\,. \tag{2.9}$$

We now observe that $f_0(u_n)/\|u_n\|_{L^2(\Omega)}$ is bounded in $L^2(\Omega)$. Thus, after multiplication with v_n in (2.9) and integration, we deduce that (v_n) is bounded in $H_0^1(\Omega)$. So, up to a subsequence,

$$v_n \to v \qquad \text{in } H_0^1(\Omega).$$

Moreover, since $\|v_n\|_{L^2(\Omega)} = 1$, we also have

$$\|v\|_{L^2(\Omega)} = 1. \tag{2.10}$$

Next, we observe that the definition of f_0 implies that there is some $C > 0$ such that $f_0(u) \le -C$ for all $u \in \mathbb{R}$. Thus, by (2.9),

$$-\Delta u_n \ge -C + \psi_n \qquad \text{in } \Omega\,.$$

So, by the maximum principle, $u_n \ge -C + \rho_n$, where $\rho_n \to 0$ in $H_0^1(\Omega)$ and hence $\rho_n \to 0$ a.e. in Ω (at a subsequence). This implies that

$$v \ge 0 \text{ a.e. in } \Omega. \tag{2.11}$$

On the other hand, our assumption (2.3) implies

$$f_0(u) \ge Au - B \qquad \text{for all } u \in \mathbb{R},$$

where $A > \lambda_1$ and $B > 0$. This implies that

$$-\Delta u_n \ge Au_n - B + \psi_n \qquad \text{in } \Omega. \tag{2.12}$$

Let $e_1 > 0$ be the first eigenfunction of the Laplace operator in $H_0^1(\Omega)$. After multiplication with e_1 in relation (2.12) we obtain

$$\lambda_1 \int_\Omega u_n\psi_n\,dx \ge A\int_\Omega u_n e_1\,dx - C + \langle \psi_n, e_n\rangle_{H^{-1},H_0^1}\,.$$

Dividing by $\|u_n\|_{L^2(\Omega)}$ and taking $n \to \infty$ we obtain $\lambda_1 \int_\Omega ve_1\,dx \ge A\int_\Omega ve_1\,dx$, which contradicts relations (2.10) and (2.11). This concludes the proof of our claim (2.7). From now on, using the same ideas as in the proof of Theorem 9 in [13], we can deduce that any bounded sequence (u_n) in $H_0^1(\Omega)$ satisfying relation (2.6) has the property that contains a strongly convergent subsequence in $H_0^1(\Omega)$. These arguments do not depend on the linear growth of the nonlinear term f. Our proof is now complete. □

Acknowledgments. The author acknowledges the support through Grant CNCSIS PCCE–55/2008 *"Sisteme diferenţiale în analiza neliniară şi aplicaţii"*.

References

[1] Ambrosetti, A., Malchiodi, A., *Nonlinear Analysis and Semilinear Elliptic Problems*, Cambridge Studies in Advanced Mathematics, vol. 104, Cambridge University Press, Cambridge, 2007.

[2] Ambrosetti, A., Rabinowitz, P., *Dual variational methods in critical point theory and applications*, J. Funct. Anal., **14** (1973), 349-381.

[3] Brezis, H., Browder, F., *Partial differential equations in the 20th century*, Adv. Math., **135** (1998), 76-144.

[4] Brezis, H., Nirenberg, L., *Remarks on finding critical points*, Comm. Pure Appl. Math., **44** (1991), 939-964.

[5] Emden, R., *Die Gaskügeln*, Teubner, Leipzig, 1907.

[6] Fowler, R. H., *Further studies of Emden and similar differential equations*, Quart. J. Math., **2** (1931), 259-288.

[7] Ghergu, M., Rădulescu, V., *Singular Elliptic Problems. Bifurcation and Asymptotic Analysis*, Oxford Lecture Series in Mathematics and Its Applications, vol. 37, Oxford University Press, 2008.

[8] Jabri, Y., *The Mountain Pass Theorem. Variants, Generalizations and Some Applications*, Encyclopedia of Mathematics and its Applications, vol. 95, Cambridge University Press, Cambridge, 2003.

[9] Kristály, A., Rădulescu, V., Varga, Cs., *Variational Principles in Mathematical Physics, Geometry, and Economics. Qualitative Analysis of Nonlinear Equations and Unilateral Problems*, Encyclopedia of Mathematics and its Applications, vol. 136, Cambridge University Press, Cambridge, 2010.

[10] Mawhin, J., Willem, M., *Critical Point Theory and Hamiltonian Systems*, Applied Mathematical Sciences, vol. 74, Springer, New York, 1989.

[11] Palais, R., *Ljusternik-Schnirelmann theory on Banach manifolds*, Topology, **5** (1966), 115-132.

[12] Palais, R., Smale, S., *A generalized Morse theory*, Bull. Amer. Math. Soc., **70** (1964), 165-171.

[13] Pucci, P., Rădulescu, V., *The impact of the mountain pass theory in nonlinear analysis: a mathematical survey*, Boll. Unione Mat. Ital., in press.

[14] Pucci, P., Serrin, J., *Extensions of the mountain pass theorem*, J. Funct. Anal., **59** (1984), 185-210.

[15] Pucci, P., Serrin, J., *A mountain pass theorem*, J. Differential Equations, **60** (1985), 142-149.

[16] Pucci, P., Serrin, J., *The structure of the critical set in the mountain pass theorem*, Trans. Amer. Math. Soc., **299** (1987), 115-132.

[17] Rabinowitz, P., *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, CBMS Reg. Conference Series in Mathematics, **65**, 1986 American Mathematical Society, Providence, RI.

[18] Schechter, M., *Minimax Systems and Critical Point Theory*, Birkhäuser, Boston, 2009.

[19] Struwe, M., *Variational Methods. Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*, Springer-Verlag, Berlin, 1990.

[20] Willem, M., *Minimax Theorems*, Progress in Nonlinear Differential Equations and their Applications, vol. 24, Birkhäuser, Boston, 1996.
[21] Zou, W., *Sign-changing Critical Point Theory*, Springer, New York, 2008.

Institute of Mathematics "Simion Stoilow" of the Romanian Academy
P.O. Box 1-764, 014700 Bucharest, Romania
Department of Mathematics, University of Craiova
200585 Craiova, Romania
E-mail address: vicentiu.radulescu@imar.ro