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GENERALIZED BIHARMONIC PROBLEMS WITH VARIABLE EXPONENT AND NAVIER BOUNDARY CONDITION

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Abstract. We study a class of biharmonic problems with Navier boundary condition and involving a generalized differential operator and competing nonlinearities with variable exponent. The main result of this paper establishes a sufficient condition for the existence of nontrivial weak solutions, in relationship with the values of a positive parameter. The proofs combine variational methods with analytic arguments. The approach developed in this paper allows the treatment of several classes of nonhomogeneous biharmonic problems with variable growth arising in applied sciences, including the capillarity equation and the mean curvature problem.

1. Introduction

The interest in recent years to the mathematical analysis of partial differential equations driven by nonhomogeneous differential operators is motivated by their numerous applications to various fields. We refer, e.g., to phenomena in the applied sciences that are characterized by the fact that the associated energy density changes its ellipticity and growth properties according to the point. Such models have been studied starting with the pioneering papers by Halsey [11] and Zhikov [29, 30], in close relationship with the qualitative mathematical analysis of strongly anisotropic materials in the context of the homogenization and nonlinear elasticity.

In the framework of materials with non-homogeneities, the standard approach based on the classical theory of L^p and $W^{1,p}$ Lebesgue and Sobolev spaces is inadequate. We refer to electrorheological (smart) fluids or to phenomena in image processing, which should enable that the exponent p is varying, see Chen, Levine and Rao [6], and Ruzicka [25]. For instance, we refer to the Winslow effect of some fluids (like lithium polymetachrylate) in which the viscosity in an electrical field is inversely proportional to the strength of the field. The field induces string-like formations in the fluid, which are parallel to the field. They can raise the viscosity by as much as five orders of magnitude. This corresponds to electrorheological (non-Newtonian) fluids, which are mathematically described by means of nonlinear equations with variable exponent. Such a study corresponds to the abstract setting of variable exponents Lebesgue and Sobolev spaces, $L^{p(x)}$ and $W^{1,p(x)}$, where

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p is a real-valued function. The theory of function spaces with variable exponent has been rigorously developed in the monograph of Diening, Hästo, Harjulehto and Ruzicka [10] while the recent book by Rădulescu and Repovš [22] is devoted to the thorough variational and topological analysis of several classes of problems with one or more variable exponents; see also the survey papers of Harjulehto, Hästö, Le and Nuortio [12] and Rădulescu [20]. We also refer to Mingione et al. [2, 8, 9], Cencelj, Rădulescu and Repovš [4] Cencelj, Repovš and Virk [5], and Repovš [24] for related results. The abstract setting of $p(x)$-biharmonic problems with singular weights has been recently considered by Kefi and Rădulescu [13] in relationship with micro-electromechanical phenomena, surface diffusion on solids, thin film theory, flow in Hele-Shaw cells and phase field models of multiphasic systems. The present paper complements some results contained in [13] to more general operators. In such a way, we extend the approach developed in Chorfi and Rădulescu [7] to generalized biharmonic operators.

The study of elliptic problems with variable exponent has been recently extended by Kim and Kim [14] to a new class of non-homogeneous differential operators. Their contribution is a step forward in the analysis of nonlinear problems with variable exponent since it enables the understanding of problems with possible lack of uniform convexity. More precisely, in [14] they studied problems of the type

$$\begin{gathered} -\operatorname{div}(\phi(x,|\nabla u|)\nabla u) = f(x,u) \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary. The nonlinear term $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition and the function $\phi(x,t)$ is of type $|t|^{p(x)-2}$ with $p : \overline{\Omega} \to (1,\infty)$ continuous.

In the special case when $\phi(x,t) = |t|^{p(x)-2}$, the operator involved in problem (1.1) reduces to the $p(x)$-Laplacian, that is,

$$\Delta_{p(x)} u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u).$$

In many papers (see, e.g., [17, Hypothesis (A4), p. 2629]), the functional Φ induced by the principal part of problem (1.1) is assumed to be uniformly convex, namely, there exists $k > 0$ such that for all $x \in \Omega$ and all $\xi, \psi \in \mathbb{R}^N$,

$$\Phi\Big(x, \frac{\xi+\psi}{2}\Big) \le \frac{1}{2}\,\Phi(x,\xi) + \frac{1}{2}\,\Phi(x,\psi) - k\,|\xi-\psi|^{p(x)}.$$

However, since the function $\Psi(x,t) = t^p$ is not uniformly convex for $t > 0$ and $1 < p < 2$, this condition is not applicable to all p-Laplacian problems. An important feature of the abstract setting developed in [14] is that the main results are obtained without any uniform convexity assumption.

In this article, we extend some results of [13] in the framework of general biharmonic operators with variable exponent, as studied in [14]. We develop the study of biharmonic problems with Navier boundary condition for equations driven by the operator $\Delta(\phi(x,|\Delta u|)\Delta u)$, where ϕ is as in (1.1). Notice that if $\phi(x,t) = |t|^{p(x)-2}$, then we obtain the $p(x)$-biharmonic operator defined by $\Delta^2_{p(x)} u = \Delta(|\Delta u|^{p(x)-2}\Delta u)$.

2. Abstract framework and preliminary results

Throughout this article we assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary. Set

$$C_+(\overline{\Omega}) = \{h \in C(\overline{\Omega}),\ h(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

Assume that $p \in C_+(\overline{\Omega})$ and let

$$p^+ = \sup_{x\in\Omega} p(x) \quad \text{and} \quad p^- = \inf_{x\in\Omega} p(x).$$

We define the Lebesgue space with variable exponent by

$$L^{p(x)}(\Omega) = \big\{u : u \text{ is measurable and } \int_\Omega |u(x)|^{p(x)}\,dx < \infty\big\}.$$

This function space is a Banach space if it is endowed with the norm

$$|u|_{p(x)} = \inf\big\{\mu > 0;\ \int_\Omega |\frac{u(x)}{\mu}|^{p(x)}\,dx \le 1\big\}.$$

This norm is also called the Luxemburg norm. Then $L^{p(x)}(\Omega)$ is reflexive if and only if $1 < p^- \le p^+ < \infty$ and continuous functions with compact support are dense in $L^{p(x)}(\Omega)$ if $p^+ < \infty$.

The standard inclusion between Lebesgue spaces generalizes to the framework of spaces with variable exponent, namely if $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponents such that $p_1 \le p_2$ in Ω then there exists the continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$.

Let $L^{p'(x)}(\Omega)$ denote the conjugate space of $L^{p(x)}(\Omega)$, where $1/p(x)+1/p'(x) = 1$. Then for all $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ the following Hölder-type inequality holds:

$$\big|\int_\Omega uv\,dx\big| \le \Big(\frac{1}{p^-} + \frac{1}{p'^-}\Big)|u|_{p(x)}|v|_{p'(x)}\,. \tag{2.1}$$

An important role in analytic arguments on Lebesgue spaces with variable exponent is played by the *modular* of $L^{p(x)}(\Omega)$, which is the map $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)}\,dx.$$

If (u_n), $u \in L^{p(x)}(\Omega)$ and $p^+ < \infty$ then the following properties hold:

$$|u|_{p(x)} > 1 \ \Rightarrow\ |u|^{p^-}_{p(x)} \le \rho_{p(x)}(u) \le |u|^{p^+}_{p(x)}, \tag{2.2}$$

$$|u|_{p(x)} < 1 \ \Rightarrow\ |u|^{p^+}_{p(x)} \le \rho_{p(x)}(u) \le |u|^{p^-}_{p(x)}, \tag{2.3}$$

$$|u_n - u|_{p(x)} \to 0 \ \Leftrightarrow\ \rho_{p(x)}(u_n - u) \to 0. \tag{2.4}$$

We define the variable exponent Sobolev space by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\}.$$

On $W^{1,p(x)}(\Omega)$ we may consider one of the following equivalent norms

$$\|u\|_{p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)}$$

or

$$\|u\|_{p(x)} = \inf\big\{\mu > 0;\ \int_\Omega \Big(|\frac{\nabla u(x)}{\mu}|^{p(x)} + |\frac{u(x)}{\mu}|^{p(x)}\Big)\,dx \le 1\big\}\,.$$

Zhikov [30] showed that smooth functions are in general not dense in $W^{1,p(x)}(\Omega)$. This property is in relationship with the *Lavrentiev phenomenon*, which asserts that there exist variational problems for which the infimum over the smooth functions is strictly greater than the infimum over all functions that satisfy the same boundary conditions. We refer to [22, pp. 12-13] for more details.

Let $W_0^{1,p(x)}(\Omega)$ denote the closure of the set of compactly supported $W^{1,p(x)}$-functions with respect to the norm $\|u\|_{p(x)}$. When smooth functions are dense, we can also use the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. Using the Poincaré inequality, the space $W_0^{1,p(x)}(\Omega)$ can be defined, in an equivalent manner, as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{p(x)} = |\nabla u|_{p(x)}.$$

The vector space $(W_0^{1,p(x)}(\Omega), \|\cdot\|)$ is a separable and reflexive Banach space. Moreover, if $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponents such that $p_1 \leq p_2$ in Ω then there exists a continuous embedding $W_0^{1,p_2(x)}(\Omega) \hookrightarrow W_0^{1,p_1(x)}(\Omega)$.

Set

$$\varrho_{p(x)}(u) = \int_\Omega |\nabla u(x)|^{p(x)}\, dx. \tag{2.5}$$

If (u_n), $u \in W_0^{1,p(x)}(\Omega)$ then the following properties hold:

$$\|u\| > 1 \;\Rightarrow\; \|u\|^{p^-} \leq \varrho_{p(x)}(u) \leq \|u\|^{p^+}\,, \tag{2.6}$$

$$\|u\| < 1 \;\Rightarrow\; \|u\|^{p^+} \leq \varrho_{p(x)}(u) \leq \|u\|^{p^-}\,, \tag{2.7}$$

$$\|u_n - u\| \to 0 \;\Leftrightarrow\; \varrho_{p(x)}(u_n - u) \to 0\,. \tag{2.8}$$

Set

$$p_*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N \\ +\infty & \text{if } p(x) \geq N. \end{cases}$$

We point out that if $p, q \in C_+(\overline{\Omega})$ and $q(x) < p_\star(x)$ for all $x \in \overline{\Omega}$ then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact.

For any positive integer k, let

$$W^{k,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega), |\alpha| \leq k\},$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$ is a multi-index, $|\alpha| = \sum_{i=1}^N \alpha_i$ and

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial^{\alpha_1} x_1 \dots \partial^{\alpha_N} x_N}.$$

Then $W^{k,p(x)}(\Omega)$ is a separable and reflexive Banach space equipped with the norm

$$\|u\|_{k,p(x)} = \sum_{|\alpha| \leq k} |D^\alpha u|_{p(x)}.$$

The space $W_0^{k,p(x)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$.

Next, we recall some properties of the space

$$\mathcal{X} := W_0^{1,p(x)}(\Omega) \cap W^{2,p(x)}(\Omega).$$

For any $u \in \mathcal{X}$ we have $\|u\| = \|u\|_{1,p(x)} + \|u\|_{2,p(x)}$, thus

$$\|u\| = |u|_{p(x)} + |\nabla u|_{p(x)} + \sum_{|\alpha|=2} |D^\alpha u|_{p(x)}.$$

In Zang and Fu [28], the equivalence of the norms was proved, and they even established that the norm $|\Delta u|_{p(x)}$ is equivalent to the norm $\|u\|$ (see [28, Theorem 4.4]). Note that $(\mathcal{X}, \|\cdot\|)$ is a separable and reflexive Banach space.

We recall that the critical Sobolev exponent is defined as

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-2p(x)}, & \text{if } p(x) < \frac{N}{2}, \\ +\infty, & \text{if } p(x) \geq \frac{N}{2}. \end{cases}$$

Assume that $q \in C^+(\overline{\Omega})$ and $q(x) < p^*(x)$ for any $x \in \Omega$. Then, by [1, Theorem 3.2], the function space $\mathcal{X}$ is continuously and compactly embedded in $L^{q(x)}(\Omega)$.

For a constant function p, the variable exponent Lebesgue and Sobolev spaces coincide with the standard Lebesgue and Sobolev spaces. As pointed out in [22], the function spaces with variable exponent have some striking properties, such as:

(i) If $1 < p^- \leq p^+ < \infty$ and $p : \overline{\Omega} \to [1, \infty)$ is smooth, then the formula

$$\int_\Omega |u(x)|^p dx = p \int_0^\infty t^{p-1} \left|\{x \in \Omega;\ |u(x)| > t\}\right| dt$$

has no variable exponent analogue.

(ii) Variable exponent Lebesgue spaces do *not* have the *mean continuity property*. More precisely, if p is continuous and nonconstant in an open ball B, then there exists a function $u \in L^{p(x)}(B)$ such that $u(x+h) \notin L^{p(x)}(B)$ for all $h \in \mathbb{R}^N$ with arbitrary small norm.

(iii) The function spaces with variable exponent are *never* translation invariant. The use of convolution is also limited, for instance the Young inequality

$$|f * g|_{p(x)} \leq C|f|_{p(x)} \|g\|_{L^1}$$

holds if and only if p is constant.

3. Main result

In this article we assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary.

Let $p \in C_+(\overline{\Omega})$ and consider the function $\phi : \Omega \times [0, \infty) \to [0, \infty)$ satisfying the following hypotheses:

(H1) the mapping $\phi(\cdot, \xi)$ is measurable on Ω for all $\xi \geq 0$ and $\phi(x, \cdot)$ is locally absolutely continuous on $[0, \infty)$ for almost all $x \in \Omega$;

(H2) there exists $b > 0$ such that

$$|\phi(x, |v|)v| \leq b|v|^{p(x)-1}$$

for almost all $x \in \Omega$ and for all $v \in \mathbb{R}^N$;

(H3) there exists $c > 0$ such that

$$\phi(x, \xi) \geq c\xi^{p(x)-2}, \quad \phi(x, \xi) + \xi \frac{\partial \phi}{\partial \xi}(x, \xi) \geq c\xi^{p(x)-2}$$

for almost all $x \in \Omega$ and for all $\xi > 0$.

An interesting consequence of theses assumptions is that ϕ satisfies a Simon-type inequality. More precisely, if we denote

$$\Omega_1 := \{x \in \Omega : 1 < p(x) < 2\} \quad \text{and} \quad \Omega_2 := \{x \in \Omega;\ p(x) \geq 2\},$$

then the following estimate holds for all $u, v \in \mathbb{R}^N$

$$\begin{aligned} &\langle \phi(x,|u|)u - \phi(x,|v|)v, u - v\rangle \\ &\geq \begin{cases} c(|u|+|v|)^{p(x)-2}|u-v|^2 & \text{if } x \in \Omega_1 \text{ and } (u,v) \neq (0,0) \\ 4^{1-p^+}c|u-v|^{p(x)} & \text{if } x \in \Omega_2, \end{cases} \end{aligned} \tag{3.1}$$

where c is the positive constant from hypothesis (H3).

Let $A : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ defined by

$$A(u) = \int_\Omega \int_0^{|\nabla u(x)|} s\phi(x,s)\,ds\,dx.$$

Inequality (3.1) was used in [14] to show that $A'\colon W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ is both a monotone operator and a mapping of type (S_+). We refer to Simon [27] for the initial version of inequality (3.1) in the framework of the p-Laplace operator.

We study the following biharmonic problem with variable growth, competing reaction terms, and Navier boundary condition

$$\begin{gathered} \Delta(\phi(x,|\Delta u|)\Delta u) + \phi(x,|u|)u = |u|^{q(x)-2}u - |u|^{r(x)-2}u \quad \text{in } \Omega, \\ u = \Delta u = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{3.2}$$

where q, r are continuous functions.

If $\phi(x,\xi) = \xi^{p(x)-2}$ then we obtain the standard $p(x)$-Laplace biharmonic operator, that is, $\Delta^2_{p(x)}u := \Delta(|\Delta u|^{p(x)-2}\Delta u)$.

Our abstract setting includes the case $\phi(x,\xi) = (1+\xi^2)^{(p(x)-2)/2}$, which corresponds to the generalized biharmonic mean curvature operator

$$\Delta[(1+|\Delta u|^2)^{(p(x)-2)/2}\Delta u].$$

The biharmonic capillarity equation corresponds to

$$\phi(x,\xi) = \Big(1 + \frac{\xi^{p(x)}}{\sqrt{1+\xi^{2p(x)}}}\Big)\xi^{p(x)-2}, \quad x \in \Omega,\ \xi > 0,$$

hence the corresponding capillary phenomenon is described by the differential operator

$$\Delta\Big[\Big(1 + \frac{|\Delta u|^{p(x)}}{\sqrt{1+|\Delta u|^{2p(x)}}}\Big)|\Delta u|^{p(x)-2}\Delta u\Big].$$

We say that u is a solution of problem (3.2) if $u \in \mathcal{X} \setminus \{0\}$ with $\Delta u = 0$ on $\partial\Omega$ and

$$\int_\Omega [\phi(x,|\Delta u|)\Delta u\Delta v + \phi(x,|u|)uv]\,dx = \int_\Omega |u|^{q(x)-2}uvdx - \int_\Omega |u|^{r(x)-2}uv,$$

for all $v \in \mathcal{X}$. The main result of this paper is the following.

Theorem 3.1. *Assume that hypotheses*(H1)–(H3) *are fulfilled and that*

$$1 < q(x) < r(x) < p(x) < p^*(x) \quad \textit{for all } x \in \overline{\Omega}. \tag{3.3}$$

Then problem (3.2) *has at least one nontrivial weak solution with negative energy.*

In the present paper, problem (3.2) is studied for the *subcritical case*, namely under the basic hypothesis (3.3), which is crucial for compactness arguments. We consider that a very interesting research direction is to study the same problem in the *almost critical* setting, hence under the following assumption: there exists $x_0 \in \Omega$ such that

$$\begin{gathered} q(x) < r(x) < p(x) < p^*(x) \quad \text{for all } x \in \Omega \setminus \{x_0\} \text{ and} \\ q(x_0) = r(x_0) = p(x_0) = p^*(x_0). \end{gathered} \tag{3.4}$$

Of course, this hypothesis is not possible if the functions p, q and r are *constant*. We conjecture that the result stated in Theorem 3.1 remains true under assumption (3.4).

4. Proof of Theorem 3.1

Denote

$$\Phi(x,t) := \int_0^t s\phi(x,s)ds \quad \text{for all } x \in \Omega.$$

The energy functional associated to problem (3.2) is $\mathcal{E} : \mathcal{X} \to \mathbb{R}$ defined by

$$\mathcal{E}(u) = \int_\Omega \Phi(x, |\Delta u|)dx + \int_\Omega \Phi(x, |u|)dx - \int_\Omega \frac{|u|^{q(x)}}{q(x)}dx + \int_\Omega \frac{|u|^{r(x)}}{r(x)}dx.$$

By hypothesis (3.3), the function space $\mathcal{X}$ is continuously embedded into $L^{q(x)}(\Omega)$, $L^{r(x)}(\Omega)$, and $L^{p(x)}(\Omega)$. We deduce that $\mathcal{E}$ is well-defined.

On the other hand, with the same arguments as in [13, Proposition 3.3], the energy functional $\mathcal{E}$ is sequentially lower semicontinuous and of class C^1. Moreover, the mapping $\mathcal{E}' : \mathcal{X} \to \mathcal{X}^*$ is a strictly monotone, bounded homeomorphism and is of type (S_+); that is, if

$$u_n \rightharpoonup u \text{ in } \mathcal{X} \text{ and } \limsup_{n\to\infty} \mathcal{E}'(u_n)(u_n - u) \le 0,$$

then $u_n \to u$ in $\mathcal{X}$.

Proof. We split the proof of Theorem 3.1 into several steps.

Step 1. The energy functional $\mathcal{E}$ is coercive. Using (H3), we have for all $u \in \mathcal{X}$

$$\mathcal{E}(u) \ge c\int_\Omega \frac{|\Delta u|^{p(x)}}{p(x)}dx + c\int_\Omega \frac{|u|^{p(x)}}{p(x)}dx - \int_\Omega \frac{|u|^{q(x)}}{q(x)}dx + \int_\Omega \frac{|u|^{r(x)}}{r(x)}dx.$$

Therefore,

$$\begin{aligned} \mathcal{E}(u) &\ge \frac{c}{p^+}\int_\Omega |\Delta u|^{p(x)}dx + \frac{c}{p^+}\int_\Omega |u|^{p(x)}dx - \frac{1}{q^-}\int_\Omega |u|^{q(x)}dx + \frac{1}{r^+}\int_\Omega |u|^{r(x)}dx \\ &\ge \frac{c}{p^+}\int_\Omega |\Delta u|^{p(x)}dx + \frac{c}{p^+}\int_\Omega |u|^{p(x)}dx - \frac{1}{q^-}\int_\Omega |u|^{q(x)}dx. \end{aligned}$$

It follows that for all $u \in \mathcal{X}$ with $\|u\| > 1$ we have

$$\mathcal{E}(u) \ge \frac{c}{p^+}\|u\|^{p^-} - \frac{1}{q^-}|u|_{q(x)}^{q^+}dx.$$

We conclude the proof of Step 1 by using hypothesis (3.3), more precisely the fact that $q^- < p^+$.

The next step shows that the energy $\mathcal{E}$ does not satisfy one of the geometric hypotheses of the mountain pass theorem. More precisely, we show that there exists a "valley" for $\mathcal{E}$ close to the origin, so not far away from the origin, as it is required by the Ambrosetti-Rabinowitz theorem.

Step 2. There exists $v \in \mathcal{X}$ such that $\mathcal{E}(tv) < 0$ for all small enough $t > 0$. Since $q^- < r^-$, let $\varepsilon > 0$ be such that $q^- + \varepsilon < r^-$. By continuity, there exists $\omega \Subset \Omega$ such that

$$|q(x) - q^-| \leq \varepsilon \quad \text{for all } x \in \omega.$$

Let $v \in C_c^\infty(\Omega)$ such that $\operatorname{supp}(v) \subset \omega$, $0 \leq v \leq 1$, and $v \equiv 1$ in a subset of $\operatorname{supp}(v)$.

Hypothesis (H2) yields that for all $u \in \mathcal{X}$,

$$\Phi(x, |\Delta u|) \leq \Big| \int_0^{|\Delta u|} b|s|^{p(x)-1} ds \Big| \leq b \frac{|\Delta u|^{p(x)}}{p(x)},$$

$$\Phi(x, |u|) \leq \left| \int_0^{|u|} b|s|^{p(x)-1} ds \right| \leq b \, \frac{|u|^{p(x)}}{p(x)}.$$

It follows that

$$\int_\Omega [\Phi(x, |\Delta u|) + \Phi(x, |u|)] dx \leq b \int_\Omega \frac{|\Delta u|^{p(x)} + |u|^{p(x)}}{p(x)} dx.$$

It follows that for all $t \in (0, 1)$ we have

$$\begin{aligned}
\int_\Omega \left[\Phi(x, |\Delta(tv)|) + \Phi(x, |tv|)\right] dx &\leq b \int_\Omega t^{p(x)} \, \frac{|\Delta v|^{p(x)} + |v|^{p(x)}}{p(x)} dx \\
&\leq b \, \frac{t^{p^-}}{p^-} \int_\Omega \Big(|\Delta v|^{p(x)} + |v|^{p(x)}| \Big) dx \\
&= C_1 t^{p^-}.
\end{aligned}$$

Next, we have

$$\int_\Omega \frac{|tv|^{q(x)}}{q(x)} dx \geq \frac{t^{q^- + \varepsilon}}{q^-} \int_\omega |v|^{q(x)} dx = C_2 \, t^{q^- + \varepsilon},$$

$$\int_\Omega \frac{|tv|^{r(x)}}{r(x)} dx \leq \frac{t^{r^-}}{r^-} \int_\Omega |v|^{r(x)} dx = C_3 \, t^{r^-}.$$

We conclude that

$$\mathcal{E}(tv) \leq C_1 t^{p^-} - C_2 t^{q^- + \varepsilon} + C_3 t^{r^-}, \tag{4.1}$$

where C_1, C_2 and C_3 are positive constants.

Since $q^- + \varepsilon < r^- < p^-$, relation (4.1) implies that $\mathcal{E}(tv) < 0$, provided that $t > 0$ is small enough. Since $\mathcal{E}$ is coercive and weakly lower semi-continuous, it admits a global minimizer u_0, which is a critical point of $\mathcal{E}$. By step 2, we have $u_0 \neq 0$.

To show that u_0 is a solution of problem (3.2), it remains to show that $\Delta u_0 = 0$ on $\partial\Omega$.

Step 3. We have $\Delta u_0 = 0$ on $\partial\Omega$.

Since u_0 verifies (3.2) in the weak sense, we deduce that u_0 satisfies, for all $v \in \mathcal{X}$,

$$\int_\Omega \phi(x, |\Delta u_0|) \Delta u_0 \Delta v dx = \int_\Omega A(x) v dx, \tag{4.2}$$

where

$$A(x) := |u_0|^{q(x)-2} u_0 - |u_0|^{r(x)-2} u_0 - \phi(x, |u_0|) u_0.$$

Let $z \in \mathcal{X}$ be the unique solution of the linear problem

$$\begin{gathered} \Delta z = A(x) \quad \text{in } \Omega \\ z = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{4.3}$$

It follows that for all $v \in \mathcal{X}$,

$$\int_\Omega \phi(x, |\Delta u_0|)\Delta u_0 \Delta v dx = \int_\Omega (\Delta z) v dx.$$

By Green's formula we deduce that for all $v \in C_c^\infty(\Omega) \subset \mathcal{X}$

$$\int_\Omega \phi(x, |\Delta u_0|)\Delta u_0 \Delta v dx = \int_\Omega z \Delta v dx. \tag{4.4}$$

For all $w \in C_c^\infty(\Omega)$, let $v \in C_c^\infty(\Omega$ be the unique solution of the problem

$$\begin{gathered} \Delta v = w \quad \text{in } \Omega \\ v = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

Returning to (4.4), we deduce that for all $w \in C_c^\infty(\Omega)$

$$\int_\Omega (\phi(x, |\Delta u_0|)\Delta u_0 - z) \, w dx = 0.$$

Applying [3, Lemma VIII.1] we conclude that

$$\phi(x, |\Delta u_0|)\Delta u_0 - z = 0 \quad \text{in } \Omega. \tag{4.5}$$

But $z = 0$ on $\partial\Omega$. Using hypothesis (H3), relation (4.5) implies that $\Delta u_0 = 0$ on $\partial\Omega$. The proof of Theorem 3.1 is now complete. □

A very interesting open problem concerns the same analysis if the left-hand side of problem (3.2) is replaced either by the differential operator

$$\Delta(\phi_1(x, |\Delta u|)\Delta u) + V(x)\Delta(\phi_2(x, |\Delta u|)\Delta u) \tag{4.6}$$

or by

$$\Delta(\phi_1(x, |\Delta u|)\Delta u) + V(x)\Delta(\phi_2(x, |\Delta u|)\Delta u) \log(e + |x|), \tag{4.7}$$

where V is a nonnegative potential and ϕ_1, ϕ_2 satisfy hypotheses (H1)–(H3) corresponding to the variable exponents $p_1(x)$, $p_2(x)$ with $p_1(x) \le p_2(x)$ in Ω. Considering two different materials with power hardening exponents $p_1(x)$ and $p_2(x)$, respectively, the coefficient $V(x)$ dictates the geometry of a composite of the two materials. When $V(x) > 0$ then $p_2(x)$-material is present, otherwise the $p_1(x)$-material is the only one making the composite. Composite materials with locally different hardening exponents $p_1(x)$ and $p_2(x)$ can be described using the energies associated to the differential operators defined in (4.6) and (4.7).

Problems of this type were also motivated by applications to elasticity, homogenization, modelling of strongly anisotropic materials, Lavrentiev phenomenon, etc. In the case of constant exponents, we refer to the pioneering papers by Marcellini [15, 16] and Mingione et al. [2, 8, 9]. Double phase problems with variable growth have been recently considered by Cencelj, Rădulescu and Repovš [4], Rădulescu and Zhang [23], and Shi, Rădulescu, Repovš and Zhang [26].

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