Concentration of normalized solutions for non-autonomous fractional Schrödinger equations

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Abstract

In the present paper, we investigate the existence, multiplicity and concentration of normalized solutions to the following fractional Schrödinger equation with potential

$$\begin{cases} (-\Delta)^s u + V(\varepsilon x)u + \lambda u = f(u),\ x \in \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} |u|^2 dx = a^2, \end{cases}$$

where $0 < s < 1$, $N \geq 2$, $a,\ \varepsilon > 0$, $V \in C(\mathbb{R}^N, \mathbb{R})$, λ is an unknown parameter that will appear as a Lagrange multiplier, f is a mass subcritical and Sobolev subcritical nonlinearity. Under fairly general assumptions about f and a global condition about V, with the aid of minimization techniques and Ljusternik-Schnirelmann category theory, we study the relation between the numbers of normalized solutions and the topology of the set where the potential V attains its minimum value. In addition, we obtain the decay behavior of normalized solutions. Finally, by using of the cut-off technique we also consider the Sobolev supercritical case that has not been considered about the study of normalized solutions.

Key words: Normalized solutions; L^2-subcritical; Sobolev supercritical growth.

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1 Introduction

In the present paper, we study the existence, multiplicity and concentration and decay behavior of normalized solutions to the following fractional Schrödinger equation with potential

$$\begin{cases} (-\Delta)^s u + V(\varepsilon x)u + \lambda u = f(u),\ x \in \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} |u|^2 dx = a^2, \end{cases} \tag{P_a}$$

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where $0 < s < 1$, $N \geq 2$, a, $\varepsilon > 0$, $V \in C(\mathbb{R}^N, \mathbb{R})$, λ is an unknown parameter that will appear as a Lagrange multiplier, f is a mass subcritical and Sobolev subcritical nonlinearity. Moreover, by using of the cut-off technique we also consider the Sobolev supercritical case to the study of normalized solutions.

The problem

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u + \lambda u = f(u), \quad x \in \mathbb{R}^N, \tag{1.1}$$

comes from the following time-dependent nonlinear Schrödinger equation

$$i\varepsilon\frac{\partial \Psi}{\partial t} = \varepsilon^{2s}(-\Delta)^s\Psi + V(x)\Psi - g(|\Psi|^2)\Psi, \tag{1.2}$$

by looking for standing wave solutions

$$\Psi(t,x) = e^{\frac{i\lambda t}{\varepsilon}}u(x),$$

with $\lambda \in \mathbb{R}$ is the frequency or the chemical potential, where $0 < s < 1$, $(-\Delta)^s$ denotes the fractional Laplacian of order s, $V : \mathbb{R}^N \to \mathbb{R}$ is an external potential function, $f(u)$ is the nonlinearity, i denotes the imaginary unit and ε is a sufficiently small parameter which corresponds to the Planck constant. Furthermore, $\Psi = \Psi(t,x) \in \mathbb{C}$ is a wave function which represents the quantum mechanical probability amplitude for a given unit mass particle to have position x at time t (the corresponding probability density is $|\Psi(t,x)|^2$), g is an appropriate nonlinearity which verifies $f(u) = g(|u|^2)u$. The equation (1.2) is the fractional nonlinear Schrödinger equation that was introduced by Laskin([22]), as a result of extending the Feynman path integral, from Brownian-like to Lévy-like quantum mechanical paths, where the Feynman path integral leads to the classical Schrödinger equation and the path integral over Lévy trajectories leads to the fractional Schrödinger equation. Such kind of equation is of particular interest in fractional quantum mechanics in the study of particles on stochastic fields modelled by Lévy processes, see [1], which are widely used in optimization, finance, phase transitions, stratified materials, crystal dislocation, flame propagation, conservation laws, materials science and water waves. It results that fractional problems are conducted extensive and deeply researched by many experts and scholars.

Equation (1.1) has attracted much attention in the community of nonlinear PDEs in the last decades. A solution $u(x)$ is referred to as a bound state of (1.1) if $u(x) \to 0$ as $|x| \to +\infty$. When $\varepsilon > 0$ is sufficiently small, bound states of (1.1) are called semiclassical states and an important feature of semiclassical states is their concentration as $\varepsilon \to 0$. Making the change of variable $\varepsilon z = x$, we can rewrite (1.1) as the following equation

$$(-\Delta)^s u + V(\varepsilon x)u + \lambda u = f(u), \quad x \in \mathbb{R}^N. \tag{1.3}$$

Until now, there exist two substantially different view of points in terms of the frequency λ in (1.3). One is to regard λ as a given constant. At this time, with the aid of the critical point theory, there are a lot of works on the existence, multiplicity and concentration of solutions for equation (1.3) with different potentials involving Sobolev subcritical, critical and supercritical growth. It seems almost impossible for us to give a complete list of references. We refer the readers to [9, 10, 32, 33, 34, 36, 41] and the references therein. The other one is to regard λ as unknown quantities to (1.3). In this case, it is natural to prescribe the value of the mass $\int_{\mathbb{R}^N} |u|^2 dx$ so that λ can be interpreted as the Lagrange multiplier. On this line, a new critical exponent appears, the L^2-critical exponent (also named mass-critical exponent): $r = 2 + \frac{4s}{N}$. It is the threshold exponent for many dynamical properties, such as global existence vs. blow-up, and the stability or instability of ground states. People call $r < 2 + \frac{4s}{N}$ as L^2-subcritical, and $r > 2 + \frac{4s}{N}$ as L^2-supercritical. In addition, the mass admits often a clear physical meaning: it represents the power supply in nonlinear optics, or the total number of atoms in Bose-Einstein condensation. They are two main fields of application of the NLS and physicists are often interested in them.

In quantum mechanics, as mentioned above, $|\Psi(t,x)|^2$ represents the probability density of the particles appearing in space x at time t. For single particle system, physicists are interested in normalized solutions, namely, solutions satisfying the normalized condition $\int_{\mathbb{R}^N} |\Psi(t,x)|^2 dx = 1$. For n body system of Bose-Einstein condensate (see [3]), the wave function for the whole condensate becomes

$\widetilde{\Psi}(t,x) = \sqrt{n}\Psi(t,x)$, and so the wave function is normalized according to the total number of the particles, i.e., $\int_{\mathbb{R}^N} |\widetilde{\Psi}(t,x)|^2 dx = n$ (see [38]). But for convenience and extension, the normalized condition in mathematics is always assumed to hold for any positive constant $a > 0$, i.e., $\int_{\mathbb{R}^N} |\Psi(t,x)|^2 dx = a^2$. Accordingly, $\int_{\mathbb{R}^N} |u|^2 dx = a^2$. What we are interested in this paper is the existence of solutions to Eq. (1.3) with prescribed L^2-norm

$$\int_{\mathbb{R}^N} |u|^2 dx = a^2.$$

Namely, for given $a > 0$, to study the solutions for (1.3) under the L^2-norm constrained manifold

$$S_a := \left\{ u \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2 dx = a^2 \right\},$$

where the definition of $H^s(\mathbb{R}^N)$ reads below. As above, physically, such type of solutions are the so-called normalized solutions to (P_a), which are critical points of the energy functional $I_\varepsilon : H^s(\mathbb{R}^N) \to \mathbb{R}$ given by

$$I_\varepsilon(u) := \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x) u^2 dx - \int_{\mathbb{R}^N} F(u) dx$$

restricted on S_a, where $F(u) := \int_0^u f(t)dt$. At this time, the frequency λ is an unknown number that can be determined as the Lagrange multiplier associated to the constraint S_a. In addition, it is well known to us that the mass is conserved along the trajectories of (1.2), i.e.,

$$\int_{\mathbb{R}^N} |\Psi(t,x)|^2 dx = \int_{\mathbb{R}^N} |u|^2 dx$$

for all $t > 0$. The study about the existence of normalized solutions is particularly relevant from a physical point of view, since it can provide a good insight of the dynamical properties (such as, orbital stability and instability) of solutions to the equation (1.2), and is becoming more and more popular among many scholars (see [12]).

When $s = 1$ and $V \equiv 0$, Jeanjean in [21] considered a semilinear elliptic equation

$$-\Delta u + \lambda u = g(u), \quad x \in \mathbb{R}^N, \tag{1.4}$$

where $N \geq 1$, $\lambda \in \mathbb{R}$, g satisfies

(g_0) $g \in C(\mathbb{R}, \mathbb{R})$ and g is odd;

(g_1) there exist $\alpha,\ \beta \in \mathbb{R}$ with $2 + \frac{4}{N} < \alpha \leq \beta < 2^*$ such that

$$0 < \alpha G(t) \leq g(t)t \leq \beta G(t)$$

for all $t \in \mathbb{R}\backslash\{0\}$, where $2^* := \frac{2N}{N-2}$ for $N \geq 3$ and $2^* := +\infty$ for $N = 1,\ 2$;

(g_2) $\widetilde{G}(t) := g(t)t - 2G(t) \in C^1(\mathbb{R}, \mathbb{R})$ and

$$\widetilde{G}'(t)t > (2 + \frac{4}{N})\widetilde{G}(t)$$

for all $t \in \mathbb{R}\backslash\{0\}$.

It is easy to see that the corresponding energy functional is unbounded from below on S_a. By using of a minimax procedure, Jeanjean showed that for each $a > 0$, (1.4) possesses at least one couple $(u_a, \lambda_a) \in H^1(\mathbb{R}^N) \times \mathbb{R}^+$ of weak solution with $\|u_a\|_2 = a$ and u_a is radial under (g_0)-(g_1) for $N \geq 2$, where $H^1(\mathbb{R}^N)$ is endowed with the usual norm $\|u\|_{H^1} = \left(\|\nabla u\|_2^2 + \|u\|_2^2\right)^{1/2}$. Furthermore, when (g_2) is also assumed, he obtained the existence of ground states for $N \geq 1$. But, afterwards, there was little progress about the study of normalized solutions for a long time. One of the main reasons is that it is hard to prove the boundedness of constrained Palais-Smale sequence when the functional is unbounded from below on the constraint manifold. More recently, problems of such type begun to receive much attention. Still under (g_0)-(g_1), by virtue of a fountain theorem type argument, Bartsch and de Valeriola [4] established infinitely many radial solutions to (1.4) with $\|u\|_2 = a > 0$. About another proof for this multiplicity result can be seen [20], and [5, 6] but requires the additional assumption (g_2). For combined power nonlinearities, Soave [35] studied the existence and properties of ground states to Eq.

$$-\Delta u + \lambda u = \mu|u|^{q-2}u + |u|^{p-2}u, \quad x \in \mathbb{R}^N$$

on S_a, where $N \geq 1$, $2 < q \leq 2 + \frac{4}{N} \leq p < 2^*$. There he gave a complete classification about the existence and nonexistence of normalized solution to L^2-subcritical, L^2-critical and L^2-supercritical cases. Which is more difficult and substantially different with purely subcritical or supercritical cases, because the interplay between subcritical, critical and supercritical nonlinearities has deep impacts on the geometry of the functional and on the existence and properties of ground states.

Inspired by [35], Chen and Liu [8] studied the asymptotic behavior of ground states for the fractional Schrödinger equation with combined L^2-critical and L^2-subcritical nonlinearities

$$(-\Delta)^s u + \lambda u = \mu|u|^{q-2}u + |u|^{p-2}u,\ x \in \mathbb{R}^N$$

with prescribed mass $\|u\|_2 = a > 0$, where $\mu \in \mathbb{R}$, $2 < q < p = 2 + \frac{4s}{N}$, $N \geq 2$. The same equation is studied by Luo and Zhang [29], and they extended the range of exponents to $2 < q < p < 2_s^* := \frac{2N}{N-2s}$. Under different assumptions on $q < p$, they obtained some existence and nonexistence results about the normalized solutions. Feng et al. [16] studied the existence and instability of normalized standing waves for the fractional Schrödinger equation

$$i\frac{\partial \psi}{\partial t} = (-\Delta)^s \psi - |\psi|^{p-2}\psi,\ x \in \mathbb{R}^N,$$

where $2 + \frac{4s}{N} < p < 2_s^*$. Relied on the construction of a minimax structure, by means of a Pohozaev' s mountain in a product space and some deformation arguments under a new version of the Palais-Smale condition introduced in [19] and [20], Cingolani, Gallo and Tanaka [7] obtained the existence of a weak solution to the following problem

$$\begin{cases} (-\Delta)^s u + \lambda u = f(u),\ x \in \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = a^2, \\ u \in H^s_{rad}(\mathbb{R}^N), \end{cases}$$

where $N \geq 2$, f is a L^2-subcritical nonlinearity, where

$$H^s_{rad}(\mathbb{R}^N) := \{u \in H^s(\mathbb{R}^N) : u \text{ is radially decreasing}\}.$$

Under different conditions on $a,\ p,\ s$ and N, Zhang and Han [42] obtained the existence of normalized solutions to the fractional Schrödinger equation with Sobolev critical growth

$$\begin{cases} (-\Delta)^s u + \lambda u = |u|^{p-2}u + |u|^{2_s^*-2}u,\ x \in \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = a^2, \end{cases}$$

where $N \geq 2$, $2 < p < 2_s^*$. Combining minimax method, barycentric functions and Brouwer degree theory, the authors [39] investigated the existence of normalized solutions to the following mass subcritical fractional Schrödinger equations in exterior domains

$$\begin{cases} (-\Delta)^s u + \lambda u = |u|^{p-2}u, & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \int_{\Omega} |u|^2 dx = a^2, \end{cases}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ is an exterior domain with smooth boundary $\partial\Omega \neq \emptyset$ such that $\mathbb{R}^N \backslash \Omega$ is bounded. By using of scaling transformation, classification discussion and concentration compactness principle, Zuo and Rădulescu [43] considered the normalized solutions for fractional Sobolev critical nonlinear Schrödinger coupled systems. Du et al. [14] studied the existence, nonexistence and mass concentration of normalized solutions for the nonlinear fractional Schrödinger equation

$$(-\Delta)^s u + V(x)u = \mu u + af(u),\ x \in \mathbb{R}^N,$$

where f is a Sobolev subcritical nonlinearity but the potential is coercive. Under the periodic potential function $V(x)$, Dinh [11] considered normalized solutions for the fractional Schrödinger equation

$$(-\Delta)^s u + V(x)u = \mu u + |u|^{p-2}u,\ x \in \mathbb{R}^N,$$

where $2 < p < 2_s^*$ if $N > 2s$ and $2 < p < +\infty$ if $N \le 2s$. Using the concentration compactness principle, he gave a complete classification for the existence and non-existence of minimizers for the problem. And he also gave a detailed description of blow-up behaviour of minimizers once the mass tends to a critical value for L^2-critical case.

It is not hard to see from the above literature that most of the results about the study of normalized solutions for fractional Schrödinger equations are performed in the absence of potential, or coercive and periodic potential. Motivated by the aforementioned papers, in this article, we focus mainly on the existence, multiplicity and concentration and decay behavior of normalized solutions to (P_a) under the potential condition (V). To the best of our knowledge, there is no result about Sobolev supercritical cases to the study of normalized solutions. In this paper, we shall fill the gap of information. To reduce the statements for main results, we list the assumptions as follows:

(V) $V \in C(\mathbb{R}^N, \mathbb{R}) \cap L^\infty(\mathbb{R}^N)$ and

$$0 < V_0 := \inf_{x\in\mathbb{R}^N} V(x) < V_\infty := \liminf_{|x|\to+\infty} V(x).$$

(f_1) $f \in C(\mathbb{R}, \mathbb{R})$ is odd and there exist $r \in (2, 2 + \frac{4s}{N})$ and $\alpha \in (0, +\infty)$ such that $\lim\limits_{t\to 0} \frac{|f(t)|}{|t|^{r-1}} = \alpha$.

(f_2) There exist two constants $C_1,\ C_2 > 0$ and $p \in (2, 2 + \frac{4s}{N})$ such that

$$|f(t)| \le C_1 + C_2|t|^{p-1},\ \forall t \in \mathbb{R}.$$

(f_3) There exists $q \in (2, 2 + \frac{4s}{N})$ such that $\frac{f(t)}{t^{q-1}}$ is an increasing function of t on $(0, +\infty)$.

Let $\mathscr{S}$ be the Schwartz space of rapidly decaying C^∞ functions in $\mathbb{R}^N$, for any $u \in \mathscr{S}$ and $s \in (0, 1)$, $(-\Delta)^s$ is defined as

$$\begin{aligned}(-\Delta)^s u(x) =& C_{N,s} P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy = C_{N,s} \lim_{\varepsilon\to 0+} \int_{\mathscr{C}B_\varepsilon(x)} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy \\ =& \frac{1}{2} C_{N,s} \int_{\mathbb{R}^N} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{N+2s}} dy,\end{aligned}$$

where $\mathscr{C}B_\varepsilon(x) := \mathbb{R}^N \setminus B_\varepsilon(x)$. The symbol P.V. stands for the Cauchy principal value and $C_{N,s}$ is a dimensional constant that depends on N and s, precisely given by

$$C_{N,s} = \left(\int_{\mathbb{R}^N} \frac{1 - cos\zeta_1}{|\zeta|^{N+2s}} d\zeta \right)^{-1},\ \zeta = (\zeta_1, \zeta_2, \cdots, \zeta_N).$$

For any $0 < s < 1$, the fractional Sobolev space $H^s(\mathbb{R}^N)$ is defined by

$$H^s(\mathbb{R}^N) := \{u \in L^2(\mathbb{R}^N) : u \in \mathcal{D}^{s,2}(\mathbb{R}^N)\}$$

endowed with the norm

$$\|u\|_{H^s} := \Big(\|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} + \|u\|^2_{L^2(\mathbb{R}^N)} \Big)^{\frac{1}{2}} = \Big(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx + \int_{\mathbb{R}^N} u^2 dx \Big)^{\frac{1}{2}}.$$

Where the homogeneous fractional Sobolev space

$$\mathcal{D}^{s,2}(\mathbb{R}^N) := \{u \in L^{2_s^*}(\mathbb{R}^N) : |\xi|^s \hat{u}(\xi) \in L^2(\mathbb{R}^N)\},$$

which is the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm

$$\|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} := \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx = \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi.$$

Moreover, from the monograph by Molica Bisci-Rădulescu-Servadei [30], we can see that

$$2C_{N,s}^{-1} \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi = 2C_{N,s}^{-1} \|(-\Delta)^{\frac{s}{2}} u\|^2_{L^2(\mathbb{R}^N)} = [u]^2_{H^s(\mathbb{R}^N)},$$

where

$$[u]_{H^s(\mathbb{R}^N)} := \Big(\int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} dxdy \Big)^{\frac{1}{2}}.$$

The best fractional critical Sobolev constant is given by

$$S := \inf_{u \in \mathcal{D}^{s,2}(\mathbb{R}^N) \setminus \{0\}} \frac{\|u\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)}}{\|u\|^2_{L^{2_s^*}(\mathbb{R}^N)}}.$$

For any $\delta > 0$, set $M_\delta := \{x \in \mathbb{R}^N : \text{dist}(x, M) \leq \delta\}$, where $M := \{x \in \mathbb{R}^N : V(x) = V_0\}$. Without loss of generality, we may assume that $0 \in M$.

We are now in a position to state the main results.

Theorem 1.1. *Let (V) and (f_1)-(f_3) hold. Then there exists $V_* > 0$ such that for each $\delta > 0$, there is a constant $\varepsilon_0 > 0$ such that (P_a) possesses at least $cat_{M_\delta}(M)$ couples $(u_j, \lambda_j) \in H^s(\mathbb{R}^N) \times \mathbb{R}^+$ of weak solutions for $0 < \varepsilon < \varepsilon_0$ and $||V||_\infty < V_*$ with $I_\varepsilon(u_j) < 0$. Furthermore, if u_ε is one of these solutions and $\xi_\varepsilon \in \mathbb{R}^N$ is a global maximum of $|u_\varepsilon|$, then*

(i) *(concentration)* $\lim_{\varepsilon \to 0} V(\varepsilon \xi_\varepsilon) = V_0$;

(ii) *(decay estimates) there exists a constant $C > 0$ such that*

$$u_\varepsilon(x) \leq C|x - \xi_\varepsilon|^{-(N+2s)}.$$

Moreover, fixed $0 < \varepsilon < \varepsilon_0$ and $\|V\|_\infty < V_*$, we also study the existence of normalized solutions for the following Schrödinger equation with Sobolev supercritical growth

$$\begin{cases} (-\Delta)^s u + V(\varepsilon x) u + \lambda u = f(u) + \eta |u|^{p-2} u, \text{ in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = a^2, \end{cases} \tag{Q_a}$$

where $p > 2_s^*$. Suppose that (V), (f_1), (f_3) and the following $(\tilde{f}_2)$ hold.

$(\tilde{f}_2)$ There exist two constants $C_1,\ C_2 > 0$ and $q \in (r, 2 + \frac{4s}{N})$ such that

$$|f(t)| \leq C_1 + C_2 |t|^{q-1},\ \forall\, t \in \mathbb{R}.$$

With the aid of the truncation technique, we obtain the following theorem.

Theorem 1.2. *Let (V) and (f_1), $(\tilde{f}_2)$ and (f_3) hold. For fixed $0 < \varepsilon < \varepsilon_0$ and $\|V\|_\infty < V_*$, there exists some $\eta_0 > 0$ such that for $\eta \in (0, \eta_0]$, problem (Q_a) admits at least $cat_{M_\delta}(M)$ couples $(u, \lambda) \in H^s(\mathbb{R}^N) \times \mathbb{R}^+$ of weak solutions.*

Remark 1.3. *As far as we know, there is no result on the normalized solutions above Sobolev supercritical case. In our work, with the aid of the truncation technique, we investigate the existence of normalized soluitions to* (Q_a)*.*

Remark 1.4. *Notice that* $(-\Delta)^s$ *on* $\mathbb{R}^N$ *with* $0 < s < 1$ *is a nonlocal operator. The nonlocal nature of the fractional Laplacian makes it difficult to study. Comparing with the classical Schrödinger equation, we encounter some new challenges due to the nonlocal nature of the fractional Laplacian. Such as, the ground state for* $(-\Delta)^s$ *decays polynomially at infinity, which is in contrast to the fact that the ground state for* $-\Delta$ *decays exponentially at infinity. Even to the existence of normalized solutions, contrast with the following fractional Schrödinger equation without potential*

$$\begin{cases} (-\Delta)^s u + \lambda u = f(u), \text{ in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = a^2, \end{cases}$$

the technique introduced in [21] by Jeanjean is not applicable at all. As a consequence, some fine estimates are necessary.

Remark 1.5. *Unlike [11] and [14], we do not assume that the potential is coercive or periodic. In [40], Yang, Yu and Tang obtained the multiplicity of normalized solutions to the following fractional Schrödinger equations in the absence of potential*

$$\begin{cases} (-\Delta)^s u + \lambda u = h(\varepsilon x) f(u), \text{ in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = a^2. \end{cases}$$

And they proved that the numbers of normalized solutions is at least the numbers of global maximum points of h *when* $\varepsilon > 0$ *is small enough. We point out that people also begin to pay attention to the study of normalized solutions for fractional Choquard equations, see [18, 26]. More results, please see [24, 25, 28].*

Remark 1.6. *When the nonlinearity* f *satisfies*

(f_1') $f(t) = o(t)$ *as* $t \to 0,$ $f(t)t > 0$ *for all* $t > 0$ *and* $f(t) = 0$ *for all* $t \leq 0$*;*

(f_2') $\frac{f(t)}{t}$ *is strictly increasing for* $t > 0$*;*

(f_3') $|f(t)| \leq C(1 + |t|^{p-1})$ *for some* $C > 0,$ *where* $2 < p < 2_s^*$*,*
and (V) *hold, Shang and Zhang [36] considered the following fixed frequency problem*

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u = |u|^{2_s^*-2}u + \lambda f(u), \ x \in \mathbb{R}^N,$$

and investigated the relation between the numbers of solutions and the topology of the set where V *attains its minimum by applying Ljusternik-Schnirelmann category theory. Our results can be seen as an expansion from the fixed frequency problem to unfixed frequency problem.*

In the sequel, let us introduce the following fractional Gagliardo-Nirenberg inequality ([23]): if $u \in H^s(\mathbb{R}^N)$ and $2 < t < 2_s^*$, then

$$\|u\|_t^t \leq C_{s,N,t} \|(-\Delta)^{\frac{s}{2}} u\|_2^{\frac{N(t-2)}{2s}} \|u\|_2^{t-\frac{N(t-2)}{2s}}. \tag{1.5}$$

By (f_1)-(f_2), there exist two constants $C_1, \ C_2 > 0$ such that

$$|f(t)| \leq C_1|t|^{r-1} + C_2|t|^{p-1}, \ \forall \, t \in \mathbb{R}. \tag{1.6}$$

In addition, by (f_3), for any $t, \ \tau > 0$ and $t \geq 1$, one has

$$F(t\tau) \geq t^q F(\tau). \tag{1.7}$$

By (f_1)-(f_2), for any $\tau > 0$, there exists a constant $C_\tau > 0$ such that

$$|f(t)| \leq \tau|t| + C_\tau|t|^{p-1}, \ \forall\, t \in \mathbb{R}. \tag{1.8}$$

By (f_1) and (f_3),

$$f(t)t \geq qF(t) \geq 2F(t) \geq 0, \ \forall\, t \in \mathbb{R}. \tag{1.9}$$

2 The autonomous case

We shall firstly consider the autonomous case. To be exact, we firstly consider the existence of normalized solutions to the following autonomous problem

$$\begin{cases} (-\Delta)^s u + \mu u + \lambda u = f(u), \ x \in \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = a^2, \end{cases} \tag{2.1}$$

where $0 < s < 1$, $N \geq 2$, $a > 0$, $\mu \geq 0$, λ is unknown, f is a mass subcritical and Sobolev subcritical nonlinearity satisfying (f_1)-(f_3).

As is known to us that solutions to the problem (2.1) are critical points of the energy functional $I_\mu : H^s(\mathbb{R}^N) \to \mathbb{R}$ given by

$$I_\mu(u) := \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} \mu u^2 dx - \int_{\mathbb{R}^N} F(u) dx$$

restricted on S_a.

Lemma 2.1. *I_μ is coercive and bounded from below on S_a.*

Proof. For any $u \in S_a$, by (1.5) and (1.6)

$$\begin{aligned} I_\mu(u) =& \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} \mu u^2 dx - \int_{\mathbb{R}^N} F(u) dx \\ \geq& \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx - C_1\int_{\mathbb{R}^N} |u|^r dx - C_2\int_{\mathbb{R}^N} |u|^p dx \\ \geq& \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx - C_1 C_{s,N,r}\|(-\Delta)^{\frac{s}{2}} u\|_2^{\frac{N(r-2)}{2s}} \|u\|_2^{r-\frac{N(r-2)}{2s}} \\ &- C_2 C_{s,N,p}\|(-\Delta)^{\frac{s}{2}} u\|_2^{\frac{N(p-2)}{2s}} \|u\|_2^{p-\frac{N(p-2)}{2s}} \\ =& \frac{1}{2}\|(-\Delta)^{\frac{s}{2}} u\|_2^2 - C_1 C_{s,N,r}\|(-\Delta)^{\frac{s}{2}} u\|_2^{\frac{N(r-2)}{2s}} a^{r-\frac{N(r-2)}{2s}} \\ &- C_2 C_{s,N,p}\|(-\Delta)^{\frac{s}{2}} u\|_2^{\frac{N(p-2)}{2s}} a^{p-\frac{N(p-2)}{2s}} \end{aligned}$$

Since $2 < r,\ p < 2 + \frac{4s}{N}$, it follows that $\frac{N(r-2)}{2s},\ \frac{N(p-2)}{2s} < 2$. Consequently, I_μ is coercive and bounded from below on S_a. ☐

Remark 2.2. *By Lemma 2.1, $m_{\mu,a} := \inf\limits_{u \in S_a} I_\mu(u)$ is well defined. Furthermore, from the proof of Lemma 2.1, together with (V) we know that I_ε is also coercive and bounded from below on S_a.*

About $m_{\mu,a}$, we have the following lemma.

Lemma 2.3. *There exists $V_* > 0$ such that $m_{\mu,a} < 0$ for $0 \leq \mu \leq V_*$.*

Proof. Let $u_0 \in S_a \cap L^\infty(\mathbb{R}^N)$ be a nonnegative function. Set

$$(\tau * u_0)(x) = e^{\frac{N\tau}{2}} u_0(e^\tau x),$$

where $x \in \mathbb{R}^N$, $\tau \in \mathbb{R}$. Then, there holds

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}(\tau * u_0)|^2 dx = e^{2\tau s} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_0|^2 dx,$$

$$\int_{\mathbb{R}^N} |\tau * u_0|^2 dx = \int_{\mathbb{R}^N} |u_0|^2 dx,$$

$$\int_{\mathbb{R}^N} F(\tau * u_0) dx = \int_{\mathbb{R}^N} F(e^{\frac{N\tau}{2}} u_0(e^\tau x)) dx = e^{-N\tau} \int_{\mathbb{R}^N} F(e^{\frac{N\tau}{2}} u_0) dx.$$

By (f_1), there exists $\delta_0 > 0$ such that

$$\frac{rF(t)}{t^r} \geq \frac{\alpha}{2}, \ \forall\, t \in [0, \delta_0].$$

Noting that when $\tau < 0$ and $|\tau|$ large enough, one has $0 \leq e^{\frac{N\tau}{2}} u_0(x) \leq \delta_0, \ \forall x \in \mathbb{R}^N$. Hence,

$$\begin{aligned}\int_{\mathbb{R}^N} F(\tau * u_0) dx =& e^{-N\tau} \int_{\mathbb{R}^N} F(e^{\frac{N\tau}{2}} u_0) dx \\ \geq& e^{-N\tau} \frac{\alpha}{2r} \int_{\mathbb{R}^N} |e^{\frac{N\tau}{2}} u_0|^r dx \\ =& \frac{\alpha}{2r} e^{N\tau(\frac{r}{2}-1)} \int_{\mathbb{R}^N} |u_0|^r dx,\end{aligned}$$

which implies that

$$I_\mu(\tau * u_0) \leq \frac{1}{2} e^{2\tau s} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_0|^2 dx + \frac{\mu}{2} a^2 - \frac{\alpha}{2r} e^{N\tau(\frac{r}{2}-1)} \int_{\mathbb{R}^N} |u_0|^r dx.$$

By the fact that $2 < r < 2 + \frac{4s}{N}$, so $N\tau(\frac{r}{2} - 1) < 2\tau s$ and we may take $|\tau|$ large enough such that

$$\frac{1}{2} e^{2\tau s} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_0|^2 dx - \frac{\alpha}{2r} e^{N\tau(\frac{r}{2}-1)} \int_{\mathbb{R}^N} |u_0|^r dx := A_\tau < 0.$$

This yields that $I_\mu(\tau * u_0) \leq A_\tau + \frac{\mu}{2} a^2$. Take $0 < V_* < -\frac{2A_\tau}{a^2}$. Therefore, if $0 \leq \mu \leq V_*$, then

$$I_\mu(\tau * u_0) \leq A_\tau + \frac{\mu}{2} a^2 \leq A_\tau + \frac{V_*}{2} a^2 < 0.$$

As a consequence, $m_{\mu,a} \leq I_\mu(\tau * u_0) < 0$. $\square$

Here we point out the proof of the above lemma yields that

$$m_{0,a} = \inf_{u \in S_a} I_0(u) \leq I_0(\tau * u_0) < 0$$

for $|\tau|$ large enough. Then, taking $V_* = -\frac{2m_{0,a}}{a^2} > 0$, one has

$$m_{\mu,a} = m_{0,a} + \frac{1}{2} \mu a^2 < 0$$

for $0 \leq \mu < V_*$.

It is easy to see that Lemma 2.3 guarantees that the following lemma holds.

Lemma 2.4. *Fix $0 \le \mu \le V_*$, let $0 < a_1 < a_2$. Then, $a_1^2 \cdot m_{\mu,a_2} < a_2^2 \cdot m_{\mu,a_1} < 0$.*

Proof. It suffices to prove the first inequality holds. Let $0 < a_1 < a_2$. Set $\xi = \frac{a_2}{a_1} > 1$. Since f is odd, we can let $\{u_n\} \subset S_{a_1}$ be a nonnegative minimizing sequence of m_{μ,a_1}. Namely, $I_\mu(u_n) \to m_{\mu,a_1}$ as $n \to \infty$. Set $v_n := \xi u_n$. Then, $v_n \in S_{a_2}$. By virtue of (1.7) we deduce that

$$
\begin{aligned}
m_{\mu,a_2} \le & I_\mu(v_n) = \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v_n|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} \mu |v_n|^2 dx - \int_{\mathbb{R}^N} F(v_n) dx \\
= & \frac{1}{2}\xi^2 \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx + \frac{\mu}{2}\xi^2 \int_{\mathbb{R}^N} |u_n|^2 dx - \int_{\mathbb{R}^N} F(\xi u_n) dx \\
= & \xi^2 I_\mu(u_n) + \xi^2 \int_{\mathbb{R}^N} F(u_n) dx - \int_{\mathbb{R}^N} F(\xi u_n) dx \\
\le & \xi^2 I_\mu(u_n) + \xi^2 \int_{\mathbb{R}^N} F(u_n) dx - \xi^q \int_{\mathbb{R}^N} F(u_n) dx \\
= & \xi^2 I_\mu(u_n) + (\xi^2 - \xi^q) \int_{\mathbb{R}^N} F(u_n) dx.
\end{aligned}
$$

We assert that there are a constant $C > 0$ and $n_0 \in \mathbb{N}$ such that

$$
\int_{\mathbb{R}^N} F(u_n) dx \ge C
$$

for all $n \ge n_0$. Otherwise, up to a subsequence, we may assume that $\int_{\mathbb{R}^N} F(u_n) dx \to 0$ as $n \to \infty$. By $I_\mu(u_n) \to m_{\mu,a_1}$ as $n \to \infty$ we derive that

$$
m_{\mu,a_1} + o_n(1) = I_\mu(u_n) \ge -\int_{\mathbb{R}^N} F(u_n) dx = o_n(1),
$$

which yields $m_{\mu,a_1} \ge 0$, a contradiction to Lemma 2.3. Hence, the assertion is proved. Thereby,

$$
\begin{aligned}
m_{\mu,a_2} \le & \xi^2 I_\mu(u_n) + (\xi^2 - \xi^q) \int_{\mathbb{R}^N} F(u_n) dx \\
\le & \xi^2 I_\mu(u_n) + (\xi^2 - \xi^q) C.
\end{aligned}
$$

Let $n \to +\infty$, it results that

$$
m_{\mu,a_2} \le \xi^2 m_{\mu,a_1} + (\xi^2 - \xi^q) C < \xi^2 m_{\mu,a_1},
$$

to wit, $a_1^2 \cdot m_{\mu,a_2} < a_2^2 \cdot m_{\mu,a_1}$. □

In the sequel, we prove the following compactness lemma on S_a, which is very important to our subsequent proof.

Lemma 2.5. *Let $\mu \in [0, V_*]$ and $\{u_n\} \subset S_a$ be a minimizing sequence of $m_{\mu,a}$. Then, one of the following conclusions holds:*

(i) $\{u_n\}$ has a strongly convergence subsequence in $H^s(\mathbb{R}^N)$;

(ii) there exists a sequence $\{y_n\} \subset \mathbb{R}^N$ with $|y_n| \to +\infty$ such that the sequence $\{v_n\}$ is strongly convergent to a function $v \in S_a$ with $I_\mu(v) = m_{\mu,a}$, where $v_n(x) := u_n(x + y_n)$.

Proof. By Lemma 2.1, $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$. Then, passing to a subsequence, there exists $u \in H^s(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^N)$. We next continue our arguments by distinguishing two cases.

Case 1: $u \ne 0$. Set $\|u\|_2 = b$. If $b \ne a$, then the weak lower semi-continuity of the norm tells us $0 < b < a$. Set $v_n = u_n - u$. By Brézis-Lieb lemma (see Lemma 1.32 in [37]),

$$
\|u_n\|_2^2 - \|v_n\|_2^2 - \|u\|_2^2 = o_n(1).
$$

Moreover,

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx = \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v_n|^2 dx + \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx + o_n(1).$$

In the sequel, we shall prove that

$$\int_{\mathbb{R}^N} F(u_n) dx = \int_{\mathbb{R}^N} F(v_n) dx + \int_{\mathbb{R}^N} F(u) dx + o_n(1). \tag{2.2}$$

Indeed, by the Young inequality and (1.6) we deduce that for any $\tau > 0$, there exists $C_\tau > 0$ such that

$$\begin{aligned}
&|F(u_n) - F(v_n) - F(u)| \\
\leq &|f(v_n + \theta(u_n - v_n))(u_n - v_n)| + |F(u)| \\
\leq &[C_1|v_n + \theta u|^{r-1} + C_2|v_n + \theta u|^{p-1}]|u| + C_1|u|^r + C_2|u|^p \\
\leq &C|v_n|^{r-1}|u| + C|u|^r + C|v_n|^{p-1}|u| + C|u|^p \\
\leq &\tau|v_n|^r + \tau|v_n|^p + C_\tau|u|^r + C_\tau|u|^p.
\end{aligned}$$

Set

$$G_{\tau,n}(x) = \max\{|F(u_n) - F(v_n) - F(u)| - \tau|v_n|^r - \tau|v_n|^p, 0\}.$$

Then

$$0 \leq G_{\tau,n}(x) \leq C_\tau|u|^r + C_\tau|u|^p \in L^1(\mathbb{R}^N)$$

and $G_{\tau,n}(x) \to 0$ a.e. on $\mathbb{R}^N$. It follows from Lebesgue dominated convergence theorem that

$$\int_{\mathbb{R}^N} G_{\tau,n}(x) dx \to 0, \text{ as } n \to \infty.$$

As a result,

$$\begin{aligned}
&\limsup_{n\to\infty} \Big| \int_{\mathbb{R}^N} [F(u_n) - F(v_n) - F(u)] dx \Big| \\
\leq &\limsup_{n\to\infty} \int_{\mathbb{R}^N} G_{\tau,n}(x) dx + \tau \limsup_{n\to\infty} \int_{\mathbb{R}^N} |v_n|^r dx + \tau \limsup_{n\to\infty} \int_{\mathbb{R}^N} |v_n|^p dx \\
\leq &C\tau.
\end{aligned}$$

By the arbitrariness of τ, (2.2) is proved.

Set $\|v_n\|_2 = d_n$. Suppose $d_n \to d$ as $n \to \infty$. Then, $a^2 = b^2 + d^2 > d^2$, which implies that $0 < d_n < a$ for large n. Consequently, together with Lemma 2.4 we deduce that

$$\begin{aligned}
m_{\mu,a} + o_n(1) = &I_\mu(u_n) = I_\mu(v_n) + I_\mu(u) + o_n(1) \\
\geq &m_{\mu,d_n} + m_{\mu,b} + o_n(1) \\
\geq &\frac{d_n^2}{a^2} m_{\mu,a} + m_{\mu,b} + o_n(1).
\end{aligned}$$

Which yields

$$\begin{aligned}
m_{\mu,a} \geq &\frac{d^2}{a^2} m_{\mu,a} + m_{\mu,b} > \frac{d^2}{a^2} m_{\mu,a} + \frac{b^2}{a^2} m_{\mu,a} \\
= &\frac{d^2 + b^2}{a^2} m_{\mu,a} = m_{\mu,a},
\end{aligned}$$

a contradiction. Consequently, $b = a$, namely, $u \in S_a$, and $\|u_n\|_2 \to \|u\|_2$ as $n \to \infty$. Since $L^2(\mathbb{R}^N)$ is reflexive,

$$u_n \to u \text{ in } L^2(\mathbb{R}^N). \tag{2.3}$$

From (1.6) and (2.3) we infer that

$$\int_{\mathbb{R}^N} F(u_n)dx \to \int_{\mathbb{R}^N} F(u)dx \tag{2.4}$$

as $n \to \infty$. Therefore,

$$\begin{aligned}
m_{\mu,a} \leq & I_\mu(u) = \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} \mu u^2 dx - \int_{\mathbb{R}^N} F(u)dx \\
= & \frac{C_{N,s}}{4}\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy + \frac{1}{2}\int_{\mathbb{R}^N} \mu u^2 dx - \int_{\mathbb{R}^N} F(u)dx \\
\leq & \frac{C_{N,s}}{4} \liminf_{n\to\infty} \int_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^{N+2s}} dxdy + \frac{1}{2}\lim_{n\to\infty}\int_{\mathbb{R}^N} \mu |u_n|^2 dx \\
& - \lim_{n\to\infty}\int_{\mathbb{R}^N} F(u_n)dx \\
\leq & \lim_{n\to\infty} I_\mu(u_n) = m_{\mu,a}.
\end{aligned}$$

As a result, $\lim\limits_{n\to\infty} I_\mu(u_n) = I_\mu(u)$. So, together (2.3) with (2.4) we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx = \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx.$$

It yields that $\lim\limits_{n\to\infty} \|u_n\|_{H^s} = \|u\|_{H^s}$.

Case 2: $u = 0$. Based on Lemma 2.3, by similar calculations as in the proof of Lemma 2.4 we can conclude that there exists $C > 0$ such that

$$\int_{\mathbb{R}^N} F(u_n)dx \geq C \tag{2.5}$$

for large n. We assert that there exist $R,\ \alpha > 0$ and $y_n \in \mathbb{R}^N$ such that

$$\int_{B_R(y_n)} |u_n|^2 dx \geq \alpha, \forall\, n \in \mathbb{N}. \tag{2.6}$$

Otherwise, by Lemma 2.3 in [36] one has $u_n \to 0$ in $L^t(\mathbb{R}^N)$ for all $t \in (2, 2_s^*)$. Which implies by (1.6),

$$\int_{\mathbb{R}^N} F(u_n)dx \to 0$$

as $n \to \infty$, a contradiction to (2.5). It follows from (2.6) that $\{y_n\}$ is unbounded in $\mathbb{R}^N$. Set $v_n(x) := u_n(x + y_n)$. Obviously, $v_n \subset S_a$ is also a minimizing sequence of $m_{\mu,a}$. As a result, there exists $v \in H^s(\mathbb{R}^N)\backslash\{0\}$ such that $v_n \rightharpoonup v$ in $H^s(\mathbb{R}^N)$ and $v_n(x) \to v(x)$ a.e. on $\mathbb{R}^N$. Then the proof follows from the same arguments used in Case 1. □

Lemma 2.6. *Let* (f_1)-(f_3) *hold and* $\mu \in [0, V_*]$. *Then, the problem* (2.1) *possesses solutions* $(u, \lambda) \in H^s(\mathbb{R}^N) \times \mathbb{R}^+$, *where* u *is positive and radial.*

Proof. By Lemma 2.1, there exists a bounded minimizing sequence $\{u_n\} \subset S_a$. Then, by Lemma 2.5, up to a subsequence, there exists $u \in S_a$ such that $u_n \to u$ in $H^s(\mathbb{R}^N)$ and $I_\mu(u) = m_{\mu,a}$. Therefore, by the Lagrange multiplier theorem, there exists $\lambda_a \in \mathbb{R}$ such that $I'_\mu(u) = -\lambda_a \psi'(u)$ in $H^{-s}(\mathbb{R}^N)$, where

$$\psi(w) := \int_{\mathbb{R}^N} |w|^2 dx,\ w \in H^s(\mathbb{R}^N).$$

Then,

$$(-\Delta)^s u + \mu u + \lambda_a u = f(u),\ x \in \mathbb{R}^N.$$

By Lemma 2.3 and (1.9), we can obtain that

$$\begin{aligned}
0 >& m_{\mu,a} = I_\mu(u) \\
=& \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} \mu u^2 dx - \int_{\mathbb{R}^N} F(u) dx \\
=& \frac{1}{2}\Big[\int_{\mathbb{R}^N} f(u) u dx - \lambda_a \int_{\mathbb{R}^N} u^2 dx\Big] - \int_{\mathbb{R}^N} F(u) dx \\
=& \int_{\mathbb{R}^N} \Big[\frac{1}{2} f(u) u - F(u)\Big] dx - \frac{1}{2}\lambda_a \int_{\mathbb{R}^N} u^2 dx \\
\geq& -\frac{1}{2}\lambda_a \int_{\mathbb{R}^N} u^2 dx,
\end{aligned}$$

to wit, $\lambda_a > 0$.

In the following, we shall prove that u is positive, radial. Indeed, since f is odd, then $I_\mu(|u|) = I_\mu(u)$. Consequently, $m_{\mu,a} \leq I_\mu(|u|) = I_\mu(u) = m_{\mu,a}$, i.e., $I_\mu(|u|) = m_{\mu,a}$. Hence, we may assume that u is nonnegative. By the strong maximum principle ([13]) we get $u > 0$ in $\mathbb{R}^N$. Furthermore, let u^* denote the symmetric radial decreasing rearrangement of u. It follows from [31] that

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u^*|^2 dx \leq \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx,$$

$$\int_{\mathbb{R}^N} F(u^*) dx = \int_{\mathbb{R}^N} F(u) dx,$$

$$\int_{\mathbb{R}^N} |u^*|^2 dx = \int_{\mathbb{R}^N} |u|^2 dx.$$

Therefore, $u^* \in S_a$ and

$$m_{\mu,a} \leq I_\mu(u^*) \leq I_\mu(u) = m_{\mu,a},$$

i.e., $I_\mu(u^*) = m_{\mu,a}$. That is to say, we can replace u by u^*. □

Remark 2.7. *By the proof of Lemma 2.6, we know that* u *satisfies* $I_\mu(u) = m_{\mu,a}$.

The following corollary is a product of Lemma 2.6.

Corollary 2.8. *Fix* $a > 0$*, let* $0 \leq \mu_1 < \mu_2 \leq V_*$*. Then,* $m_{\mu_1,a} < m_{\mu_2,a} < 0$.

Proof. Let $m_{\mu_2,a} \in S_a$ be such that $I_{\mu_2}(u_{\mu_2,a}) = m_{\mu_2,a} < 0$. Then,

$$m_{\mu_1,a} \leq I_{\mu_1}(u_{\mu_2,a}) < I_{\mu_2}(u_{\mu_2,a}) = m_{\mu_2,a} < 0.$$

□

3 The non-autonomous case

In this section, we always assume that $\|V\|_\infty \leq V_*$. Define

$$m_{V_0,a} = \inf_{u \in S_a} I_{V_0}(u),\ m_{\infty,a} = \inf_{u \in S_a} I_\infty(u),\ m_{\varepsilon,a} = \inf_{u \in S_a} I_\varepsilon(u),$$

where $I_\infty(\cdot) = I_{V_\infty}(\cdot)$. By (V) and Corollary 2.8, we can see that

$$m_{V_0,a} < m_{\infty,a} < 0. \tag{3.1}$$

The relations about $m_{V_0,a},\ m_{\infty,a}$ and $m_{\varepsilon,a}$ as follows.

Lemma 3.1. $\limsup_{\varepsilon\to 0^+} m_{\varepsilon,a} \le m_{V_0,a}$. *Furthermore, there exists $\varepsilon_0 > 0$ such that*

$$m_{\varepsilon,a} < m_{\infty,a}$$

for all $\varepsilon \in (0, \varepsilon_0)$.

Proof. By Remark 2.7, let $u_0 \in S_a$ be such that $I_{V_0}(u_0) = m_{V_0,a}$. Hence,

$$\begin{aligned}\limsup_{\varepsilon\to 0^+} m_{\varepsilon,a} \le & \limsup_{\varepsilon\to 0^+} I_\varepsilon(u_0)\\ =& \limsup_{\varepsilon\to 0^+} \left\{ \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_0|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x) u_0^2 dx - \int_{\mathbb{R}^N} F(u_0) dx \right\}\\ =& \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_0|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(0) u_0^2 dx - \int_{\mathbb{R}^N} F(u_0) dx\\ =& I_{V_0}(u_0) = m_{V_0,a}.\end{aligned}$$

The other conclusion is a consequence of (3.1). □

Set $\rho_1 := \frac{1}{2}(m_{\infty,a} - m_{V_0,a}) > 0$ by (3.1). We obtain the following two lemmas.

Lemma 3.2. *Let $\{u_n\} \subset S_a$ be such that $I_\varepsilon(u_n) \to c$ with $c < m_{V_0,a} + \rho_1 < 0$. If $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^N)$, then $u \neq 0$.*

Proof. Arguing by contradiction we assume that $u = 0$. In view of (V), for any given $\xi > 0$, there exists $R > 0$ such that

$$V(x) \ge V_\infty - \xi,\ \forall\, |x| \ge R.$$

In addition, by Remark 2.2 we know that $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$. As a result,

$$\begin{aligned}m_{V_0,a} + \rho_1 + o_n(1) >& c + o_n(1) = I_\varepsilon(u_n)\\ =& I_\infty(u_n) + \frac{1}{2}\int_{\mathbb{R}^N} [V(\varepsilon x) - V_\infty] u_n^2 dx\\ =& I_\infty(u_n) + \frac{1}{2}\int_{B_{\frac{R}{\varepsilon}}(0)} [V(\varepsilon x) - V_\infty] u_n^2 dx\\ &+ \frac{1}{2}\int_{\mathbb{R}^N \setminus B_{\frac{R}{\varepsilon}}(0)} [V(\varepsilon x) - V_\infty] u_n^2 dx\\ \ge& I_\infty(u_n) + \frac{1}{2}\int_{B_{\frac{R}{\varepsilon}}(0)} [V(\varepsilon x) - V_\infty] u_n^2 dx - \frac{1}{2}\xi \int_{\mathbb{R}^N \setminus B_{\frac{R}{\varepsilon}}(0)} u_n^2 dx\\ \ge& I_\infty(u_n) + o_n(1) - C\xi \ge m_{\infty,a} + o_n(1) - C\xi.\end{aligned}$$

Which indicates that $m_{\infty,a} \le m_{V_0,a} + \rho_1$, i.e., $\rho_1 \ge m_{\infty,a} - m_{V_0,a}$, a contradiction. □

Lemma 3.3. *Fix $\varepsilon \in (0, \varepsilon_0)$. Let $\{u_n\} \subset S_a$ be a $(PS)_c$ sequence for I_ε restricted to S_a with $c < m_{V_0,a} + \rho_1 < 0$ and $u_n \rightharpoonup u_\varepsilon$ in $H^s(\mathbb{R}^N)$, i.e., $I_\varepsilon(u_n) \to c$ and $\|I_\varepsilon|'_{S_a}(u_n)\| \to 0$ as $n \to \infty$. If $v_n := u_n - u_\varepsilon \not\to 0$ in $H^s(\mathbb{R}^N)$, then decreasing ε_0 if necessary, there exists $\beta > 0$ independent of $\varepsilon \in (0, \varepsilon_0)$ such that*

$$\liminf_{n\to\infty} \|u_n - u_\varepsilon\|_2^2 \ge \beta.$$

Proof. Set

$$\Phi(v) := \frac{1}{2}\int_{\mathbb{R}^N} |v|^2 dx, \ \forall\, v \in H^s(\mathbb{R}^N),$$

then $S_a = \Phi^{-1}(\{\frac{a^2}{2}\})$. By Proposition 5.12 in [37], there exists a sequence $\{\lambda_n\} \subset \mathbb{R}$ such that

$$\|I'_\varepsilon(u_n) - \lambda_n \Phi'(u_n)\|_{H^{-s}} \to 0 \tag{3.2}$$

as $n \to \infty$, which means that

$$(-\Delta)^s u_n + V(\varepsilon x)u_n - f(u_n) = \lambda_n u_n + o_n(1) \text{ in } H^{-s}(\mathbb{R}^N).$$

Therefore, for $\varphi \in H^s(\mathbb{R}^N)$,

$$\begin{aligned}&\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u_n (-\Delta)^{\frac{s}{2}}\varphi dx + \int_{\mathbb{R}^N} V(\varepsilon x)u_n\varphi dx - \int_{\mathbb{R}^N} f(u_n)\varphi dx\\ =&\lambda_n \int_{\mathbb{R}^N} u_n\varphi dx + o_n(1)\|\varphi\|.\end{aligned}$$

Especially,

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 dx + \int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^2 dx - \int_{\mathbb{R}^N} f(u_n)u_n dx = \lambda_n a^2 + o_n(1)\|\varphi\|.$$

By Remark 2.2, $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$. Then, we can find that $\{\lambda_n\}$ is bounded. As a consequence, up to a subsequence, there exists $\lambda_\varepsilon \in \mathbb{R}$ such that $\lambda_n \to \lambda_\varepsilon$ as $n \to \infty$. It is standard to show that $\|I'_\varepsilon(u_n) - \lambda_\varepsilon \Phi'(u_n)\|_{H^s} \to 0$ as $n \to \infty$. And so,

$$I'_\varepsilon(u_\varepsilon) - \lambda_\varepsilon \Phi'(u_\varepsilon) = 0 \tag{3.3}$$

in $H^{-s}(\mathbb{R}^N)$, see [27]. Moreover, it is not difficult to prove

$$\begin{aligned} I'_\varepsilon(u_n) &= I'_\varepsilon(u_\varepsilon) + I'_\varepsilon(v_n) + o_n(1),\\ \Phi'(u_n) &= \Phi'(u_\varepsilon) + \Phi'(v_n) + o_n(1).\end{aligned}$$

These together with (3.3) yield that

$$I'_\varepsilon(u_n) - \lambda_\varepsilon \Phi'(u_n) = I'_\varepsilon(v_n) - \lambda_\varepsilon \Phi'(v_n) + o_n(1).$$

Therefore, $\|I'_\varepsilon(v_n) - \lambda_\varepsilon \Phi'(v_n)\|_{H^{-s}} \to 0$ as $n \to \infty$. As a result, by (1.9) and (3.2) one has

$$\begin{aligned} 0 >&m_{V_0,a} + \rho_1 > c\\ =&\liminf_{n\to\infty} I_\varepsilon(u_n)\\ =&\liminf_{n\to\infty} \big[I_\varepsilon(u_n) - \frac{1}{2}I'_\varepsilon(u_n)u_n + \frac{1}{2}\lambda_n a^2\big]\\ =&\liminf_{n\to\infty} \Big\{\int_{\mathbb{R}^N} \big[\frac{1}{2}f(u_n)u_n - F(u_n)\big]dx + \frac{1}{2}\lambda_n a^2\Big\}\\ \geq&\frac{1}{2}\lambda_\varepsilon a^2,\end{aligned}$$

namely,

$$\limsup_{\varepsilon\to 0^+} \lambda_\varepsilon \leq \frac{2(m_{V_0,a} + \rho_1)}{a^2} < 0.$$

Which implies that there exists $\lambda_* < 0$ independent of ε such that

$$\lambda_\varepsilon \leq \lambda_* < 0 \tag{3.4}$$

for all $\varepsilon \in (0,\varepsilon_0)$. Moreover, together with the boundedness of $\{\|v_n\|_{H^s}\}$ we obtain

$$\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}v_n|^2dx+\int_{\mathbb{R}^N}V(\varepsilon x)|v_n|^2dx-\lambda_\varepsilon\int_{\mathbb{R}^N}|v_n|^2dx=\int_{\mathbb{R}^N}f(v_n)v_ndx+o_n(1). \tag{3.5}$$

Taking into account (3.4) and (3.5) we can deduce that

$$\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}v_n|^2dx+\int_{\mathbb{R}^N}V(\varepsilon x)|v_n|^2dx-\lambda_*\int_{\mathbb{R}^N}|v_n|^2dx\le\int_{\mathbb{R}^N}f(v_n)v_ndx+o_n(1).$$

By the means of (1.8) we can prove that

$$\|v_n\|_{H^s}^2=\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}v_n|^2dx+\int_{\mathbb{R}^N}|v_n|^2dx\le C\|v_n\|_p^p+o_n(1)\le C\|v_n\|_{H^s}^p+o_n(1).$$

By the fact that $v_n\not\to 0$ in $H^s(\mathbb{R}^N)$ we know that, up to a subsequence, $\liminf\limits_{n\to\infty}\|v_n\|_{H^s}>0$.

Consequently, $\liminf\limits_{n\to\infty}\|v_n\|_{H^s}\ge(\frac{1}{C})^{\frac{1}{p-2}}$, which yields that $\liminf\limits_{n\to\infty}\|v_n\|_p^p\ge C_1$, where $C_1>0$ is a constant that does not depend on ε. And so, making use of (1.5) we have

$$\begin{aligned}C_1\le&\liminf_{n\to\infty}\|v_n\|_p^p\\ \le&\liminf_{n\to\infty}\left[C_{s,N,p}\|(-\Delta)^{\frac{s}{2}}v_n\|_2^{\frac{N(p-2)}{2s}}\cdot\|v_n\|_2^{p-\frac{N(p-2)}{2s}}\right]\\ \le&C_2\liminf_{n\to\infty}\|v_n\|_2^{\frac{2N-p(N-2s)}{2s}},\end{aligned}$$

then, we can deduce that

$$\liminf_{n\to\infty}\|v_n\|_2^2\ge(\frac{C_1}{C_2})^{\frac{4s}{2N-p(N-2s)}}:=\beta>0.$$

□

In the sequel, in order to prove that I_ε satisfies the $(PS)_c$ condition, we fix

$$0<\rho<\min\left\{\frac{1}{2},\frac{\beta}{a^2}\right\}(m_{\infty,a}-m_{V_0,a})\le\rho_1.$$

Lemma 3.4. *For each $\varepsilon\in(0,\varepsilon_0)$, I_ε satisfies the $(PS)_c$ condition restricted to S_a for $c<m_{V_0,a}+\rho$.*

Proof. Let $\{u_n\}$ be a $(PS)_c$ sequence for I_ε restricted to S_a, i.e.,

$$u_n\in S_a,\ I_\varepsilon(u_n)\to c<m_{V_0,a}+\rho,\ \|I_\varepsilon|'_{S_a}(u_n)\|\to 0$$

as $n\to\infty$. Again, set $\Phi(v):=\frac{1}{2}\int_{\mathbb{R}^N}|v|^2dx,\ \forall v\in H^s(\mathbb{R}^N)$, then $S_a=\Phi^{-1}(\{\frac{a^2}{2}\})$. By Proposition 5.12 in [37], there exists a sequence $\{\lambda_n\}\subset\mathbb{R}$ such that

$$\|I'_\varepsilon(u_n)-\lambda_n\Phi'(u_n)\|_{H^{-s}}\to 0$$

as $n\to\infty$. Set $v_n:=u_n-u_\varepsilon$. If $v_n\not\to 0$ in $H^s(\mathbb{R}^N)$, it follows by Lemma 3.3 that there exists $\beta>0$ independent of $\varepsilon\in(0,\varepsilon_0)$ such that

$$\liminf_{n\to\infty}\|v_n\|_2^2\ge\beta>0. \tag{3.6}$$

Set $\|v_n\|_2=d_n$ and $\|u_\varepsilon\|_2=b$. Assume that $d_n\to d$ as $n\to\infty$. Then, by (3.6) one has $d^2\ge\beta>0$. By Lemma 3.2, $b>0$. By the Brézis-Lieb lemma,

$$\|v_n\|_2^2=\|u_n\|_2^2-\|u_\varepsilon\|_2^2+o_n(1).$$

Then, $a^2 = b^2 + d^2 > d^2$. Which implies that $0 < d_n < a$ for large n. Similar to the proof of Lemma 3.2, we can conclude that for any $\zeta > 0$,

$$I_\varepsilon(v_n) \geq m_{\infty,d_n} + o_n(1) - C\zeta.$$

Therefore, by (V) and Lemma 2.4 we deduce that

$$\begin{aligned} m_{V_0,a} + \rho + o_n(1) >& c + o_n(1) = I_\varepsilon(u_n) \\ =& I_\varepsilon(v_n) + I_\varepsilon(u_\varepsilon) + o_n(1) \\ \geq& m_{\infty,d_n} - C\zeta + m_{V_0,b} + o_n(1) \\ \geq& \frac{d_n^2}{a^2} m_{\infty,a} + \frac{b^2}{a^2} m_{V_0,a} + o_n(1) - C\zeta. \end{aligned}$$

Let $n \to \infty$, we get

$$\rho \geq \frac{d^2}{a^2} m_{\infty,a} - \frac{d^2}{a^2} m_{V_0,a} - C\zeta \geq \frac{\beta}{a^2}(m_{\infty,a} - m_{V_0,a}) - C\zeta.$$

By the arbitrariness of ζ, $\rho \geq \frac{\beta}{a^2}(m_{\infty,a} - m_{V_0,a})$, a contradiction.

As a consequence, $v_n \to 0$ in $H^s(\mathbb{R}^N)$, namely, $u_n \to u_\varepsilon$ in $H^s(\mathbb{R}^N)$. □

Remark 3.5. *By Lemma 3.4, $u_\varepsilon \in S_a$. By the fact that $\|I'_\varepsilon(u_n) - \lambda_n \Phi'(u_n)\|_{H^{-s}} \to 0$ as $n \to \infty$ we obtain*

$$(-\Delta)^s u_\varepsilon + V(\varepsilon x) u_\varepsilon + \lambda_\varepsilon u_\varepsilon = f(u_\varepsilon),\ x \in \mathbb{R}^N,$$

where $\lambda_\varepsilon = \lim\limits_{n\to\infty}(-\lambda_n)$. The proof is identical to that of Lemma 3.3.

4 Multiplicity result

In this section, we investigate the multiplicity of solutions for (P_a) by the Ljusternik-Schnirelmann category theory and study the behavior of its maximum points concentrating on the set M of global minima of V and decay behavior.

Suppose that $\|V\|_\infty \leq V_*$. By Lemma 2.6 and Remark 2.7, let $(w, \lambda) \in H^s(\mathbb{R}^N) \times \mathbb{R}^+$ solve the following problem

$$\begin{cases} (-\Delta)^s u + V_0 u + \lambda u = f(u),\ x \in \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = a^2, \end{cases} \tag{4.1}$$

and $I_{V_0}(w) = m_{V_0,a}$. Simultaneously, let η be a smooth nonincreasing cut-off function defined in $[0,\infty)$ with $\eta(t) = 1$ if $0 \leq t \leq \frac{1}{2}$ and $\eta(t) = 0$ if $t \geq 1$ and $|\nabla \eta| \leq 1$. For each $y \in M$, let

$$\Psi_{\varepsilon,y}(x) = \eta(|\varepsilon x - y|) w(\frac{\varepsilon x - y}{\varepsilon})$$

and

$$\widetilde{\Psi}_{\varepsilon,y}(x) = a \cdot \frac{\Psi_{\varepsilon,y}(x)}{\|\Psi_{\varepsilon,y}(x)\|_2}.$$

Then for small $0 < \varepsilon < 1$, one has $\Psi_{\varepsilon,y} \in H^s(\mathbb{R}^N) \setminus \{0\}$ for all $y \in M$. Namely, there exists $\varepsilon^* > 0$ such that $\Psi_{\varepsilon,y} \in H^s(\mathbb{R}^N) \setminus \{0\}$ for every $\varepsilon \in (0, \varepsilon^*)$. Frow now on, we assume that $\varepsilon \in (0, \varepsilon^*)$. For $y \in M$, let $\Phi_\varepsilon(y) = \widetilde{\Psi}_{\varepsilon,y}$. Then $\Phi_\varepsilon : M \to S_a$, and $\Phi_\varepsilon(y)$ has a compact support for any $y \in M$. Moreover, we have the following fact for Φ_ε.

Lemma 4.1. $\lim\limits_{\varepsilon\to 0} I_\varepsilon(\Phi_\varepsilon(y)) = m_{V_0,a}$ *uniformly in $y \in M$.*

Proof. Suppose that the conclusion is false. Then, there exist $\zeta_0 > 0,\ \{y_n\} \subset M$ with $y_n \to y \in M$ and $\varepsilon_n \to 0$ such that

$$|I_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - m_{V_0,a}| \geq \zeta_0 > 0.$$

By the means of Lebesgue dominated convergence theorem, we obtain the following relations

$$\begin{aligned}
&\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} \Phi_{\varepsilon_n}(y_n)|^2 dx \\
=&\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} \widetilde{\Psi}_{\varepsilon_n,y_n}|^2 dx \\
=&\frac{C_{N,s}}{2}\int_{\mathbb{R}^{2N}} \frac{|\widetilde{\Psi}_{\varepsilon_n,y_n}(x) - \widetilde{\Psi}_{\varepsilon_n,y_n}(z)|^2}{|x-z|^{N+2s}} dxdz \\
=&\frac{C_{N,s}}{2}\int_{\mathbb{R}^{2N}} \frac{\big|a\frac{1}{\|\Psi_{\varepsilon_n,y_n}(x)\|_2}\eta(|\varepsilon_n x|)w(x) - a\frac{1}{\|\Psi_{\varepsilon_n,y_n}(z)\|_2}\eta(|\varepsilon_n z|)w(z)\big|^2}{|x-z|^{N+2s}} dxdz \\
\to&\frac{C_{N,s}}{2}\int_{\mathbb{R}^{2N}} \frac{|w(x)-w(z)|^2}{|x-z|^{N+2s}} dxdz \\
=&\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} w|^2 dx,
\end{aligned}$$

$$\begin{aligned}
\int_{\mathbb{R}^N} F(\Phi_{\varepsilon_n}(y_n))dx = \int_{\mathbb{R}^N} F\big(a\frac{\eta(|\varepsilon_n x|)w(x)}{\|\Psi_{\varepsilon_n,y_n}\|_2}\big)dx \to \int_{\mathbb{R}^N} F(w)dx,\\
\int_{\mathbb{R}^N} V(\varepsilon_n x)\Phi^2_{\varepsilon_n}(y_n)dx =& a^2\int_{\mathbb{R}^N} V(\varepsilon_n x + y_n)\frac{\eta^2(|\varepsilon_n x|)w^2(x)}{\|\Psi_{\varepsilon_n,y_n}\|_2^2}dx \\
&\to \int_{\mathbb{R}^N} V(y)w^2 dx = \int_{\mathbb{R}^N} V_0 w^2 dx
\end{aligned}$$

as $n \to \infty$. As a consequence,

$$\begin{aligned}
I_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) =& \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}\Phi_{\varepsilon_n}(y_n)|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon_n x)\Phi^2_{\varepsilon_n}(y_n)dx \\
&- \int_{\mathbb{R}^N} F(\Phi_{\varepsilon_n}(y_n))dx \\
\to& \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} w|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V_0 w^2 dx - \int_{\mathbb{R}^N} F(w)dx \\
=& I_{V_0}(w) = m_{V_0,a},
\end{aligned}$$

a contradiction. □

Let $\rho = \rho(\delta) > 0$ be such that $M_\delta \subset B_\rho(0)$. Consider the mapping $\chi : \mathbb{R}^N \to \mathbb{R}^N$ defined by $\chi(x) = x$ for $|x| \leq \rho$ and $\chi(x) = \frac{\rho x}{|x|}$ for $|x| \geq \rho$. Moreover, we also consider the map $\beta_\varepsilon : S_a \to \mathbb{R}^N$ defined by

$$\beta_\varepsilon(v) = \frac{\int_{\mathbb{R}^N} \chi(\varepsilon x)v^2 dx}{\int_{\mathbb{R}^N} v^2 dx}.$$

We have the following facts.

Lemma 4.2. $\lim_{\varepsilon\to 0}\beta_\varepsilon(\Phi_\varepsilon(y)) = y$ *uniformly in* $y \in M$.

Proof. Suppose by contradiction that there exist $\delta_0 > 0$, $\{y_n\} \subset M$ and $\varepsilon_n \to 0$ such that

$$|\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - y_n| \geq \delta_0,\ \forall\, n \in \mathbb{N}.$$

It is easy to calculate that

$$\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = y_n + \frac{\int_{\mathbb{R}^N}[\chi(\varepsilon_n x + y_n) - y_n]\eta^2(|\varepsilon_n x|)w^2(x)dx}{\int_{\mathbb{R}^N}\eta^2(|\varepsilon_n x|)w^2(x)dx}.$$

Since $\{y_n\} \subset M \subset B_\rho(0)$, by Lebesgue dominated convergence theorem we have

$$|\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - y_n| \to 0$$

as $n \to \infty$, a contradiction. □

Lemma 4.3. *Let $\varepsilon_n \to 0$ and $\{u_n\} \subset S_a$ be such that $I_\varepsilon(u_n) \to m_{0,a}$ as $n \to \infty$. Then, there exists $\{y_n\} \subset \mathbb{R}^N$ such that $v_n(x) := u_n(x + y_n) \to v$ in $H^s(\mathbb{R}^N)\backslash\{0\}$ as $n \to \infty$. Furthermore, up to a subsequence, $\tilde{y}_n := \varepsilon_n y_n \to y \in M$ as $n \to \infty$.*

Proof. By Remark 2.2, $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$. We claim that there exist $R,\ \alpha > 0$ and $y_n \in \mathbb{R}^N$ such that

$$\int_{B_R(y_n)} |u_n|^2 dx \geq \alpha,\ \forall n \in \mathbb{N}.$$

Otherwise, by Lemma 2.3 in [36] we have $u_n \to 0$ in $L^t(\mathbb{R}^N)$ for all $t \in (2, 2_s^*)$. It follows from (1.6) that $\int_{\mathbb{R}^N} F(u_n)dx \to 0$ as $n \to \infty$. Then,

$$\begin{aligned} m_{V_0,a} &= \lim_{n\to\infty} I_{\varepsilon_n}(u_n) \\ &= \lim_{n\to\infty}\Big[\frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u_n|^2dx + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon_n x)|u_n|^2dx - \int_{\mathbb{R}^N} F(u_n)dx\Big] \\ &\geq 0, \end{aligned}$$

a contradiction to Lemma 2.3. Set $v_n(x) := u_n(x + y_n)$. Then, up to a subsequence, there exists $v \in H^s(\mathbb{R}^N)\backslash\{0\}$ such that $v_n \rightharpoonup v$ in $H^s(\mathbb{R}^N)$. Noting that

$$m_{V_0,a} \leq I_{V_0}(v_n) = I_{V_0}(u_n) \leq I_{\varepsilon_n}(u_n) \to m_{V_0,a}$$

as $n \to \infty$, that is to say, $I_{V_0}(v_n) \to m_{V_0,a}$ as $n \to \infty$. By Lemma 2.5, $v_n \to v$ in $H^s(\mathbb{R}^N)$ and $v \in S_a$. Set $\tilde{y}_n = \varepsilon_n y_n$. In the following, we prove that $\{\tilde{y}_n\}$ is bounded in $\mathbb{R}^N$. Indeed, if the conclusion is false, up to a subsequence, we may assume that $|\tilde{y}_n| \to +\infty$ as $n \to \infty$. Then, there holds that

$$\begin{aligned} m_{V_0,a} &= \lim_{n\to\infty} I_{\varepsilon_n}(u_n) \\ &= \lim_{n\to\infty}\Big[\frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u_n|^2dx + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon_n x)|u_n|^2dx - \int_{\mathbb{R}^N} F(u_n)dx\Big] \\ &= \lim_{n\to\infty}\Big[\frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}v_n|^2dx + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon_n x + \tilde{y}_n)|v_n|^2dx - \int_{\mathbb{R}^N} F(v_n)dx\Big] \\ &= \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}v|^2dx + \frac{1}{2}\int_{\mathbb{R}^N} V_\infty|v|^2dx - \int_{\mathbb{R}^N} F(v)dx \\ &= I_\infty(v) \geq m_{\infty,a}, \end{aligned}$$

which contradicts to (3.1). Consequently, up to a subsequence, there exists $y \in \mathbb{R}^N$ such that $\tilde{y}_n \to y$ in $\mathbb{R}^N$. With a similar arguments as the above inequality we obtain

$$\begin{aligned} m_{V_0,a} &\geq \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}v|^2dx + \frac{1}{2}V(y)\int_{\mathbb{R}^N} v^2dx - \int_{\mathbb{R}^N} F(v)dx \\ &= I_{V(y)}(v) \geq m_{V(y),a}. \end{aligned}$$

Noting that $V(y) \geq V_0$. If $V(y) > V_0$, by Corollary 2.8, $m_{V(y),a} > m_{V_0,a}$, a contradiction. Therefore, $V(y) = V_0$, and so $y \in M$. □

Let $h(\tau)$ be a positive function tending to 0 as $\tau \to 0$. Define the set

$$\widetilde{S}_a = \{u \in S_a : I_\varepsilon(u) \le m_{V_0,a} + h(\varepsilon)\}.$$

For each $y \in M$, we can use Lemma 4.1 to deduce that $h(\varepsilon) := |I_\varepsilon(\Phi_\varepsilon(y)) - m_{V_0,a}|$ satisfying $h(\varepsilon) \to 0$ as $\varepsilon \to 0$. Then, $\Phi_\varepsilon(y) \in \widetilde{S}_a$ and $\widetilde{S}_a \neq \emptyset$ for $\varepsilon > 0$.

Lemma 4.4. *For any $\delta > 0$, there holds that* $\lim\limits_{\varepsilon\to 0} \sup\limits_{v\in\widetilde{S}_a} \inf\limits_{y\in M_\delta} |\beta_\varepsilon(v) - y| = 0.$

Proof. By the definition of supremum, let $\varepsilon_n \to 0$ and $u_n \in \widetilde{S}_a$ be such that

$$dist(\beta_{\varepsilon_n}(u_n), M_\delta) = \inf_{z\in M_\delta} |\beta_{\varepsilon_n}(u_n) - z| = \sup_{u\in\widetilde{S}_a} \inf_{z\in M_\delta} |\beta_{\varepsilon_n}(u) - z| + o_n(1).$$

Then, it suffices to find a sequence $\{y_n\} \subset M_\delta$ such that

$$\lim_{n\to\infty} |\beta_{\varepsilon_n}(u_n) - y_n| = 0.$$

Indeed, since $u_n \in \widetilde{S}_a$,

$$m_{V_0,a} \le I_{V_0}(u_n) \le I_{\varepsilon_n}(u_n) \le m_{V_0,a} + h(\varepsilon_n), \forall\, n \in \mathbb{N}.$$

Hence, $u_n \in S_a$, $I_{\varepsilon_n}(u_n) \to m_{V_0,a}$. From the above lemma, we can see that there exists $\{y_n\} \subset \mathbb{R}^N$ such that $\tilde{y}_n := \varepsilon_n y_n \to y \in M$ as $n \to \infty$ and $v_n(x) := u_n(x + y_n) \to v$ in $H^s(\mathbb{R}^N)\backslash\{0\}$ as $n \to \infty$. As a result, $\{\tilde{y}_n\} \subset M_\delta$ for large n and

$$\beta_{\varepsilon_n}(u_n) - \tilde{y}_n = \frac{\int_{\mathbb{R}^N} [\chi(\varepsilon_n z + \tilde{y}_n) - \tilde{y}_n]|v_n|^2 dz}{a^2} \to 0$$

as $n \to \infty$. □

Proof of Theorem 1.1. (i) We divide the proof into two parts.

Part 1): Multiplicity of solutions.

For any $y \in M$, by Lemmas 4.1 and 4.4, we know that there exists $\varepsilon_\delta > 0$ such that for any $\varepsilon \in (0, \varepsilon_\delta)$, the diagram

$$\beta_\varepsilon \circ \Phi_\varepsilon : M \to S_a \to M_\delta$$

is well defined and $\beta_\varepsilon \circ \Phi_\varepsilon$ is homotopically equivalent to the inclusion map $id : M \to M_\delta$. Then, by Lemma 4.3 of [2] we get $cat_{M_\delta}(M_\delta) \ge cat_{M_\delta}(M)$. Furthermore, by Lemma 3.4 we obtain that I_ε satisfies the $(PS)_c$ condition for $c \in (m_{V_0,a}, m_{V_0,a} + h(\varepsilon))$. Consequently, standard Ljusternik-Schnirelmann category theory (Refs. [17]) gives that I_ε admits at least $cat_{M_\delta}(M)$ critical points on S_a.

Part 2): Concentration of the maximum points.

For any $\{\varepsilon_n\} \to 0^+$, let $(u_{\varepsilon_n}, \lambda_n) \in \widetilde{S}_a \times \mathbb{R}^+$ solve (P_a) and $\xi_n \in \mathbb{R}^N$ be a global maximum of $|u_{\varepsilon_n}|$. Then,

$$m_{V_0,a} \le I_{V_0}(u_{\varepsilon_n}) \le I_{\varepsilon_n}(u_{\varepsilon_n}) \le m_{V_0,a} + h(\varepsilon_n),$$

i.e., $I_{\varepsilon_n}(u_{\varepsilon_n}) \to m_{V_0,a}$ as $n \to \infty$. By Lemma 4.3, there exists $\{y_n\} \subset \mathbb{R}^N$ with $\tilde{y}_n := \varepsilon_n y_n \to y \in M$ such that $v_n(x) := u_{\varepsilon_n}(x + y_n) \to v \in H^s(\mathbb{R}^N)\backslash\{0\}$. It is easy to see that

$$(-\Delta)^s v_n + V(\varepsilon_n x + \tilde{y}_n)v_n + \lambda_n v_n = f(v_n),\ x \in \mathbb{R}^N.$$

Similar to the proof of Lemma 3.3, by (1.9) we get $\lim\limits_{n\to\infty} \lambda_n \ge -\frac{2m_{V_0,a}}{a^2} > 0$. Since $v_n \to v$ in $H^s(\mathbb{R}^N)$, we can prove that $\lim\limits_{|x|\to+\infty} v_n(x) = 0$ uniformly in $n \in \mathbb{N}$. Then, for given $\tau > 0$, there exist $R_1 > 0$

and $n_0 \in \mathbb{N}^+$ such that $|v_n(x)| \leq \tau$ for $|x| \geq R_1$ and $n \geq n_0$. Clearly, $\|v_n\|_\infty \not\to 0$ as $n \to \infty$. Otherwise, $v_n \to 0$ in $H^s(\mathbb{R}^N)$, which contradicts to $v_n \in S_a$. In the following, let us fix $\tau > 0$ such that $\|v_n\|_\infty \geq 2\tau$ and $z_n \in \mathbb{R}^N$ satisfying $|v_n(z_n)| = \|v_n\|_\infty$ for all $n \in \mathbb{N}$. Then, $|z_n| \leq R_1$, $\xi_n = z_n + y_n$ and

$$\lim_{n\to\infty} V(\varepsilon_n \xi_n) = \lim_{n\to\infty} V(\varepsilon_n z_n + \varepsilon_n y_n) = V(y) = V_0.$$

(ii) In what follows, we shall study the decay behavior of u_ε. Noting that by Lemma 4.3 in [15], there exists a function w such that

$$0 < w(x) \leq \frac{C}{1+|x|^{N+2s}},$$

and

$$(-\Delta)^s w + \frac{V_0}{2} w \geq 0, \ \forall |x| \geq \bar{R}, \tag{4.2}$$

where $\bar{R} > 0$ is a suitable constant. Assume that $\xi_\varepsilon \in \mathbb{R}^N$ is the global maximum of $|u_\varepsilon|$. Set $v_\varepsilon(x) := u_\varepsilon(x + \xi_\varepsilon)$. Then, $\lim\limits_{|x|\to+\infty} v_\varepsilon(x) = 0$, it follows from (f_1) and (V) that there exists some large $R_1 > 0$ such that

$$\begin{aligned}
(-\Delta)^s v_\varepsilon + \frac{V_0}{2} v_\varepsilon =& (-\Delta)^s v_\varepsilon + V(\varepsilon x + \varepsilon \xi_\varepsilon) v_\varepsilon - [V(\varepsilon x + \varepsilon \xi_\varepsilon) - \frac{V_0}{2}] v_\varepsilon \\
=& f(v_\varepsilon) - \lambda_\varepsilon v_\varepsilon - \frac{V_0}{2} v_\varepsilon \\
\leq& f(v_\varepsilon) - \frac{V_0}{2} v_\varepsilon \\
\leq& 0
\end{aligned} \tag{4.3}$$

for $x \in \mathbb{R}^N \backslash B_{R_1}(0)$. Now we take $R_2 := \max\{\bar{R}, R_1\}$ and set

$$b := \inf_{B_{R_2}(0)} w > 0, \ \ z_\varepsilon := (k+1)w - b v_\varepsilon,$$

where $k := \sup \|v_\varepsilon\|_{L^\infty} < +\infty$. We assert that $z_\varepsilon \geq 0$ in $\mathbb{R}^N$. Indeed, if the conclusion is false, there exists $x_{j,\varepsilon} \in \mathbb{R}^N$ such that

$$\inf_{x\in\mathbb{R}^N} z_\varepsilon(x) = \lim_{j\to+\infty} z_\varepsilon(x_{j,\varepsilon}) < 0. \tag{4.4}$$

Since

$$\lim_{|x|\to+\infty} w(x) = 0 = \lim_{|x|\to+\infty} v_\varepsilon(x) = 0,$$

then, we have that

$$\lim_{|x|\to+\infty} z_\varepsilon(x) = 0.$$

Consequently, $\{x_{j,\varepsilon}\}$ is bounded in $\mathbb{R}^N$. Up to a subsequence, we may assume that $x_{j,\varepsilon} \to x_\varepsilon \in \mathbb{R}^N$ as $j \to +\infty$. Hence, by (4.4) one has

$$\inf_{x\in\mathbb{R}^N} z_\varepsilon(x) = \lim_{j\to+\infty} z_\varepsilon(x_{j,\varepsilon}) = z_\varepsilon(x_\varepsilon) < 0. \tag{4.5}$$

Taking into account the continuity property of x_ε and the integral representation of the fractional Laplacian of z_ε at the point x_ε we deduce that

$$(-\Delta)^s z_\varepsilon(x_\varepsilon) = \frac{1}{2} C_{N,s} \int_{\mathbb{R}^{2N}} \frac{2z_\varepsilon(x_\varepsilon) - z_\varepsilon(x_\varepsilon + y) - z_\varepsilon(x_\varepsilon - y)}{|y|^{N+2s}} dy \leq 0 \tag{4.6}$$

Therefore,

$$z_\varepsilon = (k+1)w - b v_\varepsilon \geq kb + w - kb = w > 0$$

in $B_{R_2}(0)$. By use of (4.5) we have $x_\varepsilon \in \mathbb{R}^N \backslash B_{R_2}(0)$. It follow from (4.2) and (4.3) that

$$\begin{aligned}(-\Delta)^s z_\varepsilon + \frac{V_0}{2} z_\varepsilon =& (-\Delta)^s[(k+1)w - bv_\varepsilon] + \frac{V_0}{2}[(k+1)w - bv_\varepsilon] \\ =& (k+1)(-\Delta)^s w + (k+1)\frac{V_0}{2} w - b(-\Delta)^s v_\varepsilon - b\frac{V_0}{2} v_\varepsilon \\ =& (k+1)[(-\Delta)^s w + \frac{V_0}{2} w] - b[(-\Delta)^s v_\varepsilon + \frac{V_0}{2} v_\varepsilon] \\ \geq & 0\end{aligned}$$

in $\mathbb{R}^N \backslash B_{R_2}(0)$. As a result, by (4.5) and (4.6) we get that

$$0 \leq (-\Delta)^s z_\varepsilon(x_\varepsilon) + \frac{V_0}{2} z_\varepsilon(x_\varepsilon) < 0$$

a contradiction. As a consequence, $z_\varepsilon(x) \geq 0$ in $\mathbb{R}^N$. Namely,

$$v_\varepsilon(x) \leq \frac{k+1}{b} w \leq \frac{C}{1+|x|^{N+2s}}.$$

And so

$$u_\varepsilon(x) = v_\varepsilon(x - \xi_\varepsilon) \leq \frac{C}{1+|x-\xi_\varepsilon|^{N+2s}}.$$

□

5 Sobolev supercritical case

In this section, we consider the Sobolev supercritical case. To the best of our knowledge, there is no paper considering the Sobolev supercritical case to the study of normalized solutions. Precisely, we shall investigate the existence and multiplicity of normalized solutions for the following fractional Schrödinger equation with Sobolev supercritical growth

$$\begin{cases}(-\Delta)^s u + V(\varepsilon x)u + \lambda u = f(u) + \eta |u|^{p-2}u, \text{ in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = a^2,\end{cases} \tag{Q_a}$$

where $p > 2_s^*$. Suppose that (V), (f_1), $(\tilde{f_2})$ and (f_3) hold. It is easy to see that the solutions of problem (Q_a) are critical points of the energy functional

$$J_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x) u^2 dx - \int_{\mathbb{R}^N} F(u) dx - \frac{\eta}{p}\int_{\mathbb{R}^N} |u|^p dx$$

restricted on S_a. But the functional J_ε is not well defined on $H^s(\mathbb{R}^N)$, since $p > 2_s^*$. To do this, we will introduce a cutoff function and use the truncation technique to overcome the difficulty caused by the Sobolev supercritical growth.

Define the following cutoff function

$$\phi(t) = \begin{cases} |t|^{p-2}t, & |t| \leq M, \\ M^{p-q}|t|^{q-2}t, & |t| > M,\end{cases}$$

where $M > 0$. Then $\phi \in C(\mathbb{R}, \mathbb{R})$, $\phi(t)t \geq q\Phi(t) := q\int_0^t \phi(s)ds \geq 0$ and $|\phi(t)| \leq M^{p-q}|t|^{q-1}$ for all $t \in \mathbb{R}$. Set $h_\eta(t) = \eta\phi(t) + f(t)$ for all $t \in R$. Then $h_\eta(t)$ possesses the following properties:

(h_1) $h_\eta \in C(\mathbb{R}, \mathbb{R})$ is odd and $\lim_{t\to 0} \frac{|h_\eta(t)|}{|t|^{r-1}} = \alpha$.

(h_2) $|h_\eta(t)| \leq \eta M^{p-q}|t|^{q-1} + C_1 + C_2|t|^{q-1}$ for all $t \in \mathbb{R}$.

(h_3) $\frac{h_\eta(t)}{t^{q-1}}$ is an increasing function of t on $(0,+\infty)$.

(h_4) $h_\eta(t)t \geq qH_\eta(t) := q\int_0^t h_\eta(\tau)d\tau \geq 0$ for all $t \in \mathbb{R}$.

Fixed $0 < \varepsilon < \varepsilon_0$ and $\|V\|_\infty < V_*$, where ε_0 and V_* appear in Theorem 1.1, taking into account (V) and (h_1)-(h_3) and Theorem 1.1, the following problem

$$\begin{cases}(-\Delta)^s u + V(\varepsilon x)u + \lambda u = h_\eta(u), \text{ in } \mathbb{R}^N,\\ \int_{\mathbb{R}^N} |u|^2 dx = a^2,\end{cases} \tag{5.1}$$

admits at least $cat_{M_\delta}(M)$ couples $(u_\eta, \lambda) \in H^s(\mathbb{R}^N) \times \mathbb{R}^+$ of weak solutions. Let

$$\begin{aligned} J_{\varepsilon,\eta}(u) =& \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x)u^2 dx - \int_{\mathbb{R}^N} H_\eta(u)dx \\ =& \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon x)u^2 dx - \eta\int_{\mathbb{R}^N} \Phi(u)dx - \int_{\mathbb{R}^N} F(u)dx. \end{aligned}$$

Then $J_{\varepsilon,\eta}|'_{S_a}(u_\eta) = 0$ and $J_{\varepsilon,\eta}(u_\eta) = m_{V_0,a,\eta} := \inf\limits_{u\in S_a} J_{V_0,\eta}(u)$, where

$$J_{V_0,\eta}(u) = \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V_0 u^2 dx - \int_{\mathbb{R}^N} H_\eta(u)dx.$$

Furthermore, it is easy to see that $m_{V_0,a,\eta} \leq m_{V_0,a}$.

Lemma 5.1. *The solution u_η satisfies $\|(-\Delta)^{\frac{s}{2}}u_\eta\|_2^2 \leq \frac{2(qm_{V_0,a}+\lambda a^2)}{q-2}$.*

Proof. By (5.1), we can infer that

$$\begin{aligned} 0 =& \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u_\eta|^2 dx + \int_{\mathbb{R}^N} V(\varepsilon x)|u_\eta|^2 dx + \lambda\int_{\mathbb{R}^N} |u_\eta|^2 dx - \int_{\mathbb{R}^N} h_\eta(u_\eta)u_\eta dx \\ =& \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u_\eta|^2 dx + \int_{\mathbb{R}^N} V(\varepsilon x)|u_\eta|^2 dx + \lambda\int_{\mathbb{R}^N} |u_\eta|^2 dx - \eta\int_{\mathbb{R}^N} \phi(u_\eta)u_\eta dx \\ & - \int_{\mathbb{R}^N} f(u_\eta)u_\eta dx. \end{aligned}$$

It follows from (1.9) that

$$\begin{aligned} qm_{V_0,a} \geq qm_{V_0,a,\eta} =& \frac{q}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u_\eta|^2 dx + \frac{q}{2}\int_{\mathbb{R}^N} V(\varepsilon x)|u_\eta|^2 dx - \eta q\int_{\mathbb{R}^N} \Phi(u_\eta)dx \\ & - q\int_{\mathbb{R}^N} F(u_\eta)dx - \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u_\eta|^2 dx - \int_{\mathbb{R}^N} V(\varepsilon x)|u_\eta|^2 dx \\ & - \lambda\int_{\mathbb{R}^N} |u_\eta|^2 dx + \eta\int_{\mathbb{R}^N} \phi(u_\eta)u_\eta dx + \int_{\mathbb{R}^N} f(u_\eta)u_\eta dx \\ =& \frac{q-2}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u_\eta|^2 dx + \frac{q-2}{2}\int_{\mathbb{R}^N} V(\varepsilon x)|u_\eta|^2 dx - \lambda\int_{\mathbb{R}^N} |u_\eta|^2 dx \\ & + \eta\int_{\mathbb{R}^N} [\phi(u_\eta)u_\eta - q\Phi(u_\eta)]dx + \int_{\mathbb{R}^N} [f(u_\eta)u_\eta - qF(u_\eta)]dx \\ \geq& \frac{q-2}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u_\eta|^2 dx - \lambda\int_{\mathbb{R}^N} |u_\eta|^2 dx \\ =& \frac{q-2}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u_\eta|^2 dx - \lambda a^2, \end{aligned}$$

which implies that the lemma holds. □

Lemma 5.2. *There exist two constants $B, D > 0$ independent on η such that $\|u_\eta\|_{L^\infty} \le B(1+\eta)^D$.*

Proof. For any $L > 0$ and $\beta > 1$, set

$$\gamma(u_\eta) := \gamma_{\eta,L}(u_\eta) = u_\eta |u_{\eta,L}|^{2(\beta-1)} \in H^s(\mathbb{R}^N),$$

where $u_{\eta,L} := \min\{u_\eta, L\}$. Since γ is an increasing function, we can deduce that

$$(a-b)[\gamma(a) - \gamma(b)] \ge 0, \ \forall\, a,\ b \in \mathbb{R}.$$

Set $\Psi(t) = \frac{|t|^2}{2}$ and $\Gamma(t) = \int_0^t (\gamma'(\tau))^{\frac{1}{2}} d\tau$ for $t \ge 0$. We claim that

$$\Psi'(a-b)[\gamma(a)-\gamma(b)] \ge |\Gamma(a) - \Gamma(b)|^2$$

for all $a,\ b \in \mathbb{R}$. Indeed, if $a > b$ we obtain

$$\begin{aligned}\Psi'(a-b)[\gamma(a)-\gamma(b)] =&(a-b)[\gamma(a)-\gamma(b)] = (a-b)\int_b^a \gamma'(t)dt \\ =&(a-b)\int_b^a (\Gamma'(t))^2 dt \ge (\int_b^a \Gamma'(t)dt)^2 \\ =&|\Gamma(a)-\Gamma(b)|^2.\end{aligned}$$

If $a \le b$, the proof is similar. Hence, the assertion is true. It results that

$$|\Gamma(u_\eta)(x) - \Gamma(u_\eta)(y)|^2 \le [(u_\eta)(x) - (u_\eta)(y)] \cdot [(u_\eta|u_{\eta,L}|^{2(\beta-1)})(x) - (u_\eta|u_{\eta,L}|^{2(\beta-1)})(y)]. \quad (5.2)$$

As a consequence, taking $\gamma(u_\eta) = u_\eta|u_{\eta,L}|^{2(\beta-1)}$ as a test function, in the light of (5.2) we see that

$$\begin{aligned}&\frac{C_{N,s}}{2}[\Gamma(u_\eta)]^2_{H^s(\mathbb{R}^N)} + \int_{\mathbb{R}^N} V(\varepsilon x) u_\eta^2 |u_{\eta,L}|^{2(\beta-1)}dx \\ \le&\frac{C_{N,s}}{2}\int_{\mathbb{R}^{2N}} \frac{[(u_\eta)(x) - (u_\eta)(y)] \cdot [(u_\eta|u_{\eta,L}|^{2(\beta-1)})(x) - (u_\eta|u_{\eta,L}|^{2(\beta-1)})(y)]}{|x-y|^{N+2s}}dxdy \\ &+ \int_{\mathbb{R}^N} V(\varepsilon x) u_\eta^2 |u_{\eta,L}|^{2(\beta-1)}dx + \lambda \int_{\mathbb{R}^N} u_\eta^2|u_{\eta,L}|^{2(\beta-1)}dx \\ =&\int_{\mathbb{R}^N} h_\eta(u_\eta) u_\eta |u_{\eta,L}|^{2(\beta-1)}dx.\end{aligned} \quad (5.3)$$

It follow from (h_1) and (h_2),for fixed $\eta > 0$ we can prove

$$|h_\eta(t)| = |f(t) + \eta\phi(t)| \le V_0|t| + (1+\eta)C|t|^{q-1} \quad (5.4)$$

for all $t \in \mathbb{R}$. Simultaneously, $|\Gamma(u_\eta)| \ge \frac{1}{\beta} u_\eta |u_{\eta,L}|^{\beta-1}$ and

$$\frac{C_{N,s}}{2}[\Gamma(u_\eta)]^2_{H^s(\mathbb{R}^N)} = \|\Gamma(u_\eta)\|^2_{\mathcal{D}^{s,2}(\mathbb{R}^N)} \ge S\|\Gamma(u_\eta)\|^2_{2_s^*} \ge \frac{1}{\beta^2} S \|u_\eta|u_{\eta,L}|^{\beta-1}\|^2_{2_s^*}. \quad (5.5)$$

Therefore, taking into account (5.3)-(5.5) and (V) we infer that

$$\begin{aligned}&\frac{1}{\beta^2} S\|u_\eta|u_{\eta,L}|^{\beta-1}\|^2_{2_s^*} \le \frac{C_{N,s}}{2}[\Gamma(u_\eta)]^2_{H^s(\mathbb{R}^N)} \\ \le&\int_{\mathbb{R}^N}[f(u_\eta) + \eta\phi(u_\eta)]u_\eta|u_{\eta,L}|^{2(\beta-1)}dx - \int_{\mathbb{R}^N} V(\varepsilon x) u_\eta^2|u_{\eta,L}|^{2(\beta-1)}dx \\ \le&\int_{\mathbb{R}^N} V_0 u_\eta^2 |u_{\eta,L}|^{2(\beta-1)}dx + (1+\eta)C\int_{\mathbb{R}^N} |u_\eta|^q |u_{\eta,L}|^{2(\beta-1)}dx \\ &- \int_{\mathbb{R}^N} V(\varepsilon x) u_\eta^2 |u_{\eta,L}|^{2(\beta-1)}dx \\ \le&C(1+\eta)\int_{\mathbb{R}^N} |u_\eta|^q |u_{\eta,L}|^{2(\beta-1)}dx.\end{aligned}$$

Consequently,

$$\|u_\eta |u_{\eta,L}|^{\beta-1}\|_{2_s^*}^2 \leq C(1+\eta)\beta^2 \int_{\mathbb{R}^N} |u_\eta|^q |u_{\eta,L}|^{2(\beta-1)}dx.$$

Set $w_{\eta,L} = u_\eta |u_{\eta,L}|^{\beta-1}$, by virtue of the Hölder inequality one has

$$\begin{aligned}\|w_{\eta,L}\|_{2_s^*}^2 &\leq C(1+\eta)\beta^2 \int_{\mathbb{R}^N} |u_\eta|^{q-2}|u_\eta|^2 |u_{\eta,L}|^{2(\beta-1)}dx \\ &\leq C(1+\eta)\beta^2 (\int_{\mathbb{R}^N} |u_\eta|^{2_s^*}dx)^{\frac{q-2}{2_s^*}} \cdot (\int_{\mathbb{R}^N} |w_{\eta,L}|^{\alpha_s^*}dx)^{\frac{2}{\alpha_s^*}},\end{aligned}$$

where $\alpha_s^* = \frac{22_s^*}{2_s^*-(q-2)} \in (2, 2_s^*)$.

By Lemma 5.1 one has

$$\|w_{\eta,L}\|_{2_s^*}^2 \leq C(1+\eta)\beta^2 \|w_{\eta,L}\|_{\alpha_s^*}^2. \tag{5.6}$$

Now we observe that if $u_\eta^\beta \in L^{\alpha_s^*}(\mathbb{R}^N)$, from the definition of $\{u_{\eta,L}\}$ and by using of the fact $|u_{\eta,L}| \leq |u_\eta|$ and (5.6) we see that

$$\|w_{\eta,L}\|_{2_s^*}^2 \leq C(1+\eta)\beta^2 (\int_{\mathbb{R}^N} |u_\eta|^{\beta\alpha_s^*}dx)^{\frac{2}{\alpha_s^*}} < +\infty.$$

Let $L \to +\infty$, it follows from the Fatou lemma that

$$\|u_\eta\|_{\beta 2_s^*} \leq C^{\frac{1}{\beta}} (\sqrt{1+\eta})^{\frac{1}{\beta}} \beta^{\frac{1}{\beta}} \|u_\eta\|_{\beta\alpha_s^*}, \tag{5.7}$$

whenever $|u_\eta|^{\beta\alpha_s^*} \in L^1(\mathbb{R}^N)$.

Now, set $\beta := \frac{2_s^*}{\alpha_s^*} > 1$. By the fact that $u_\eta \in L^{2_s^*}(\mathbb{R}^N)$ we know that the above inequality holds for this choice of β. Then, observing that $\beta^2\alpha_s^* = \beta 2_s^*$, it follows that (5.7) holds with β replaced by β^2. Therefore,

$$\begin{aligned}\|u_\eta\|_{\beta^2 2_s^*} &\leq C^{\frac{1}{\beta^2}} (\sqrt{1+\eta})^{\frac{1}{\beta^2}} \beta^{\frac{2}{\beta^2}} \|u_\eta\|_{\beta^2\alpha_s^*} \\ &= C^{\frac{1}{\beta^2}} (\sqrt{1+\eta})^{\frac{1}{\beta^2}} \beta^{\frac{2}{\beta^2}} \|u_\eta\|_{\beta 2_s^*} \\ &\leq C^{\frac{1}{\beta^2}} (\sqrt{1+\eta})^{\frac{1}{\beta^2}} \beta^{\frac{2}{\beta^2}} C^{\frac{1}{\beta}} (\sqrt{1+\eta})^{\frac{1}{\beta}} \beta^{\frac{1}{\beta}} \|u_\eta\|_{\beta\alpha_s^*} \\ &= C^{\frac{1}{\beta}+\frac{1}{\beta^2}} (\sqrt{1+\eta})^{\frac{1}{\beta}+\frac{1}{\beta^2}} \beta^{\frac{1}{\beta}+\frac{2}{\beta^2}} \|u_\eta\|_{\beta\alpha_s^*}.\end{aligned}$$

Interating this process, and recalling that $\beta\alpha_s^* = 2_s^*$, we can conclude that for every $m \in \mathbb{N}$

$$\|u_\eta\|_{\beta^m 2_s^*} \leq C^{\sum_{i=1}^m \frac{1}{\beta^i}} (\sqrt{1+\eta})^{\sum_{i=1}^m \frac{1}{\beta^i}} \beta^{\sum_{i=1}^m \frac{i}{\beta^i}} \|u_\eta\|_{2_s^*}.$$

Set

$$d_m = \sum_{i=1}^m \frac{1}{\beta^i} \text{ and } e_m = \sum_{i=1}^m \frac{i}{\beta^i}.$$

Then, we have $d_m \to \sigma_1 > 0$ and $e_m \to \sigma_2 > 0$ as $m \to \infty$. Let $m \to +\infty$, by Lemma 5.1 we get

$$\|u_\eta\|_{L^\infty} \leq C^{\sigma_1}(\sqrt{1+\eta})^{\sigma_1}\beta^{\sigma_2}A^{\frac{1}{2}} := B(1+\eta)^D,$$

where $B := C^{\sigma_1}\beta^{\sigma_2}A^{\frac{1}{2}} > 0$ and $D := \frac{\sigma_1}{2}$ are independent on η. □

Proof of Theorem 1.2. For large $M > 0$, we can choose small $\eta_0 > 0$ such that

$$\|u_\eta\|_{L^\infty} \leq B(1+\eta)^D \leq M \text{ for all } \eta \in (0, \eta_0].$$

Hence, $h_\eta(u_\eta) = \eta|u_\eta|^{p-2}u_\eta + f(u_\eta)$ for all $\eta \in (0, \eta_0]$. As a consequence, problem (Q_a) possesses at least $cat_{M_\delta}(M)$ couples $(u_\eta, \lambda) \in H^s(\mathbb{R}^N) \times \mathbb{R}^+$ of weak solutions. □

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Conflict of interest statement

Authors state that there is no conflict of interest.

Data availability statements

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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