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Concentration of solutions for non-autonomous double-phase problems with lack of compactness

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Abstract. The present paper is devoted to the study of the following double-phase equation

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u+\mu_\varepsilon(x)|\nabla u|^{q-2}\nabla u)+V_\varepsilon(x)(|u|^{p-2}u+\mu_\varepsilon(x)|u|^{q-2}u)=f(u)\quad\text{in}\quad\mathbb{R}^N,$$

where $N\geq 2$, $1<p<q<N$, $q<p^*$ with $p^*=\frac{Np}{N-p}$, $\mu:\mathbb{R}^N\to\mathbb{R}$ is a continuous non-negative function, $\mu_\varepsilon(x)=\mu(\varepsilon x)$, $V:\mathbb{R}^N\to\mathbb{R}$ is a positive potential satisfying a local minimum condition, $V_\varepsilon(x)=V(\varepsilon x)$, and the nonlinearity $f:\mathbb{R}\to\mathbb{R}$ is a continuous function with subcritical growth. Under natural assumptions on μ, V and f, by using penalization methods and Lusternik–Schnirelmann theory we first establish the multiplicity of solutions, and then, we obtain concentration properties of solutions.

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1. Introduction

In the present paper, we focus on the study of the multiplicity and concentration of solutions for the following double-phase problem

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u+\mu_\varepsilon(x)|\nabla u|^{q-2}\nabla u)+V_\varepsilon(x)(|u|^{p-2}u+\mu_\varepsilon(x)|u|^{q-2}u)=f(u)\quad\text{in}\quad\mathbb{R}^N,\tag{1.1}$$

where $N\geq 2$, $1<p<q<N$, $q<p^*$ with $p^*=\frac{Np}{N-p}$, $\mu_\varepsilon(x)=\mu(\varepsilon x)$, $V_\varepsilon(x)=V(\varepsilon x)$, μ and V satisfy the basic assumptions below:

(A_1) $\mu:\mathbb{R}^N\to\mathbb{R}$ is a continuous and non-negative function and $\mu\in L^\infty(\mathbb{R}^N)$;
(A_2) there exists $V_0>0$ fulfilling $V_0:=\inf_{x\in\mathbb{R}^N}V(x)$;
(A_3) there exists a bounded subset $\Lambda\subset\mathbb{R}^N$ such that

$$V_0=\inf_{x\in\Lambda}V(x)<\min_{x\in\partial\Lambda}V(x);$$

(A_4) there exists $x_{\min}\in\Lambda$ such that $V_0=V(x_{\min})$ and $\mu(x_{\min})=\inf_{\mathbb{R}^N}\mu(x):=\mu_0\geq 0$.

Moreover, we assume that $f:\mathbb{R}\to\mathbb{R}$ is a continuous function, $f(t)=0$ if $t\leq 0$ and satisfies the following assumptions:

(f_1) $\lim_{t\to 0^+}\frac{f(t)}{t^{p-1}}=0$;
(f_2) there exists $r\in(q,p^*)$ such that $\lim_{t\to+\infty}\frac{f(t)}{t^{r-1}}=0$, here $p^*=\frac{Np}{N-p}$;
(f_3) there exists $\theta\in(q,p^*)$ such that

$$0<\theta F(t):=\theta\int_0^t f(\tau)\mathrm{d}\tau\leq f(t)t\quad\text{for any}\quad t>0;$$

(f_4) for any $t \in (0, \infty)$, $\frac{f(t)}{t^{q-1}}$ is increasing.

Since the content of the paper is closely concerned with unbalanced growth, we briefly introduce in what follows the related background, pioneering contributions and related applications.

Historical background

Equation (1.1) is driven by a differential operator with unbalanced growth due to the presence of the (p, q)-Laplace operator. This type of problem comes from a general reaction–diffusion system:

$$u_t = \text{div}[A(\nabla u)\nabla u] + c(x, u), \quad \text{and} \quad A(\nabla u) = |\nabla u|^{p-2} + |\nabla u|^{q-2},$$

where the function u is a state variable and describes the density or concentration of multicomponent substances, $\text{div}[A(\nabla u)\nabla u]$ corresponds to the diffusion with coefficient $A(\nabla u)$, and $c(x, u)$ is the reaction and relates to source and loss processes. Originally, the idea to treat such operators comes from Zhikov [53] who introduced such classes to provide models of strongly anisotropic materials, see also the monograph of Zhikov et al. [54]. We refer to the remarkable pioneering papers by Marcellini [11,36–38], where the author investigated the regularity and existence of solutions of elliptic equations with unbalanced growth conditions.

The (p, q)-Laplacian equation (1.1) is also motivated by numerous models arising in mathematical physics. For instance, we can refer to the following Born–Infeld equation [12] that appears in electromagnetism, electrostatics and electrodynamics as a model based on a modification of Maxwell's Lagrangian density:

$$-\text{div}\left(\frac{\nabla u}{(1 - 2|\nabla u|^2)^{\frac{1}{2}}}\right) = h(u) \quad \text{in } \Omega.$$

Indeed, by the Taylor formula, we have

$$(1 - x)^{-\frac{1}{2}} = 1 + \frac{x}{2} + \frac{3}{2 \cdot 2^2}x^2 + \frac{5!!}{3! \cdot 2^3}x^3 + \cdots + \frac{(2n-3)!!}{(n-1)! \cdot 2^{n-1}}x^{n-1} + \cdots \quad \text{for } |x| < 1.$$

Taking $x = 2|\nabla u|^2$ and adopting the first-order approximation, we obtain problem (1.1) for $p = 2$ and $q = 4$. Furthermore, the n-th-order approximation problem is driven by the multi-phase differential operator

$$-\Delta u - \Delta_4 u - \frac{3}{2}\Delta_6 u - \cdots - \frac{(2n-3)!!}{(n-1)!}\Delta_{2n} u.$$

We also refer to the following fourth-order relativistic operator

$$u \mapsto \text{div}\left(\frac{|\nabla u|^2}{(1 - |\nabla u|^4)^{\frac{3}{4}}}\nabla u\right),$$

which describes large classes of phenomena arising in relativistic quantum mechanics. Again, by Taylor's formula, we have

$$x^2(1 - x^4)^{-\frac{3}{4}} = x^2 + \frac{3x^6}{4} + \frac{21x^{10}}{32} + \cdots .$$

This shows that the fourth-order relativistic operator can be approximated by the following operator

$$u \mapsto \Delta_4 u + \frac{3}{4}\Delta_8 u.$$

For more details on the physical backgrounds and other applications, we refer to Bahrouni et al. [9](for phenomena associated with transonic flows) and to Benci et al. [10](for models arising in quantum physics).

The double-phase operator

$$\text{div}(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u) \tag{1.2}$$

was originally introduced by Zhikov [52] to characterize the models of strongly anisotropic materials. Moreover, Zhikov found that its hardening properties drastically change with the point. This is called the Lavrentiev's phenomenon. He considered the functional

$$\int_{\Omega} (|\nabla u|^p + \mu(x)|\nabla u|^q)\mathrm{d}x$$

to describe that the integrands change their ellipticity rate according to the point. The coefficient $a(x)$ was used to regulating the mixture between two different materials, with power hardening of rates p and q, respectively. The main features of operator (1.2) are that it is non-homogeneous and the function $\mu : \mathbb{R}^N \to \mathbb{R}$ is degenerate. It is clear that this operator is a generalization of p-Laplacian (as $\mu = 0$) and $p\&q$-Laplacian (as $\mu = 1$).

An interested phenomenon is that the relevant bound assumed

$$q < p^* := \frac{Np}{N-p} \tag{1.3}$$

is equivalent to the condition on the ratio q/p

$$\frac{q}{p} < 1 + \frac{p}{N-p} = 1 + O(N)$$

Up to change N with $N-1$ and the strict inequality "$<$", then the relevant assumption (1.3) made in this manuscript, which is connected with compactness properties, is well comparable with its opposite inequality

$$\frac{q}{p} > 1 + \frac{N-1}{N-1-p},$$

which is exactly the condition to show the existence of counterexamples to regularity, see [20,37]. Colombo and Mingione [16, 17] considered the regularity of solution with some proper restrictions on p and q, which seems to be the first research result about the solution of (1.3). Recently Colombo and Mingione [16,17] gave a strong impulse with the introduction of the terminology (and not only terminology, but also fine results) of double-phase integrals. However the necessity to impose "some proper restrictions on p and q" and the first regularity results for double-phase integrals (which is a particular case of the p,q-growth conditions) have been first proposed and proved in the reference [37,39]. For more results on this topic, we see [20,21].

In the last decade, many researchers investigated the existence and multiplicity of solutions for the double-phase problem

$$\mathrm{div}(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u) = f(x,u) \quad \text{in} \quad \Omega, \tag{1.4}$$

where Ω is a bounded domain, see [16,17,25,32,35,45,46]. More precisely, Liu and Dai [32] dealt with the solutions of (1.4) by establishing a Musielak–Orlicz–Sobolev space and then obtained the existences of solutions and infinite many solutions with Dirichlet boundary condition, under the conditions that $1 < p < q < N$, $\frac{q}{p} < 1 + \frac{1}{N}$ and $\mu : \overline{\Omega} \to [0,\infty)$ is Lipschitz continuous. They also investigated some basic properties of the double-phase operator and the corresponding spaces. After that, the research to solutions of (1.4) by using the variational methods has become a hot topic.

In the case $\varepsilon = 1$, equation (1.1) boils down the equation

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u) + V(x)(|u|^{p-2}u + \mu(x)|u|^{q-2}u) = f(u) \quad \text{in} \quad \mathbb{R}^N.$$

There are few works to deal with this problem. When $V \equiv 1$, the existence of infinitely many solutions and some basic properties of the corresponding Musielak–Orlicz–Sobolev spaces have been studied by Liu and Dai [33]. Furthermore, Liu and Winkert [34] investigated the existence of two non-negative solutions with singular nonlinearity. Moreover, by using the Fountain and Dual Fountain Theorem, Stegliński [41] researched the existence of infinitely many solutions.

We point out when $p = 2$ and $\mu \equiv 0$, by the change of variable $x \to \frac{x}{\varepsilon}$, equation (1.1) turns into the following Schrödinger equation

$$-\varepsilon^2 \Delta u + V(x)u = f(u) \quad \text{in} \quad \mathbb{R}^N.$$

Under a global minimum assumption or a local minimum assumption on V, the existence, multiplicity and concentration of solutions have been studied by a number of authors, we only refer the readers to [2,3,14,22,29,30] and the references therein.

It is worth noting that the multiplicity and concentration of solutions for the $p\&q$ type problem

$$-\Delta_p u - \Delta_q u + V_\varepsilon(x)(|u|^{p-2}u + |u|^{q-2}u) = f(u) \quad \text{in} \quad \mathbb{R}^N \tag{1.5}$$

have aroused attentions of some researchers, where $\Delta_r u = \operatorname{div}(|\nabla u|^{r-2}\nabla u)$, $r \in \{p, q\}$. By using perturbation techniques and Lusternik–Schnirelmann theory, Ambrosio and Repovš [7] considered equation (1.5) under the conditions that f is continuous, subcritical growth and V satisfies the global minimum assumption

$$V_\infty = \liminf_{|x|\to\infty} V(x) > \inf_{x\in\mathbb{R}^N} V(x) = V_0 > 0, \text{ where } V_\infty \le \infty.$$

After that, Zhang, Zhang, Rădulescu [48] considered the Choquard nonlinearity which is non-local. They in [47] studied the case of competing potentials. Zhang, Zuo and Zhao [51] investigated the impact of Kirchhoff term and derived a general verifying the compactness of the associated variational functional. Now, we shortly introduce partial researches that when V satisfies the local minimum assumption (A_2) and (A_3). Costa and Figueiredo [18] investigated the case that f admits critical growth, and Ambrosio and Isernia [5] studied equation (1.5) driven by a Kirchhoff term under (A_2) and (A_3). Also, if the nonlinearity f fulfills the Berestycki–Lions condition, the existence and concentration of positive solution were investigated by Ambrosio [4]. In recent years, a number of researchers put their sight on the existence, multiplicity and concentration of fractional $p\&q$ type problem. For the details, we just refer to [6,49,50] and the reference therein.

Main result

Motivated by [5,7,18,31], we consider the multiplicity and concentration of solutions for equation (1.1). Firstly, let us review the definition of the Lusternik–Schnirelmann category. Define

$$M = \{x \in \mathbb{R}^N : V(x) = V_0, \quad \mu(x) = \mu_0\} \quad \text{and} \quad M_\delta = \{x \in \mathbb{R}^N : \operatorname{dist}(x, M) \le \delta\},$$

where $\delta > 0$. Letting Y be a closed subset of topological space X, then the Lusternik–Schnirelmann category of Y in X is the least number of closed and contractible sets in X which cover Y, denoted by $\operatorname{cat}_X(Y)$.

Our main result establishes the following multiplicity and concentration property of solutions.

Theorem 1.1. *Suppose that (A_1)–(A_4) and (f_1)–(f_4) hold. Then for any $\delta > 0$ such that $M_\delta \subset \Lambda$, there exists $\varepsilon_\delta > 0$ such that for any $\varepsilon \in (0, \varepsilon_\delta)$, problem (1.1) admits at least $cat_{M_\delta}(M)$ non-negative solutions. Let u_ε denote one of the solutions and η_ε be a global maximum point of u_ε. Then,*

$$\lim_{\varepsilon\to 0} V(\varepsilon\,\eta_\varepsilon) = V_0.$$

We use the variational methods and Lusternik–Schnirelmann theory to show Theorem 1.1. To the best of the authors' knowledge, this is the first research result on the multiplicity and concentration of solutions for equation (1.1).

We point out that problem (1.5) is considered in the Sobolev space. However, since the principal operator in problem (1.1) is degenerate, this problem cannot be considered in general Sobolev space anymore. Hence it is difficult to exploit the approaches in [2,5,47,48]. Here, we will introduce a special

Sobolev space named Musielak–Orlicz–Sobolev spaces to tackle (1.1). This space is more complex than usual Sobolev space and some basic properties of this space must be established to investigate the solutions. To prove Theorem 1.1, at the first, we shall modify the nonlinearity in a suitable way, and we shall handle an autonomous problem. As well, some accurate analysis are used to verify that the variational functional $\mathcal{J}_\varepsilon$ of the modified problem satisfies the Palais–Smale condition for any $c \in \mathbb{R}$. Then, since we want to obtain the multiplicity of solutions and f is only continuous, we utilize an abstract critical point theorem developed in [42]. Note that the techniques also appear in [2,6,7] to investigate the $p\&q$ type problem. But, the appearance of function μ makes the process rather untoward. Finally, we show that the solutions of the modified problem are solutions of equation (1.1), where a Moser type iteration is applied to obtain the L^∞-estimates and decaying estimates at infinity of solutions for the modified problem. Since the double-phase operator is non-homogeneous and degenerate, we stress that it seems impossible to get the decaying estimates of the solutions by using the skills in [2,5,47,48]. In this paper, a testing function is constructed to demonstrate the uniformly decaying estimates of solutions and several new analysis techniques are applied, which are main novelty of our paper.

In this text, let $C, C_1, C_2, \cdots$ denote some fixed constants possibly different in different places; B_R denote $B_R(0)$; $o_n(1)$ represent $o_n(1) \to 0$ as $n \to \infty$, and $\rightharpoonup$ and $\to$ denote the weak convergence and the strong convergence in the corresponding spaces, respectively.

This paper is organized as follows. In Sect. 2, we introduce Musielak–Orlicz–Sobolev spaces. In Sect. 3, we consider the modified problem. In Sect. 4, we work with the autonomous problem. The last section is devoted to showing Theorem 1.1.

2. Preliminaries

In this section, we start with the definition and some basic preliminary properties of Musielak–Orlicz–Sobolev spaces. For detailed introduction on Musielak–Orlicz–Sobolev spaces, we refer to [15,27,33,40].

For any $s \in [1, \infty]$ and $\Omega \subset \mathbb{R}^N$, we denote by $\|u\|_{L^s(\mathbb{R}^N)}$ the standard norm of the usual Lebesgue space $L^s(\mathbb{R}^N)$, and for any $s \in (1, \infty)$, we denote by $W^{1,s}(\mathbb{R}^N)$ the Sobolev space

$$W^{1,s}(\mathbb{R}^N) = \{u : \mathbb{R}^N \to \mathbb{R} \quad \text{measurable} : \int\limits_{\mathbb{R}^N} (|u|^s + |\nabla u|^s) \mathrm{d}x < \infty\},$$

which is equipped with the norm

$$\|u\|_{1,s} = \left(\|\nabla u\|^s_{L^s(\mathbb{R}^N)} + \|u\|^s_{L^s(\mathbb{R}^N)}) \mathrm{d}x \right)^{\frac{1}{s}},$$

where $\|\nabla u\|_{L^s(\mathbb{R}^N)} = \||\nabla u|\|_{L^s(\mathbb{R}^N)}$.

The following basic properties of Sobolev spaces are very important.

Lemma 2.1. (see [1]) *If $p \in (1, N)$, then $W^{1,p}(\mathbb{R}^N)$ is continuous embedded in $L^t(\mathbb{R}^N)$ for any $t \in [p, p^*]$ and compactly embedded in $L^t_{loc}(\mathbb{R}^N)$ for any $t \in [p, p^*)$.*

The following Lions type result is very useful to investigate the existence of solution for the limit problem associated with (1.1).

Lemma 2.2. (see [2]) *If $1 < p < N$, let $\{u_n\}$ be a bounded sequence in $W^{1,p}(\mathbb{R}^N)$ satisfying*

$$\lim_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int\limits_{B_R(y)} |u_n|^p \mathrm{d}x = 0, \tag{2.1}$$

where $R > 0$, then $u_n \to 0$ in $L^t(\mathbb{R}^N)$ for all $t \in (p, p^*)$.

Let $1<p<q<N$, $q<p^*$ with $p^*=\frac{Np}{N-p}$. We define the functions $\mathcal{H}_{\mu_\varepsilon}:\mathbb{R}^N\times[0,\infty)\to[0,\infty)$ and $\mathcal{H}_{\mu_\varepsilon,V_\varepsilon}:\mathbb{R}^N\times[0,\infty)\to[0,\infty)$ as

$$\mathcal{H}_{\mu_\varepsilon}(x,t)=t^p+\mu_\varepsilon(x)t^q \quad \text{and} \quad \mathcal{H}_{\mu_\varepsilon,V_\varepsilon}(x,t)=V_\varepsilon(x)(t^p+\mu_\varepsilon(x)t^q).$$

Let $L^{\mathcal{H}_{\mu_\varepsilon}}(\mathbb{R}^N)$ be the Musielak–Orlicz–Lebesgue space

$$L^{\mathcal{H}_{\mu_\varepsilon}}(\mathbb{R}^N)=\{u:\mathbb{R}^N\to\mathbb{R} \quad \text{measurable}: \int\limits_{\mathbb{R}^N}\mathcal{H}_{\mu_\varepsilon}(x,|u|)\mathrm{d}x<\infty\}$$

with respect to the Luxemburg norm

$$\|u\|_{\mathcal{H}_{\mu_\varepsilon}}=\inf\{\lambda>0:\int\limits_{\mathbb{R}^N}\mathcal{H}_{\mu_\varepsilon}(x,|\frac{u}{\lambda}|)\mathrm{d}x\le 1\},$$

and let $L^{\mathcal{H}_{\mu_\varepsilon,V_\varepsilon}}(\mathbb{R}^N)$ be the Musielak–Orlicz–Lebesgue space

$$L^{\mathcal{H}_{\mu_\varepsilon,V_\varepsilon}}(\mathbb{R}^N)=\{u:\mathbb{R}^N\to\mathbb{R} \quad \text{measurable}: \int\limits_{\mathbb{R}^N}\mathcal{H}_{\mu_\varepsilon,V_\varepsilon}(x,|u|)\mathrm{d}x<\infty\}$$

equipped with the Luxemburg norm

$$\|u\|_{\mathcal{H}_{\mu_\varepsilon,V_\varepsilon}}=\inf\{\lambda>0:\int\limits_{\mathbb{R}^N}\mathcal{H}_{\mu_\varepsilon,V_\varepsilon}(x,|\frac{u}{\lambda}|)\mathrm{d}x\le 1\}.$$

We introduce the weighted Musielak–Orlicz–Sobolev space

$$\mathbb{X}_\varepsilon=\{u:\mathbb{R}^N\to\mathbb{R} \quad \text{measurable}: u\in L^{\mathcal{H}_{\mu_\varepsilon,V_\varepsilon}}(\mathbb{R}^N) \quad \text{and} \quad |\nabla u|\in L^{\mathcal{H}_{\mu_\varepsilon}}(\mathbb{R}^N)\},$$

whose norm is equipped as

$$\|u\|_\varepsilon=\||\nabla u|\|_{\mathcal{H}_{\mu_\varepsilon}}+\|u\|_{\mathcal{H}_{\mu_\varepsilon,V_\varepsilon}}.$$

From ([41], Theorem 6), we know that $\mathbb{X}_\varepsilon$ is a separable and reflexive Banach space.

The following embedding results hold.

Lemma 2.3. (see ([33], Theorem 2.7)) *$\mathbb{X}_\varepsilon$ is continuously embedded in $W^{1,p}(\mathbb{R}^N)$. Hence, $\mathbb{X}_\varepsilon$ is continuously embedded in $L^s(\mathbb{R}^N)$ for any $s\in[p,p^*]$ and compactly embedded in $L^s_{loc}(\mathbb{R}^N)$ for any $s\in[1,p^*)$.*

Let

$$\varrho_\varepsilon(u)=\|u\|^p_{p,\varepsilon}+\|u\|^q_{q,\varepsilon,\mu},$$

where we give

$$\|u\|^p_{p,\varepsilon}=\int\limits_{\mathbb{R}^N}(|\nabla u|^p+V_\varepsilon(x)|u|^p)\mathrm{d}x \quad \text{and} \quad \|u\|^q_{q,\varepsilon,\mu}=\int\limits_{\mathbb{R}^N}\mu_\varepsilon(x)(|\nabla u|^q+V_\varepsilon(x)|u|^q)\mathrm{d}x.$$

The norm $\|\cdot\|_\varepsilon$ and the modular ϱ_ε have the following relationships.

Lemma 2.4. (([8], Proposition 2.1) or ([32], Proposition 2.1)) *Let (A_1) and (A_2) hold. Then, one has that:*

(i) if $u\neq 0$, then $\|u\|_\varepsilon=\lambda$ if and only if $\varrho_\varepsilon(\frac{u}{\lambda})=1$;
(ii) $\|u\|_\varepsilon<1$ (resp. >1, $=1$) if and only if $\varrho_\varepsilon(u)<1$ (resp. >1, $=1$);
(iii) if $\|u\|_\varepsilon<1$, then $\|u\|^q_\varepsilon\le\varrho_\varepsilon(u)\le\|u\|^p_\varepsilon$;
(iv) if $\|u\|_\varepsilon>1$, then $\|u\|^p_\varepsilon\le\varrho_\varepsilon(u)\le\|u\|^q_\varepsilon$;
(v) $\|u\|_\varepsilon\to 0$ if and only if $\varrho_\varepsilon(u)\to 0$;
(vi) $\|u\|_\varepsilon\to\infty$ if and only if $\varrho_\varepsilon(u)\to\infty$.

3. The modified problem

In this section, we consider a modified problem by using the penalization method proposed by del Pino and Felmer [22].

Without loss of generality, we may suppose that

$$0 \in \Lambda \quad \text{and} \quad V_0 = V(0).$$

Take $a > 0$ and $k > p$ such that $\frac{f(a)}{a^{p-1}} = \frac{V_0}{k}$. We denote the modified function $\tilde{f} : \mathbb{R} \to \mathbb{R}$ as

$$\tilde{f}(t) = \begin{cases} f(t) & t \leq a \\ \frac{V_0}{k} t^{p-1} & t > a. \end{cases}$$

Let χ_Λ denote the characteristic function. Furthermore, we define the modified function

$$g(x,t) := \chi_\Lambda(x) f(t) + (1 - \chi_\Lambda(x))\tilde{f}(t) \quad \text{ for } (x,t) \in \mathbb{R}^N \times \mathbb{R},$$

Clearly, letting $G(x,t) = \int_0^t g(x,\tau)\mathrm{d}\tau$, from (f_1)–(f_4), we conclude that g fulfills the following properties.

- (g_1) $\lim_{t\to 0^+} \frac{g(x,t)}{t^{p-1}} = 0$ uniformly for $x \in \mathbb{R}^N$;
- (g_2) $g(x,t) \leq f(t)$ for any $x \in \mathbb{R}^N$ and $t \geq 0$;
- (g_3) (i) $0 < \theta G(x,t) \leq g(x,t)t$ for any $x \in \Lambda$ and $t > 0$, (ii) $0 \leq pG(x,t) \leq g(x,t)t \leq \frac{V_0}{k}t^p$ for any $x \in \Lambda^c$ and $t > 0$;
- (g_4) (i) for any $x \in \Lambda$, $\frac{g(x,t)}{t^{q-1}}$ is increasing, (ii) for any $x \in \Lambda^c$, $t \to \frac{g(x,t)}{t^{p-1}}$ is increasing in $t \in (0,a)$.

Now, we introduce the modified problem

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u + \mu_\varepsilon(x)|\nabla u|^{q-2}\nabla u) + V_\varepsilon(x)(|u|^{p-2}u + \mu_\varepsilon(x)|u|^{q-2}u) = g_\varepsilon(x,u) \quad \text{in} \quad \mathbb{R}^N, \tag{3.1}$$

here $g_\varepsilon(x,u) = g(\varepsilon x, u)$. Suppose that u is a solution of equation (3.1) and satisfies that $u(x) < a$ in Λ_ε^c with $\Lambda_\varepsilon = \{x \in \mathbb{R}^N : \varepsilon x \in \Lambda\}$. Then, we say that u is a solution of equation (1.1).

The variational functional of equation (3.1) is given by

$$\mathcal{J}_\varepsilon(u) = \frac{1}{p}\|u\|_{p,\varepsilon}^p + \frac{1}{q}\|u\|_{q,\varepsilon,\mu_\varepsilon}^q - \int_{\mathbb{R}^N} G_\varepsilon(x,u)\mathrm{d}x.$$

It is easy to deduce that $\mathcal{J}_\varepsilon \in C^1(\mathbb{X}_\varepsilon, \mathbb{R})$ and for any u, $v \in \mathbb{X}_\varepsilon$, its derivative is expressed as

$$\langle \mathcal{J}_\varepsilon'(u), v\rangle = \int_{\mathbb{R}^N} \Big(|\nabla u|^{p-2}\nabla u \nabla v + \mu_\varepsilon(x)|\nabla u|^{q-2}\nabla u\nabla v + V_\varepsilon(x)(|u|^{p-2}uv + \mu_\varepsilon(x)|u|^{q-2}uv)\Big)\mathrm{d}x - \int_{\mathbb{R}^N} g_\varepsilon(x,u)v\mathrm{d}x.$$

Next we verify the condition of mountain pass theorem to $\mathcal{J}_\varepsilon$.

Lemma 3.1. *$\mathcal{J}_\varepsilon$ fulfills mountain pass geometry, that is*

(i) there exist γ, $\beta > 0$ such that $\mathcal{J}_\varepsilon(u) \geq \beta$ for any $u \in \mathbb{X}_\varepsilon$ with $\|u\|_\varepsilon = \gamma$;
(ii) there exists $e \in \mathbb{X}_\varepsilon$ fulfilling $\|e\|_\varepsilon > \gamma$ such that $\mathcal{J}_\varepsilon(e) < 0$.

Proof. In view of (g_1)–(g_2), one has that for any $\xi > 0$, there exists $C_\xi > 0$ such that

$$|G_\varepsilon(x,t)| \leq \xi|t|^p + C_\xi|t|^r \quad \text{for any} \quad x \in \mathbb{R}^N,\ t \in \mathbb{R}. \tag{3.2}$$

By Lemma 2.3, Lemma 2.4 and (3.2), we conclude that for any $u \in \mathbb{X}_\varepsilon$ with $\|u\|_\varepsilon < 1$,

$$\mathcal{J}_\varepsilon(u) \geq \frac{1}{p}\|u\|_{p,\varepsilon}^p + \frac{1}{q}\|u\|_{q,\varepsilon,\mu_\varepsilon}^q - \int_{\mathbb{R}^N} (\xi|u|^p + C_\xi|u|^r)\mathrm{d}x$$

$$\geq \frac{1}{2p}\|u\|_{p,\varepsilon}^p + \frac{1}{q}\|u\|_{q,\varepsilon,\mu_\varepsilon}^q - C_1 \int_{\mathbb{R}^N} |u|^r \mathrm{d}x$$
$$\geq \frac{1}{2q}\|u\|_\varepsilon^q - C_2\|u\|_\varepsilon^r,$$

here we took $\xi < \frac{1}{2qC_1}$. Thanks to $q < r$, then (i) holds.

It is easy from (f_3) to deduce that there exists $C > 0$ such that

$$F(t) \geq t^q - C \quad \text{for any} \quad t \geq 0. \tag{3.3}$$

Choose $u_0 \in \mathbb{X}_\varepsilon$ such that $\mathrm{supp}(u_0) \subset \Lambda_\varepsilon$ and $u_0 \geq 0$. From (3.3), there holds that

$$\mathcal{J}_\varepsilon(tu_0) \leq \frac{t^p}{p}\|u_0\|_{p,\varepsilon}^p + \frac{t^q}{q}\|u_0\|_{q,\varepsilon,\mu_\varepsilon}^q - \int_{\Lambda_\varepsilon} (|tu_0|^\theta - C)\mathrm{d}x$$
$$= \frac{t^p}{p}\|u_0\|_{p,\varepsilon}^p + \frac{t^q}{q}\|u_0\|_{q,\varepsilon,\mu_\varepsilon}^q - t^\theta \int_{\Lambda_\varepsilon} |u_0|^\theta \mathrm{d}x - C\mathrm{meas}(\Lambda_\varepsilon).$$

Since $\theta > q$, $\mathcal{J}_\varepsilon(tu_0) \to -\infty$ as $t \to \infty$. Letting $t > 0$ be large enough and taking $e = tu_0$, then we have that $\|e\|_\varepsilon > \gamma$ and $\mathcal{J}_\varepsilon(e) < 0$. □

We say that $\{u_n\} \subset \mathbb{X}_\varepsilon$ is a $(PS)_c$ sequence (Palais–Smale sequence) for $\mathcal{J}_\varepsilon$ if $\mathcal{J}_\varepsilon(u_n) \to c$ and $\mathcal{J}_\varepsilon'(u_n) \to 0$. Recall that $\mathcal{J}_\varepsilon$ satisfies $(PS)_c$ condition (Palais–Smale condition) if any $(PS)_c$ sequence has a convergent subsequence.

Next, we establish the boundedness of the $(PS)_c$ sequences.

Lemma 3.2. *For any $c \in \mathbb{R}$, then any $(PS)_c$ sequence of $\mathcal{J}_\varepsilon$ is bounded.*

Proof. Let $\{u_n\}$ be a $(PS)_c$ sequence. Then, from (g_3) we have that

$$c + 1 + o_n(1)\|u_n\|_\varepsilon$$
$$\geq \mathcal{J}_\varepsilon(u_n) - \frac{1}{\theta}\langle \mathcal{J}_\varepsilon'(u_n), u_n\rangle$$
$$= \left(\frac{1}{p} - \frac{1}{\theta}\right)\|u_n\|_{p,\varepsilon}^p + \left(\frac{1}{q} - \frac{1}{\theta}\right)\|u_n\|_{q,\varepsilon,\mu_\varepsilon}^q + \frac{1}{\theta}\int_{\mathbb{R}^N} (g_\varepsilon(x,u_n)u_n - \theta G_\varepsilon(x,u_n))\mathrm{d}x$$
$$\geq \left(\frac{1}{p} - \frac{1}{\theta}\right)\|u_n\|_{p,\varepsilon}^p + \left(\frac{1}{q} - \frac{1}{\theta}\right)\|u_n\|_{q,\varepsilon,\mu_\varepsilon}^q + \frac{1}{\theta}\int_{\Lambda_\varepsilon^c} (g_\varepsilon(x,u_n)u_n - \theta G_\varepsilon(x,u_n))\mathrm{d}x$$
$$\geq \left(\frac{1}{p} - \frac{1}{\theta}\right)\|u_n\|_{p,\varepsilon}^p + \left(\frac{1}{q} - \frac{1}{\theta}\right)\|u_n\|_{q,\varepsilon,\mu_\varepsilon}^q + \frac{1}{\theta}\int_{\Lambda_\varepsilon^c} (p-\theta)G_\varepsilon(x,u_n)\mathrm{d}x$$
$$\geq \left(\frac{1}{p} - \frac{1}{\theta}\right)\|u_n\|_{p,\varepsilon}^p + \left(\frac{1}{q} - \frac{1}{\theta}\right)\|u_n\|_{q,\varepsilon,\mu_\varepsilon}^q - \left(\frac{\theta-p}{\theta}\right)\frac{V_0}{kp}\int_{\mathbb{R}^N} |u_n|^p\mathrm{d}x$$
$$\geq \left(\frac{1}{p} - \frac{1}{\theta}\right)(1 - \frac{p}{k})\|u_n\|_{p,\varepsilon}^p + \left(\frac{1}{q} - \frac{1}{\theta}\right)\|u_n\|_{q,\varepsilon,\mu_\varepsilon}^q$$
$$\geq C_1 \min\{\|u_n\|_\varepsilon^p, \|u_n\|_\varepsilon^q\},$$

where we have used Lemma 2.4. Due to $1 < p < q$, then $\{u_n\}$ is bounded in $\mathbb{X}_\varepsilon$. □

The Nehari manifold of equation (1.1) is denoted as

$$\mathcal{N}_\varepsilon = \{u \in \mathbb{X}_\varepsilon \setminus \{0\} : \langle \mathcal{J}_\varepsilon'(u), u\rangle = 0\}.$$

Let $c_\varepsilon = \inf_{u\in\mathcal{N}_\varepsilon} \mathcal{J}_\varepsilon(u)$. We introduce the following sets

$$\mathbb{X}_\varepsilon^+ = \{u \in \mathbb{X}_\varepsilon : \text{meas}(\{u^+\} \cap \Lambda_\varepsilon) > 0\}, \quad \text{and} \quad \mathbb{S}_\varepsilon^+ = \mathbb{S}_\varepsilon \cap \mathbb{X}_\varepsilon^+,$$

where $\mathbb{S}_\varepsilon$ represents the unit sphere in $\mathbb{X}_\varepsilon$. Then, $\mathbb{S}_\varepsilon^+$ is an incomplete $C^{1,1}$ manifold of codimension one and $\mathbb{X}_\varepsilon = T_u\mathbb{S}_\varepsilon^+ \oplus \mathbb{R}u$, where

$$T_u\mathbb{S}_\varepsilon^+ = \left\{v \in \mathbb{X}_\varepsilon : \int_{\mathbb{R}^N} \Big((|\nabla u|^{p-2}\nabla u + \mu_\varepsilon(x)|\nabla u|^{q-2}\nabla u)\nabla v + V_\varepsilon(x)(|u|^{p-2}u + \mu_\varepsilon(x)|u|^{q-2}u)v\Big)\mathrm{d}x = 0\right\}.$$

The following results play a key role in obtaining multiple solutions of (3.1).

Lemma 3.3. *Suppose that (A_1)–(A_4) and (f_1)–(f_4) hold. Then*

(i) for any $u \in \mathbb{X}_\varepsilon^+$, we define $h_u : [0,\infty) \to \mathbb{R}$ as $h_u(t) := \mathcal{J}_\varepsilon(tu)$. Then, there is the unique $t_u > 0$ such that $h_u'(t) > 0$ in $(0, t_u)$ and $h_u'(t) < 0$ in $(t_u, +\infty)$;

(ii) there is $\tau > 0$, independent on u, such that $t_u \geq \tau$ for every $u \in \mathbb{S}_\varepsilon^+$. Moreover, for each compact set $\mathcal{K} \subset \mathbb{S}_\varepsilon^+$, there is $C_\mathcal{K} > 0$ such that $t_u \leq C_\mathcal{K}$ for every $u \in \mathcal{K}$;

(iii) define the map $\hat{m}_\varepsilon : \mathbb{X}_\varepsilon^+ \to \mathcal{N}_\varepsilon$ as $\hat{m}_\varepsilon(u) := t_u u$. Then $\hat{m}_\varepsilon$ is continuous and $m_\varepsilon := \hat{m}_\varepsilon|_{\mathbb{S}_\varepsilon^+}$ is a homeomorphism between $\mathbb{S}_\varepsilon^+$ and $\mathcal{N}_\varepsilon$. Moreover, $m_\varepsilon^{-1}(u) = \frac{u}{\|u\|_\varepsilon}$;

(iv) let $\{u_n\} \subset \mathbb{S}_\varepsilon^+$ be a sequence such that $dist(u_n, \partial\mathbb{S}_\varepsilon^+) \to 0$. Then, $\|m_\varepsilon(u_n)\|_\varepsilon \to \infty$ and $\mathcal{J}_\varepsilon(m_\varepsilon(u_n)) \to \infty$.

Proof. (i) It follows from (g_2) and (g_3) that for any $u \in \mathbb{X}_\varepsilon^+$, $h_u(t) \to 0^+$ as $t \to 0^+$ and $h_u(t) \to -\infty$ as $t \to \infty$. Then, h_u has a maximum point $t_u \in (0,\infty)$ such that $h_u'(t_u) = 0$. To finish the proof of (i), we only need to show that there exists a unique positive number t_u such that $h_u'(t_u) = 0$. In contrary, if there exists $t_1 > t_2 > 0$ such that $h_u'(t_1) = h_u'(t_2) = 0$, then

$$t_1^p\|u\|_{p,\varepsilon}^p + t_1^q\|u\|_{q,\varepsilon,\mu_\varepsilon}^q = \int_{\mathbb{R}^N} g_\varepsilon(x, t_1u)t_1u\mathrm{d}x \tag{3.4}$$

and

$$t_2^p\|u\|_{p,\varepsilon}^p + t_2^q\|u\|_{q,\varepsilon,\mu_\varepsilon}^q = \int_{\mathbb{R}^N} g_\varepsilon(x, t_2u)t_2u\mathrm{d}x. \tag{3.5}$$

By using (3.4) and (3.5), one has that

$$\begin{aligned}
&\left(\frac{1}{t_1^{q-p}} - \frac{1}{t_2^{q-p}}\right)\|u\|_{p,\varepsilon}^p \\
&= \int_{\mathbb{R}^N} \left(\frac{g_\varepsilon(x,t_1u)}{t_1^{q-1}}u - \frac{g_\varepsilon(x,t_2u)}{t_2^{q-1}}u\right)\mathrm{d}x \\
&= \int_{\{u>0\}} \left(\frac{g_\varepsilon(x,t_1u)}{(t_1u)^{q-1}} - \frac{g_\varepsilon(x,t_2u)}{(t_2u)^{q-1}}\right)u^q\mathrm{d}x \\
&\geq \Bigg(\int_{\{u>0\}\cap\Lambda_\varepsilon^c\cap\{t_1u<a\}} + \int_{\{u>0\}\cap\Lambda_\varepsilon^c\cap\{t_1u\geq a\geq t_2u\}} \\
&\quad + \int_{\{u>0\}\cap\Lambda_\varepsilon^c\cap\{t_2u>a\}}\Bigg)\left(\frac{g_\varepsilon(x,t_1u)}{(t_1u)^{q-1}} - \frac{g_\varepsilon(x,t_2u)}{(t_2u)^{q-1}}\right)u^q\mathrm{d}x \\
&= \int_{\{u>0\}\cap\Lambda_\varepsilon^c\cap\{t_1u<a\}} \left(\frac{f(t_1u)}{(t_1u)^{q-1}} - \frac{f(t_2u)}{(t_2u)^{q-1}}\right)u^q\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&+\int\limits_{\{u>0\}\cap\Lambda_\varepsilon^c\cap\{t_1u\geq a\geq t_2u\}}\left(\frac{V_0}{k}\frac{1}{(t_1u)^{q-p}}-\frac{f(t_2u)}{(t_2u)^{q-1}}\right)u^q\mathrm{d}x\\
&+\int\limits_{\{u>0\}\cap\Lambda_\varepsilon^c\cap\{t_2u>a\}}\left(\frac{V_0}{k}\frac{1}{(t_1u)^{q-p}}-\frac{V_0}{k}\frac{1}{(t_2u)^{q-p}}\right)u^q\mathrm{d}x\\
\geq&\int\limits_{\{u>0\}\cap\Lambda_\varepsilon^c\cap\{t_1u\geq a\geq t_2u\}}\left(\frac{V_0}{k}\frac{1}{(t_1u)^{q-p}}-\frac{V_0}{k}\frac{1}{(t_2u)^{q-p}}\right)u^q\mathrm{d}x\\
&+\frac{V_0}{k}\left(\frac{1}{t_1^{q-p}}-\frac{1}{t_2^{q-p}}\right)\int\limits_{\{u>0\}\cap\Lambda_\varepsilon^c\cap\{t_2u>a\}}u^p\mathrm{d}x\\
=&\frac{V_0}{k}\left(\frac{1}{t_1^{q-p}}-\frac{1}{t_2^{q-p}}\right)\int\limits_{\{u>0\}\cap\Lambda_\varepsilon^c\cap\{t_1u\geq a\geq t_2u\}}u^p\mathrm{d}x+\frac{V_0}{k}\left(\frac{1}{t_1^{q-p}}-\frac{1}{t_2^{q-p}}\right)\int\limits_{\{u>0\}\cap\Lambda_\varepsilon^c\cap\{t_2u>a\}}u^p\mathrm{d}x\\
\geq&\frac{1}{k}\left(\frac{1}{t_1^{q-p}}-\frac{1}{t_2^{q-p}}\right)\|u\|_{p,\varepsilon}^p,
\end{aligned}
$$

here we applied the fact that for $x\in\Lambda_\varepsilon^c$ and $t_2u(x)\leq a$,

$$\frac{f(t_2u)}{(t_2u)^{q-1}}=\frac{f(t_2u)}{(t_2u)^{p-1}}\frac{1}{(t_2u)^{q-p}}\leq\frac{V_0}{k}\frac{1}{(t_2u)^{q-p}}.$$

Hence we obtain that $\|u\|_{p,\varepsilon}^p\leq\frac{1}{k}\|u\|_{p,\varepsilon}^p$, which is a contradiction.

(ii) For any $u\in\mathbb{S}_\varepsilon^+$, it follows from (g_1) and (g_2) that for any $\xi>0$, there exists $C_\xi>0$ such that

$$|G_\varepsilon(x,t)|\leq\xi|t|^p+C_\xi|t|^r\quad\text{for any}\quad x\in\mathbb{R}^N,\ t\in\mathbb{R}. \tag{3.6}$$

Putting together $\langle\mathcal{J}_\varepsilon'(t_uu),t_uu\rangle=0$ and (3.6), there holds

$$
\begin{aligned}
t_u^p\|u\|_{p,\varepsilon}^p+t_u^q\|u\|_{q,\varepsilon,\mu_\varepsilon}^q&=\int\limits_{\mathbb{R}^N}g_\varepsilon(x,t_uu)t_uu\mathrm{d}x\\
&\leq\int\limits_{\mathbb{R}^N}(\xi|t_uu|^p+C_\xi|t_uu|^r)\mathrm{d}x\\
&\leq\frac{\xi}{V_0}t_u^p\|u\|_{p,\varepsilon}^p+C_1C_\xi t_u^r\|u\|_\varepsilon^r.
\end{aligned}\tag{3.7}
$$

Setting $\xi=\frac{V_0}{2}$, applying Lemma 2.3 and (3.7), there holds

$$C_0\min\{t_u^p\|u\|_\varepsilon^p,t_u^q\|u\|_\varepsilon^q\}\leq\frac{t_u^p}{2}\|u\|_{p,\varepsilon}^p+t_u^q\|u\|_{q,\varepsilon,\mu_\varepsilon}^q\leq C_2t_u^r\|u\|_\varepsilon^r.$$

Therefore there is $\tau>0$ independent of u, such that $t_u\geq\tau$ for every $u\in\mathbb{S}_\varepsilon^+$.

Let $\mathcal{K}\subset\mathbb{S}_\varepsilon^+$ be a compact set. By contradiction, suppose that there exists a sequence $\{u_n\}\subset\mathcal{K}$ such that $t_n:=t_{u_n}\to\infty$. By the compactness of $\mathcal{K}$, there exists $u\in\mathbb{S}_\varepsilon^+$ such that $u_n\to u$ in $\mathbb{X}_\varepsilon$. In view of the proof Lemma 3.1-(ii), one has that

$$\mathcal{J}_\varepsilon(t_nu_n)\to-\infty\quad\text{as}\quad n\to\infty. \tag{3.8}$$

Since $\langle\mathcal{J}_\varepsilon'(t_nu_n),t_nu_n\rangle=0$, $u_n\to u$ in $\mathbb{X}_\varepsilon$ and $t_n\to\infty$, then

$$\mathcal{J}_\varepsilon(t_nu_n)=\mathcal{J}_\varepsilon(t_nu_n)-\frac{1}{\theta}\langle\mathcal{J}_\varepsilon'(t_nu_n),t_nu_n\rangle\geq C_0\min\{t_n^p\|u_n\|_\varepsilon^p,t_n^q\|u_n\|_\varepsilon^q\}\to\infty. \tag{3.9}$$

Comparing (3.8) and (3.9), we obtain a contradiction.

(iii) From (i), we know that $\hat{m}_\varepsilon$ and m_ε are well defined. Now, we show that m_ε^{-1} is well defined. In fact, for any $u \in \mathcal{N}_\varepsilon$, we can deduce that $u \in \mathbb{X}_\varepsilon^+$. In contrary, if $u \notin \mathbb{X}_\varepsilon^+$, using $u \in \mathcal{N}_\varepsilon$ and (ii) of (g_3), one has

$$\|u\|_{p,\varepsilon}^p + \|u\|_{q,\varepsilon,\mu_\varepsilon}^q = \int_{\mathbb{R}^N} g_\varepsilon(x,u)u\mathrm{d}x \le \frac{1}{k}\|u\|_{p,\varepsilon}^p.$$

Then, $(1-\frac{1}{k})\|u\|_{p,\varepsilon}^p + \|u\|_{q,\varepsilon,\mu_\varepsilon}^q = 0$, which is a contradiction due to $k > 1$. Thereby, there holds $m_\varepsilon^{-1}(u) = \frac{u}{\|u\|_\varepsilon} \in \mathbb{S}_\varepsilon^+$. Then, m_ε^{-1} is well defined, continuous and a bijection.

For any $u \in \mathbb{S}_\varepsilon^+$, we deduce that

$$m_\varepsilon^{-1}(m_\varepsilon(u)) = m_\varepsilon^{-1}(t_u u) = \frac{t_u u}{\|t_u u\|_\varepsilon} = \frac{u}{\|u\|_\varepsilon} = u.$$

Then, m is a bijection. Next, we show that $\hat{m}_\varepsilon$ is a continuous function. Suppose that there exist $\{u_n\} \subset \mathbb{X}_\varepsilon^+$ and $u \in \mathbb{X}_\varepsilon^+$ fulfilling $u_n \to u$ in $\mathbb{X}_\varepsilon$. Note that for any $v \in \mathbb{X}_\varepsilon^+$, $\hat{m}_\varepsilon(v) = \hat{m}_\varepsilon(tv)$ for any $v \in \mathbb{X}_\varepsilon^+$ and any $t > 0$. Then, we may assume that $\|u_n\|_\varepsilon = \|u\|_\varepsilon = 1$ for any $n \in \mathbb{N}$. It follows from (ii) that there exists $t_0 > 0$ such that $t_n := t_{u_n} \to t_0$. Due to $\{t_n u_n\} \subset \mathcal{N}_\varepsilon$, then

$$t_n^p\|u_n\|_{p,\varepsilon}^p + t_n^q\|u_n\|_{q,\varepsilon,\mu_\varepsilon}^q = \int_{\mathbb{R}^N} g_\varepsilon(x,t_n u_n)t_n u_n\mathrm{d}x.$$

Letting $n \to \infty$ in this formula, since $u_n \to u$ in $\mathbb{X}_\varepsilon$ and $t_n \to t_0$, one has

$$t_0^p\|u\|_{p,\varepsilon}^p + t_0^q\|u\|_{q,\varepsilon,\mu_\varepsilon}^q = \int_{\mathbb{R}^N} g_\varepsilon(x,t_0 u)t_0 u\mathrm{d}x.$$

Hence, $t_0 u \in \mathcal{N}_\varepsilon$. According to (i), then $t_0 = t_u$, which implies that $\hat{m}_\varepsilon$ is continuous. Then, m_ε is continuous.

(iv) Let $\{u_n\} \subset \mathbb{S}_\varepsilon^+$ be such that $\mathrm{dist}(u_n, \partial\mathbb{S}_\varepsilon^+) \to 0$. Then, for any $s \in [p, p^*]$, one has

$$\|u_n\|_{L^s(\Lambda_\varepsilon)} \le \inf_{v\in\partial\mathbb{S}_\varepsilon^+}\|u_n - v\|_{L^s(\Lambda_\varepsilon)} \le C_s \inf_{v\in\partial\mathbb{S}_\varepsilon^+}\|u_n - v\|_\varepsilon = C_s\mathrm{dist}(u_n, \partial\mathbb{S}_\varepsilon^+) \to 0. \tag{3.10}$$

Applying (g_3) and (3.10), we deduce that for any $t > 0$,

$$\begin{aligned}\int_{\mathbb{R}^N} G_\varepsilon(x,tu_n)\mathrm{d}x &= \int_{\Lambda_\varepsilon^c} G_\varepsilon(x,tu_n)\mathrm{d}x + \int_{\Lambda_\varepsilon} G_\varepsilon(x,tu_n)\mathrm{d}x \\ &\le \frac{V_0}{kp}\int_{\Lambda_\varepsilon^c}|tu_n|^p\mathrm{d}x + \int_{\Lambda_\varepsilon} F(tu_n)\mathrm{d}x \\ &\le \frac{t^p}{kp}\|u\|_{p,\varepsilon}^p + C_1\int_{\Lambda_\varepsilon}|tu_n|^p\mathrm{d}x + C_2\int_{\Lambda_\varepsilon}|tu_n|^r\mathrm{d}x \\ &\le \frac{t^p}{kp}\|u\|_{p,\varepsilon}^p + C_3 t^p\mathrm{dist}(u_n,\partial\mathbb{S}_\varepsilon^+)^p + C_4 t^r\mathrm{dist}(u_n,\partial\mathbb{S}_\varepsilon^+)^r.\end{aligned}$$

Consequently,

$$\int_{\mathbb{R}^N} G_\varepsilon(x,tu_n)\mathrm{d}x \le \frac{t^p}{kp}\|u\|_{p,\varepsilon}^p + o_n(1). \tag{3.11}$$

It follows from (3.11) that for any $t > 1$,

$$\mathcal{J}_\varepsilon(tu_n) = \frac{t^p}{p}\|u_n\|_{p,\varepsilon}^p + \frac{t^q}{q}\|u_n\|_{q,\varepsilon,\mu_\varepsilon}^q - \int_{\mathbb{R}^N} G_\varepsilon(x,tu_n)\mathrm{d}x$$

$$\geq \frac{t^p}{p}\left(1-\frac{1}{k}\right)\|u_n\|_{p,\varepsilon}^p + \frac{t^q}{q}\|u_n\|_{q,\varepsilon,\mu_\varepsilon}^q + o_n(1)$$
$$\geq C_0 t^p + o_n(1),$$

where $C_0 = \min\{\frac{1}{p}(1-\frac{1}{k}), \frac{1}{q}\}$. Then for any $t > 1$,

$$\liminf_{n\to\infty} \mathcal{J}_\varepsilon(tu_n) \geq C_0 t^p. \tag{3.12}$$

From (3.12), one has that for any $t > 1$,

$$\liminf_{n\to\infty} \mathcal{J}_\varepsilon(m_\varepsilon(u_n)) \geq \liminf_{n\to\infty} \mathcal{J}_\varepsilon(tu_n) \geq C_0 t^p.$$

Since $t > 1$ is arbitrary, $\liminf_{n\to\infty} \mathcal{J}_\varepsilon(m_\varepsilon(u_n)) = \infty$. Observe that

$$\frac{1}{p}\|m_\varepsilon(u_n)\|_{p,\varepsilon}^p + \frac{1}{q}\|m_\varepsilon(u_n)\|_{q,\varepsilon,\mu_\varepsilon}^q \geq \mathcal{J}_\varepsilon(m_\varepsilon(u_n)) \to \infty \quad \text{as} \quad n \to \infty,$$

which combined with Lemma 2.4 suggests that $\|m_\varepsilon(u_n)\|_\varepsilon \to \infty$. □

Let us denote the maps

$$\hat{\psi}_\varepsilon : \mathbb{X}_\varepsilon^+ \to \mathbb{R} \quad \text{and} \quad \psi_\varepsilon : \mathbb{S}_\varepsilon^+ \to \mathbb{R}$$

as

$$\hat{\psi}_\varepsilon = \mathcal{J}_\varepsilon(\hat{m}_\varepsilon(u)) \quad \text{and} \quad \psi_\varepsilon = \hat{\psi}_\varepsilon|_{\mathbb{S}_\varepsilon^+}.$$

As in ([42],Corollary 10), by virtue of Lemma 3.3, we can directly obtain the next result.

Proposition 3.1. *Suppose that (A_1)–(A_4) and (f_1)–(f_4) hold. Then,*

(i) $\hat{\psi}_\varepsilon \in C^1(\mathbb{X}_\varepsilon^+, \mathbb{R})$ *and*

$$\langle \hat{\psi}_\varepsilon'(u), v\rangle = \frac{\|\hat{m}_\varepsilon(u)\|_\varepsilon}{\|u\|_\varepsilon}\langle \mathcal{J}_\varepsilon'(\hat{m}_\varepsilon(u)), v\rangle \quad \text{for every } u \in \mathbb{X}_\varepsilon^+ \text{ and } v \in \mathbb{X}_\varepsilon;$$

(ii) $\psi_\varepsilon \in C^1(\mathbb{S}_\varepsilon^+, \mathbb{R})$ *and* $\langle \psi_\varepsilon'(u), v\rangle = \|m_\varepsilon(u)\|_\varepsilon \langle \mathcal{J}_\varepsilon'(m_\varepsilon(u)), v\rangle$ *for every* $v \in T_u\mathbb{S}_\varepsilon^+$;

(iii) if $\{u_n\}$ *is a* $(PS)_d$ *sequence for* ψ_ε, *then* $\{m_\varepsilon(u_n)\}$ *is a* $(PS)_d$ *sequence for* $\mathcal{J}_\varepsilon$. *Moreover, if* $\{u_n\} \subset \mathcal{N}_\varepsilon$ *is a bounded* $(PS)_d$ *sequence for* $\mathcal{J}_\varepsilon$, *then* $\{m_\varepsilon^{-1}(u_n)\}$ *is a* $(PS)_d$ *sequence for* ψ_ε;

(iv) u *is a critical point of* ψ_ε *if and only if* $m_\varepsilon(u)$ *is a non-trivial critical point of* $\mathcal{J}_\varepsilon$. *Moreover, the corresponding critical value coincides and*

$$\inf_{u\in\mathbb{S}_\varepsilon^+} \psi_\varepsilon(u) = \inf_{u\in\mathcal{N}_\varepsilon} \mathcal{J}_\varepsilon(u).$$

Remark 3.1. Clearly, from Lemma 3.3 and Proposition 3.1, $\mathcal{J}_\varepsilon$ satisfies the following property

$$c_\varepsilon := \inf_{u\in\mathcal{N}_\varepsilon} \mathcal{J}_\varepsilon(u) = \inf_{u\in\mathbb{X}_\varepsilon^+} \max_{t\geq 0} \mathcal{J}_\varepsilon(tu) = \inf_{u\in\mathbb{S}_\varepsilon^+} \max_{t\geq 0} \mathcal{J}_\varepsilon(tu).$$

The following result is a consequence of ([23],Lemma 3) or the proof of ([32], (ii) of Proposition 3.1).

Lemma 3.4. *Let* $\{u_n\} \subset \mathbb{X}_\varepsilon$ *be a bounded* $(PS)_c$ *sequence. Then, up to a subsequence, there exists* $u \in \mathbb{X}_\varepsilon$ *such that* $\nabla u_n \to \nabla u$ *a.e. in* $\mathbb{R}^N$.

At the end of this section, we demonstrate the compactness of $\mathcal{J}_\varepsilon$ and ψ_ε. Let $\tilde{\varrho}_\varepsilon(u) = |\nabla u|^p + \mu_\varepsilon(x)|\nabla u|^q + V_\varepsilon(x)(|u|^p + \mu_\varepsilon(x)|u|^q)$.

Lemma 3.5. $\mathcal{J}_\varepsilon$ *satisfies* $(PS)_c$ *condition for any* $c \in \mathbb{R}$.

Proof. Suppose that $\{u_n\}$ is a $(PS)_c$ sequence. Then, by Lemma 3.2, we deduce that there exists $u \in \mathbb{X}_\varepsilon$ such that $u_n \rightharpoonup u$ in $\mathbb{X}_\varepsilon$. Using Lemma 3.4, we may assume that up to a subsequence,

$$u_n \to u \quad \text{in} \quad L^r_{\mathrm{loc}}(\mathbb{R}^N), \quad u_n \to u \quad \text{a.e. in} \quad \mathbb{R}^N \quad \text{and} \quad \nabla u_n \to \nabla u \quad \text{a.e. in} \quad \mathbb{R}^N. \tag{3.13}$$

To complete the proof, we only demand to show that

$$\varrho_\varepsilon(u_n) \to \varrho_\varepsilon(u) \quad \text{as} \quad n \to \infty. \tag{3.14}$$

We claim that there exists $R = R(\xi) > 0$ such that

$$\limsup_{n\to\infty} \int\limits_{B_R^c} \tilde{\varrho}_\varepsilon(u_n)\mathrm{d}x \le \xi. \tag{3.15}$$

For any $R > 0$, take $\varphi_R \in C^\infty(\mathbb{R}^N)$ such that $\varphi_R = 0$ in $B_{\frac{R}{2}}$, $\varphi_R = 1$ in B_R^c, $0 \le \varphi_R \le 1$ and $|\nabla\varphi_R| \le \frac{C}{R}$. Since $\{u_n\varphi_R\}$ is bounded in $\mathbb{X}_\varepsilon$, taking R is large such that $\Lambda_\varepsilon \subset B_{\frac{R}{2}}$, then from $\langle \mathcal{J}'_\varepsilon(u_n), u_n\varphi_R\rangle = o_n(1)$, Hölder inequality and (g_3), there holds that

$$\begin{aligned}
&\int\limits_{\mathbb{R}^N} \tilde{\varrho}_\varepsilon(u_n)\varphi_R\mathrm{d}x \\
&= -\int\limits_{\mathbb{R}^N} u_n|\nabla u_n|^{p-2}\nabla u_n\nabla\varphi_R\mathrm{d}x - \int\limits_{\mathbb{R}^N} \mu_\varepsilon(x)u_n|\nabla u_n|^{q-2}\nabla u_n\nabla\varphi_R\mathrm{d}x + \int\limits_{\mathbb{R}^N} g_\varepsilon(x,u_n)u_n\varphi_R\mathrm{d}x + o_n(1) \\
&\le \frac{C}{R}\int\limits_{\mathbb{R}^N} |u_n||\nabla u_n|^{p-1}\mathrm{d}x + \frac{C}{R}\int\limits_{\mathbb{R}^N} \mu_\varepsilon(x)|u_n||\nabla u_n|^{q-1}\mathrm{d}x + \frac{1}{k}\int\limits_{\mathbb{R}^N} V_\varepsilon(x)|u_n|^p\varphi_R\mathrm{d}x + o_n(1) \\
&\le \frac{C}{R}\Bigg(\int\limits_{\mathbb{R}^N} |u_n|^p\mathrm{d}x\Bigg)^{\frac{1}{p}}\Bigg(\int\limits_{\mathbb{R}^N} |\nabla u_n|^p\mathrm{d}x\Bigg)^{\frac{p-1}{p}} \\
&\quad + \frac{C}{R}\Bigg(\int\limits_{\mathbb{R}^N} \mu_\varepsilon(x)|u_n|^q\mathrm{d}x\Bigg)^{\frac{1}{q}}\Bigg(\int\limits_{\mathbb{R}^N} \mu_\varepsilon(x)|\nabla u_n|^q\mathrm{d}x\Bigg)^{\frac{q-1}{q}} + \frac{1}{k}\int\limits_{\mathbb{R}^N} V_\varepsilon(x)|u_n|^p\varphi_R\mathrm{d}x + o_n(1) \\
&\le \frac{C}{R} + \frac{1}{k}\int\limits_{\mathbb{R}^N} \varrho_\varepsilon(u_n)\varphi_R\mathrm{d}x + o_n(1).
\end{aligned}$$

Consequently,

$$\left(1 - \frac{1}{k}\right)\int\limits_{B_R^c} \tilde{\varrho}_\varepsilon(u_n)\mathrm{d}x \le \left(1 - \frac{1}{k}\right)\int\limits_{\mathbb{R}^N} \tilde{\varrho}_\varepsilon(u_n)\varphi_R\mathrm{d}x \le \frac{C}{R} + o_n(1).$$

Then, for any $\xi > 0$, we can take R large enough such that (3.15) holds.

Now, we prove that $u_n \to u$ in $L^p(\mathbb{R}^N)$. Since $u_n \to u$ in $L^p(B_R)$, using (3.14), we deduce that

$$\begin{aligned}
\|u_n - u\|^p_{L^p(\mathbb{R}^N)} &= \|u_n - u\|^p_{L^p(B_R)} + \|u_n - u\|^p_{L^p(B_R^c)} \\
&\le \xi + 2^{p-1}\|u_n\|^p_{L^p(B_R^c)} + 2^{p-1}\|u\|^p_{L^p(B_R^c)} \\
&\le 2\xi + \frac{2^{p-1}}{V_0}\int\limits_{B_R^c} \varrho_\varepsilon(u_n)\mathrm{d}x \\
&\le 3\xi.
\end{aligned}$$

By the arbitrariness of ξ, we have $u_n \to u$ in $L^p(\mathbb{R}^N)$. Then, $u_n \to u$ in $L^r(\mathbb{R}^N)$. From (g_2), we conclude that

$$\int_{\mathbb{R}^N} g_\varepsilon(x, u_n)u_n \mathrm{d}x \to \int_{\mathbb{R}^N} g_\varepsilon(x, u)u \mathrm{d}x. \tag{3.16}$$

We can easily obtain from (3.13) and $\mathcal{J}'_\varepsilon(u_n) \to 0$ that $\mathcal{J}'_\varepsilon(u) = 0$. Then,

$$\varrho_\varepsilon(u) = \int_{\mathbb{R}^N} g_\varepsilon(x, u)u \mathrm{d}x. \tag{3.17}$$

Since $\langle \mathcal{J}'_\varepsilon(u_n), u_n\rangle \to 0$, there holds that

$$\varrho_\varepsilon(u_n) = \int_{\mathbb{R}^N} g_\varepsilon(x, u_n)u_n \mathrm{d}x + o_n(1). \tag{3.18}$$

Putting together (3.16), (3.17) and (3.18), then (3.14) holds. □

Proposition 3.2. *ψ_ε satisfies $(PS)_c$ condition for any $c \in \mathbb{R}$.*

Proof. Suppose that $\{u_n\} \subset \mathbb{S}^+_\varepsilon$ is a $(PS)_c$ sequence for $\mathcal{J}_\varepsilon$, that is

$$\psi_\varepsilon(u_n) \to c \quad \text{and} \quad \psi'_\varepsilon(u_n) \to 0 \quad \text{in} \quad (T_{u_n}\mathbb{S}^+_\varepsilon)^*.$$

Recalling (iii) of Proposition 3.1, we have that $\{m_\varepsilon(u_n)\}$ is a $(PS)_c$ sequence for $\mathcal{J}_\varepsilon$. From Lemma 3.5 and (iii) of Lemma 3.3, we conclude that there exists $u \in \mathbb{S}^+_\varepsilon$ such that $m_\varepsilon(u_n) \to m_\varepsilon(u)$ in $\mathbb{X}_\varepsilon$. It follows from (iii) of Lemma 3.3 that $u_n \to u$ in $\mathbb{X}_\varepsilon$. □

4. The autonomous problem

In this section, we consider the autonomous problem

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u + \mu_0|\nabla u|^{q-2}\nabla u) + V_0(|u|^{p-2}u + \mu_0|u|^{q-2}u) = f(u) \quad \text{in} \quad \mathbb{R}^N. \tag{4.1}$$

Let $\mathbb{Y}_{\mu_0,V_0}$ denote the space $W^{1,p}(\mathbb{R}^N)$ if $\mu_0 = 0$ and $\mathbb{Y}_{\mu_0,V_0}$ denote the space $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ if $\mu_0 > 0$, which is equipped with the norm

$$\|u\|_{\mu_0,V_0} = \|u\|_{p,V_0} + \mu_0\|u\|_{q,V_0},$$

where

$$\|u\|^p_{p,V_0} = \int_{\mathbb{R}^N} \Big(|\nabla u|^p + V_0|u|^p\Big)\mathrm{d}x \quad \text{and} \quad \|u\|^q_{q,V_0} = \int_{\mathbb{R}^N} \Big(|\nabla u|^q + V_0|u|^q\Big)\mathrm{d}x.$$

Observe that the corresponding variational functional for equation (4.1) is expressed as

$$\mathcal{I}_{\mu_0,V_0}(u) = \frac{1}{p}\|u\|^p_{p,V_0} + \frac{\mu_0}{q}\|u\|^q_{q,V_0} - \int_{\mathbb{R}^N} F(u)\mathrm{d}x.$$

From (f_1) and (f_2), we can easily deduce that $\mathcal{I}_{\mu_0,V_0} \in C^1(\mathbb{Y}_{\mu_0,V_0}, \mathbb{R})$ and for any u, $v \in \mathbb{Y}_{\mu_0,V_0}$,

$$\begin{aligned}\langle \mathcal{I}'_{\mu_0,V_0}(u), v\rangle = &\int_{\mathbb{R}^N} (|\nabla u|^{p-2}\nabla u\nabla v + V_0|u|^{p-2}uv)\mathrm{d}x \\ &+ \mu_0\int_{\mathbb{R}^N} (|\nabla u|^{q-2}\nabla u\nabla v + V_0|u|^{q-2}uv)\mathrm{d}x - \int_{\mathbb{R}^N} f(u)v\mathrm{d}x.\end{aligned}$$

We define the following Nehari manifold

$$\mathcal{M}_{\mu_0,V_0} = \{u \in \mathbb{Y}_{\mu_0,V_0}\backslash\{0\} : \langle \mathcal{I}'_{\mu_0,V_0}(u), u\rangle = 0\}.$$

As the above section, we define that $c_\mu = \inf_{u\in\mathcal{M}_{\mu_0,V_0}} \mathcal{I}_{\mu_0,V_0}(u)$ and

$$\mathbb{Y}^+_{\mu_0,V_0} = \{u \in \mathbb{Y}_{\mu_0,V_0} : \mathrm{meas}(\{u^+\}) > 0\}.$$

Let $\mathbb{S}^+_{\mu_0,V_0} = \mathbb{S}_{\mu_0,V_0} \cap \mathbb{Y}^+_{\mu_0,V_0}$, where $\mathbb{S}_{\mu_0,V_0}$ represents the unit sphere in $\mathbb{Y}_{\mu_0,V_0}$. We know that $\mathbb{S}^+_{\mu_0,V_0}$ is an incomplete $C^{1,1}$ manifold of codimension one and $\mathbb{Y}_{\mu_0,V_0} = T_u\mathbb{S}^+_{\mu_0,V_0} \oplus \mathbb{R}u$, where

$$T_u\mathbb{S}^+_{\mu_0,V_0} = \left\{v \in \mathbb{Y}_{\mu_0,V_0} : \int_{\mathbb{R}^N} \left((|\nabla u|^{p-2}\nabla u + \mu_0|\nabla u|^{q-2}\nabla u)\nabla v + V_0(|u|^{p-2}u + \mu_0|u|^{q-2}u)v\right)\mathrm{d}x = 0\right\}.$$

It is easy to deduce that any $(PS)_c$ sequence for $\mathcal{I}_{\mu_0,V_0}$ is bounded due to (f_3).

Proceeding as in the previous section, we can set up the following conclusion.

Lemma 4.1. *Suppose that $\mu_0 \geq 0$, $V_0 > 0$ and (f_1)–(f_4) hold. Then,*

(i) for any $u \in \mathbb{Y}^+_{\mu_0,V_0}$, we define $h_u : [0,\infty) \to \mathbb{R}$ as $h_u(t) := \mathcal{I}_{\mu_0,V_0}(tu)$. Then, there is the unique $t_u > 0$ such that $h'_u(t) > 0$ in $(0,t_u)$ and $h'_u(t) < 0$ in $(t_u,+\infty)$;

(ii) there is $\tau > 0$ independent on u, such that $t_u \geq \tau$ for every $u \in \mathbb{S}^+_{\mu_0,V_0}$. Moreover, for each compact set $\mathcal{K} \subset \mathbb{S}^+_{\mu_0,V_0}$, there is $C_\mathcal{K} > 0$ such that $t_u \leq C_\mathcal{K}$ for every $u \in \mathcal{K}$;

(iii) define the map $\hat{m}_{\mu_0,V_0} : \mathbb{Y}^+_{\mu_0,V_0} \to \mathcal{M}_{\mu_0,V_0}$ as $\hat{m}_{\mu_0,V_0}(u) := t_u u$. Then, $\hat{m}_{\mu_0,V_0}$ is continuous and $m_{\mu_0,V_0} := \hat{m}_{\mu_0,V_0}|_{\mathbb{S}^+_{\mu_0,V_0}}$ is a homeomorphism between $\mathbb{S}^+_{\mu_0,V_0}$ and $\mathcal{M}_{\mu_0,V_0}$. Moreover, $m^{-1}_{\mu_0,V_0}(u) = \frac{u}{\|u\|_{\mu_0,V_0}}$;

(iv) let $\{u_n\} \subset \mathbb{S}^+_{\mu_0,V_0}$ be a sequence such that dist$(u_n, \partial\mathbb{S}^+_{\mu_0,V_0}) \to 0$. Then, $\|m_{\mu_0,V_0}(u_n)\|_{\mu_0,V_0} \to \infty$ and $\mathcal{I}_{\mu_0,V_0}(m_{\mu_0,V_0}(u_n)) \to \infty$.

Now, we define the functionals

$$\hat{\psi}_{\mu_0,V_0} : \mathbb{Y}^+_{\mu_0,V_0} \to \mathbb{R} \quad \text{and} \quad \psi_{\mu_0,V_0} : \mathbb{S}^+_{\mu_0,V_0} \to \mathbb{R}$$

as

$$\hat{\psi}_{\mu_0,V_0} = \mathcal{I}_{\mu_0,V_0}(\hat{m}_{\mu_0,V_0}(u)) \quad \text{and} \quad \psi_{\mu_0,V_0} = \hat{\psi}_{\mu_0,V_0}|_{\mathbb{S}^+_{\mu_0,V_0}}.$$

It follows from Lemma 4.1 that the following relationships hold.

Proposition 4.1. *Suppose that $\mu_0 \geq 0$, $V_0 > 0$ and (f_1)–(f_4) hold. Then,*

(i) $\hat{\psi}_{\mu_0,V_0} \in C^1(\mathbb{Y}^+_{\mu_0,V_0}, \mathbb{R})$ and

$$\langle \hat{\psi}'_{\mu_0,V_0}(u), v\rangle = \frac{\|\hat{m}_{\mu_0,V_0}(u)\|_{\mu_0,V_0}}{\|u\|_{\mu_0,V_0}} \langle \mathcal{I}'_{\mu_0,V_0}(\hat{m}_{\mu_0,V_0}(u)), v\rangle \quad \textit{for every } u \in \mathbb{Y}^+_{\mu_0,V_0} \textit{ and } v \in \mathbb{Y}_{\mu_0,V_0};$$

(ii) $\psi_{\mu_0,V_0} \in C^1(\mathbb{S}^+_{\mu_0,V_0}, \mathbb{R})$ and $\langle \psi'_{\mu_0,V_0}(u), v\rangle = \|m_{\mu_0,V_0}(u)\|_{\mu_0,V_0} \langle \mathcal{I}'_{\mu_0,V_0}(m_{\mu_0,V_0}(u)), v\rangle$ for every $v \in T_u\mathbb{S}^+_{\mu_0,V_0}$;

(iii) if $\{u_n\}$ is a $(PS)_d$ sequence for ψ_{μ_0,V_0}, then $\{m_{\mu_0,V_0}(u_n)\}$ is a $(PS)_d$ sequence for $\mathcal{I}_{\mu_0,V_0}$. Moreover, if $\{u_n\} \subset \mathcal{M}_{\mu_0,V_0}$ is a bounded $(PS)_d$ sequence for $\mathcal{M}_{\mu_0,V_0}$, then $\{m^{-1}_{\mu_0,V_0}(u_n)\}$ is a $(PS)_d$ sequence for ψ_{μ_0,V_0};

(iv) u is a critical point of ψ_{μ_0,V_0} if and only if $m_{\mu_0,V_0}(u)$ is a non-trivial critical point of $\mathcal{I}_{\mu_0,V_0}$. Moreover, the corresponding critical value coincides and

$$\inf_{u\in\mathbb{S}^+_{\mu_0,V_0}} \psi_{\mu_0,V_0}(u) = \inf_{u\in\mathcal{M}_{\mu_0,V_0}} \mathcal{I}_{\mu_0,V_0}(u).$$

Remark 4.1. As Remark 3.1, the following relationship holds:

$$c_{\mu_0,V_0} := \inf_{u\in\mathcal{M}_{\mu_0,V_0}} \mathcal{I}_{\mu_0,V_0}(u) = \inf_{u\in\mathbb{Y}^+_{\mu_0,V_0}} \max_{t\geq 0} \mathcal{I}_{\mu_0,V_0}(tu) = \inf_{u\in\mathbb{S}^+_{\mu_0,V_0}} \max_{t\geq 0} \mathcal{I}_{\mu_0,V_0}(tu).$$

The following alternative lemma is particular important for deriving the existence of ground state solution to equation (4.1).

Lemma 4.2. *Let $\{u_n\} \subset \mathbb{Y}_{\mu_0,V_0}$ be a $(PS)_c$ sequence for $\mathcal{I}_{\mu_0,V_0}$ at the level $c \in \mathbb{R}$. Then, one of the following alternatives holds:*

(i) $u_n \to 0$ in $\mathbb{Y}_{\mu_0,V_0}$;
(ii) there exist a sequence $\{y_n\} \subset \mathbb{R}^N$ and constants R, $\beta > 0$ such that

$$\liminf_{n\to\infty} \int\limits_{B_R(y_n)} |u_n|^p \mathrm{d}x \geq \beta.$$

Proof. By the assumptions of this lemma, one derives that $\{u_n\}$ is bounded in $\mathbb{Y}_{\mu_0,V_0}$,

$$\mathcal{I}_{\mu_0,V_0}(u_n) \to c_{\mu_0,V_0} \quad \text{and} \quad \langle \mathcal{I}'_{\mu_0,V_0}(u_n), u_n\rangle = o_n(1). \tag{4.2}$$

We suppose that (ii) cannot occur. Then, for any $R > 0$,

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int\limits_{B_R(y)} |u_n|^p \mathrm{d}x = 0, \tag{4.3}$$

Combined with Lemma 2.2 and (4.3), there holds that for any $s \in (p, p^*)$,

$$u_n \to 0 \quad \text{in} \quad L^s(\mathbb{R}^N). \tag{4.4}$$

Obviously, one can derive from (f_1) and (f_2) that for any $\in \mathbb{R}$,

$$|f(t)t| \leq C(|t|^q + |t|^r).$$

This together with $p < q < r < p^*$ and (4.4) imply that

$$\int\limits_{\mathbb{R}^N} |f(u_n)u_n| \mathrm{d}x \leq \|u_n\|^q_{L^q(\mathbb{R}^N)} + \|u_n\|^r_{L^r(\mathbb{R}^N)} = o_n(1).$$

Hence,

$$\int\limits_{\mathbb{R}^N} f(u_n)u_n \mathrm{d}x = o_n(1). \tag{4.5}$$

In the light of (4.2) and (4.5), we have

$$\|u_n\|^p_{p,V_0} + \mu_0\|u_n\|^q_{q,V_0} = \int\limits_{\mathbb{R}^N} f(u_n)u_n \mathrm{d}x = o_n(1).$$

So (i) holds. □

Lemma 4.3. *Problem (4.1) admits a positive ground state solution.*

Proof. Arguing directly as Lemma 3.1, $\mathcal{I}_{\mu_0,V_0}$ has a mountain pass geometry (see [44]). Then, there exists $\{u_n\} \subset \mathbb{X}_\varepsilon$ such that

$$\mathcal{I}_{\mu_0,V_0}(u_n) \to c_{\mu_0,V_0} \quad \text{and} \quad \mathcal{I}'_{\mu_0,V_0}(u_n) \to 0. \tag{4.6}$$

Observing that $c_{\mu_0,V_0}>0$, we conclude from Lemma 4.2 that there exists a sequence $\{y_n\}\subset\mathbb{R}^N$ and constants R, $\beta>0$ such that

$$\liminf_{n\to\infty}\int_{B_R(y_n)}|u_n|^p\mathrm{d}x\geq\beta. \tag{4.7}$$

Otherwise, we have from Lemma 4.2 that $\|u_n\|_{\mu_0,V_0}\to 0$. Then $\mathcal{I}_{\mu_0,V_0}(u_n)\to 0$. This contradicts to (4.6) thanks to $c_{\mu_0,V_0}>0$. Let $v_n=u_n(\cdot+y_n)$. Then, $\{v_n\}$ is bounded in $\mathbb{Y}_{\mu_0,V_0}$ and there exists $v\in\mathbb{Y}_{\mu_0,V_0}$ such that $v_n\rightharpoonup v$ in $\mathbb{Y}_{\mu_0,V_0}$. Moreover, we can obtain that

$$v_n\to v\quad\text{in}\quad L^p_{\mathrm{loc}}(\mathbb{R}^N)\quad\text{and}\quad v_n\to v\quad\text{a.e. in}\quad\mathbb{R}^N.$$

One can derive from (4.7) that $v\neq 0$. Applying (4.6), we deduce that

$$\mathcal{I}_{\mu_0,V_0}(v_n)\to c_{\mu_0,V_0}\quad\text{and}\quad\mathcal{I}'_{\mu_0,V_0}(v_n)\to 0. \tag{4.8}$$

It is easy to derive that $\mathcal{I}'_{\mu_0,V_0}(v)=0$ due to (4.8). Hence, $v\in\mathcal{M}_{\mu_0,V_0}$. By Fatou's lemma, $v\in\mathcal{M}_{\mu_0,V_0}$, (f_3) and (4.8), we conclude that

$$\begin{aligned}
c_{\mu_0,V_0}&=\liminf_{n\to\infty}\left(\mathcal{I}_{\mu_0,V_0}(v_n)-\frac{1}{\theta}\langle\mathcal{I}'_{\mu_0,V_0}(v_n),v_n\rangle\right)\\
&=\liminf_{n\to\infty}\left(\left(\frac{1}{p}-\frac{1}{\theta}\right)\|v_n\|^p_{p,V_0}+\mu_0\left(\frac{1}{q}-\frac{1}{\theta}\right)\|v_n\|^q_{q,V_0}+\frac{1}{\theta}\int_{\mathbb{R}^N}(f(v_n)\,v_n-\theta F(v_n))\mathrm{d}x\right)\\
&\geq\left(\frac{1}{p}-\frac{1}{\theta}\right)\|v\|^p_{p,V_0}+\mu_0\left(\frac{1}{q}-\frac{1}{\theta}\right)\|v\|^q_{q,V_0}+\frac{1}{\theta}\int_{\mathbb{R}^N}(f(v)v-\theta F(v))\mathrm{d}x\\
&=\mathcal{I}_{\mu_0,V_0}(v)-\frac{1}{\theta}\langle\mathcal{I}'_{\mu_0,V_0}(v),v\rangle\\
&\geq c_{\mu_0,V_0}.
\end{aligned}$$

This suggests that $v_n\to v$ in $\mathbb{Y}_{\mu_0,V_0}$. Then, from (4.8) one has that $\mathcal{I}_{\mu_0,V_0}(v)=c_{\mu_0,V_0}$ and $\mathcal{I}'_{\mu_0,V_0}(v)=0$. So v is a ground state solution of (4.1). Since $f(t)=0$ for $t\leq 0$ and $\langle\mathcal{I}'_{\mu_0,V_0}(v),v^-\rangle=0$, we conclude that $v\geq 0$ in $\mathbb{R}^N$. It can be deduced from the regularity results (see [28]) that $v\in L^\infty(\mathbb{R}^N)\cap C^1_{\mathrm{loc}}(\mathbb{R}^N)$. Further, we deduce from the Harnack inequality (see [43]) that $v>0$ in $\mathbb{R}^N$. □

Lemma 4.4. *Assume that* $\{u_n\}\subset\mathcal{M}_{\mu_0,V_0}$, $\mathcal{I}_{\mu_0,V_0}(u_n)\to c_{\mu_0,V_0}$ *and* $u_n\rightharpoonup u$ *in* $\mathbb{Y}_{\mu_0,V_0}$. *If* $u\neq 0$, *then* $u_n\to u$ *in* $\mathbb{Y}_{\mu_0,V_0}$.

Proof. Let $v_n=m^{-1}_{\mu_0,V_0}(u_n)=\frac{u_n}{\|u_n\|_{\mu_0,V_0}}\in\mathbb{S}^+_{\mu_0,V_0}$. By using the facts that $\{u_n\}\subset\mathcal{M}_{\mu_0,V_0}$, $\mathcal{I}_{\mu_0,V_0}(u_n)\to c_{\mu_0,V_0}$ and Remark 4.1, we have that

$$\psi_{\mu_0,V_0}(v_n)=\mathcal{I}_{\mu_0,V_0}(u_n)\to c_{\mu_0,V_0}=\inf_{v\in\mathbb{S}^+_{\mu_0,V_0}}\psi_{\mu_0,V_0}.$$

We define the functional $\Phi_{\mu_0,V_0}:\overline{\mathbb{S}}^+_{\mu_0,V_0}\to[-\infty,\infty]$ as

$$\Phi_{\mu_0,V_0}=\begin{cases}\psi_{\mu_0,V_0}(v) & \text{if}\quad v\in\mathbb{S}^+_{\mu_0,V_0}\\ +\infty & \text{if}\quad v\in\partial\mathbb{S}^+_{\mu_0,V_0}.\end{cases}$$

Observe that

- $(\overline{\mathbb{S}}^+_{\mu_0,V_0},d_{\mu_0,V_0})$ with $d_{\mu_0,V_0}(u,v)=\|u-v\|_{\mu_0,V_0}$ is a complete metric space;
- $\psi_{\mu_0,V_0}\in C(\overline{\mathbb{S}}^+_{\mu_0,V_0},[-\infty,\infty])$, see Lemma 4.1-$(iv)$;
- ψ_{μ_0,V_0} is bounded below, see Proposition 4.1-(iv).

By the Ekeland variational principle in [24], then there exists a sequence $\{\tilde{v}_n\} \subset \mathbb{S}^+_{\mu_0,V_0}$ such that

$$\|\tilde{v}_n - v_n\|_{\mu_0,V_0} \to 0, \quad \psi_{\mu_0,V_0}(\tilde{v}_n) \to m_{\mu_0,V_0} \quad \text{and} \quad \psi_{\mu_0,V_0} \to 0 \quad \text{in} \quad (T_{\tilde{v}_n}\mathbb{S}^+_{\mu_0,V_0})^*.$$

Proceeding as in the proof of Proposition 3.2, by Lemma 4.1 and Proposition 4.1, we can conclude that $\{u_n\}$ admits a convergent subsequence. □

5. The multiplicity of solutions for the modified problem

In this section, by using the Lusternik–Schnirelmann category theory, we establish the multiplicity of non-negative solutions for the modified problem (3.1).

Let $\delta > 0$ such that

$$M_\delta = \{x \in \mathbb{R}^N : \mathrm{dist}(x, M) \le \delta\} \subset \Lambda, \tag{5.1}$$

and take $\eta \in C^\infty([0,\infty),[0,1])$ being non-increasing and satisfying $\eta(t) = 1$ for $t \in [0, \frac{\delta}{2}]$, $\eta(t) = 0$ for $t \in [\delta,\infty)$ and $|\eta'(t)| \le \frac{4}{\delta}$. Suppose that w is a positive ground state solution of problem (4.1). For any $y \in M$, we denote

$$\Psi_{\varepsilon,y}(x) = \eta(|\,\varepsilon\, x - y|) w\left(\frac{\varepsilon\, x - y}{\varepsilon}\right).$$

Then, there exists the unique $t_\varepsilon > 0$ such that

$$\mathcal{J}_\varepsilon(t_\varepsilon \Psi_{\varepsilon,y}) = \max_{t\ge 0} \mathcal{J}_\varepsilon(t\Psi_{\varepsilon,y}).$$

Let us define the map $\Phi_\varepsilon : M \to \mathcal{N}_\varepsilon$ as $\Phi_\varepsilon = t_\varepsilon \Psi_{\varepsilon,y}$.

To obtain the multiplicity of solutions, we give some preliminary results.

Lemma 5.1. *There holds that*

$$\lim_{\varepsilon\to\infty} \mathcal{J}_\varepsilon(\Phi_\varepsilon(y)) = c_{\mu_0,V_0} \quad \textit{uniformly in} \quad y \in M.$$

Proof. Arguing by contradiction, there exist $\beta_0 > 0$, $\{y_n\} \subset M$ and $\varepsilon_n \to 0$ such that

$$|\mathcal{J}_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - c_{\mu_0,V_0}| \ge \beta_0 \quad \text{for any} \quad n \in \mathbb{N}. \tag{5.2}$$

By the change of variable $z = \frac{\varepsilon_n\, x - y_n}{\varepsilon_n}$ and the dominated convergence theorem, one has that

$$\|\Psi_{\varepsilon_n,y_n}\|^p_{p,\varepsilon_n} = \int_{\mathbb{R}^N} \Big(|\nabla(\eta(|\,\varepsilon_n\, z|) w(z))|^p + V(\varepsilon_n\, z + y_n)|\eta(|\,\varepsilon_n\, z|)w(z)|^p \Big) \mathrm{d}z \to \|w\|^p_{p,V_0}. \tag{5.3}$$

Similarly, we also conclude that

$$\begin{aligned} &\|\Psi_{\varepsilon_n,y_n}\|^q_{q,\varepsilon_n,\mu_{\varepsilon_n}} \\ &= \int_{\mathbb{R}^N} \mu(\varepsilon_n\, z + y_n) \Big(|\nabla(\eta(|\,\varepsilon_n\, z|) w(z))|^q + V(\varepsilon_n\, z + y_n)|\eta(|\,\varepsilon_n\, z|)w(z)|^q \Big) \mathrm{d}z \to \mu_0 \|w\|^q_{q,V_0}. \end{aligned} \tag{5.4}$$

Now, we prove the boundedness of $\{t_{\varepsilon_n}\}$. Otherwise, we may suppose that $t_{\varepsilon_n} \to \infty$. Since $\langle \mathcal{J}'_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)), \Phi_{\varepsilon_n}(y_n)\rangle = 0$, by the change of variable $z = \frac{\varepsilon_n\, x - y_n}{\varepsilon_n}$, we conclude that

$$\begin{aligned} t^p_{\varepsilon_n} \|\Psi_{\varepsilon_n,y_n}\|^p_{p,\varepsilon_n} + t^q_{\varepsilon_n} \|\Psi_{\varepsilon_n,y_n}\|^q_{q,\varepsilon_n,\mu_{\varepsilon_n}} &= \int_{\mathbb{R}^N} g(\varepsilon_n\, z + y_n, t_{\varepsilon_n}\eta(|\,\varepsilon_n\, z|)w(z)) t_{\varepsilon_n}\eta(|\,\varepsilon_n\, z|)w(z)\mathrm{d}z \\ &= \int_{\mathbb{R}^N} f(t_{\varepsilon_n}\eta(|\,\varepsilon_n\, z|)w(z)) t_{\varepsilon_n}\eta(|\,\varepsilon_n\, z|)w(z)\mathrm{d}z, \end{aligned} \tag{5.5}$$

here we used the fact that $\varepsilon_n z + y_n \in M_\delta \subset \Lambda$ if $|\varepsilon_n z| \leq \delta$. Recalling (f_2), we conclude that there exists $C > 0$ such that

$$f(t)t \geq t^\theta - C. \tag{5.6}$$

One can derive from (5.6) that

$$\begin{aligned}\int_{\mathbb{R}^N} f(t_{\varepsilon_n}\eta(|\varepsilon_n z|)w(z))t_{\varepsilon_n}\eta(|\varepsilon_n z|)w(z)\mathrm{d}z &\geq \int_{B_{\frac{\delta}{2}}} f(t_{\varepsilon_n}w(z))t_{\varepsilon_n}w(z)\mathrm{d}z \\ &\geq t_{\varepsilon_n}^\theta \int_{B_{\frac{\delta}{2}}} w(z)^\theta \mathrm{d}z - C\mathrm{meas}(B_{\frac{\delta}{2}}).\end{aligned} \tag{5.7}$$

Putting together (5.5) and (5.7), we have that

$$\frac{1}{t_{\varepsilon_n}^{q-p}}\|\Psi_{\varepsilon_n,y_n}\|_{p,\varepsilon_n}^p + \|\Psi_{\varepsilon_n,y_n}\|_{q,\varepsilon_n,\mu_{\varepsilon_n}}^q \geq t_{\varepsilon_n}^{\theta-q}\int_{B_{\frac{\delta}{2}}} w(z)^\theta \mathrm{d}z - \frac{1}{t_{\varepsilon_n}^q} C\mathrm{meas}(B_{\frac{\delta}{2}}). \tag{5.8}$$

Since $\theta > q$, using (5.3), (5.4) and $t_{\varepsilon_n} \to \infty$, we get a contradiction in (5.8). Thereby, $\{t_{\varepsilon_n}\}$ is bounded. Then, there exists $t_0 \geq 0$ such that $t_{\varepsilon_n} \to t_0$. In the light of $\langle \mathcal{J}'_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)), \Phi_{\varepsilon_n}(y_n)\rangle = 0$, (5.3), (5.4), (f_1) and (f_2), one can derive that $t_0 > 0$. Again using the dominated convergence theorem, there holds

$$\int_{\mathbb{R}^N} f(t_{\varepsilon_n}\eta(|\varepsilon_n z|)w(z))t_{\varepsilon_n}\eta(|\varepsilon_n z|)w(z)\mathrm{d}z \to \int_{\mathbb{R}^N} f(t_0 w)t_0 w \mathrm{d}z. \tag{5.9}$$

By virtue of (5.3), (5.4), (5.5), (5.9) and $t_{\varepsilon_n} \to t_0$, we have

$$\|t_0 w\|_{p,V_0}^p + \|t_0 w\|_{q,V_0}^q = \int_{\mathbb{R}^N} f(t_0 w)t_0 w \mathrm{d}z.$$

So $t_0 w \in \mathcal{M}_{\mu_0,V_0}$. Then, we conclude that $t_0 = 1$ due to $w \in \mathcal{M}_{\mu_0,V_0}$. by the change of variable $z = \frac{\varepsilon_n x - y_n}{\varepsilon_n}$, it can be deduced from the dominated convergence theorem and $t_{\varepsilon_n} \to 1$ that

$$\int_{\mathbb{R}^N} G(\varepsilon_n x, \Phi_{\varepsilon_n}(y_n))\mathrm{d}x \to \int_{\mathbb{R}^N} F(w)\mathrm{d}w.$$

This combined with (5.3), (5.4) and $t_{\varepsilon_n} \to 1$ implies that

$$\mathcal{J}_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) \to \mathcal{I}_{\mu_0,V_0} = c_{\mu_0,V_0},$$

which contradicts to (5.2). □

For $\delta > 0$ fulfilling (5.1), we take $\rho > 0$ such that $M_\delta \subset B_\rho$, and we introduce the map $\Upsilon : \mathbb{R}^N \to \mathbb{R}^N$ as

$$\Upsilon(x) = \begin{cases} x & \text{if } |x| < \rho \\ \frac{\rho x}{|x|} & \text{if } |x| \geq \rho. \end{cases}$$

The barycenter map $\beta_\varepsilon : \mathcal{N}_\varepsilon \to \mathbb{R}^N$ is defined as

$$\beta_\varepsilon(u) = \frac{\displaystyle\int_{\mathbb{R}^N} \Upsilon(\varepsilon x)|u(x)|^2 \,\mathrm{d}x}{\displaystyle\int_{\mathbb{R}^N} |u(x)|^2 \,\mathrm{d}x}.$$

Lemma 5.2. *We have the following limits:*

$$\lim_{\varepsilon\to 0}\beta_\varepsilon(\Phi_\varepsilon(y)) = y \quad \textit{uniformly in} \quad y\in M.$$

Proof. By contradiction, we suppose that there exist $\varepsilon_n \to 0$, $\{y_n\}\subset M$ and $C_0>0$ such that

$$|\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - y_n| \geq C_0. \tag{5.10}$$

Noting that $\varepsilon_n z + y_n \in M_\delta$ for $|\varepsilon_n z| \leq \frac{\delta}{2}$, then applying the definition of $\Phi_{\varepsilon_n}(y_n)$ and the change of variable $z = \frac{\varepsilon_n x - y_n}{\varepsilon_n}$, one can derive that

$$\begin{aligned}\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) &= y_n + \frac{\int_{\mathbb{R}^N}[\Upsilon(\varepsilon_n z + y_n) - y_n]|\eta(|\varepsilon_n z|)|^2|w(z)|^2\mathrm{d}z}{\int_{\mathbb{R}^N}|\eta(|\varepsilon_n z|)|^2|w(z)|^2\,\mathrm{d}z}\\ &= y_n + \frac{\int_{\mathbb{R}^N}\varepsilon_n z|\eta(|\varepsilon_n z|)|^2|w(z)|^2\mathrm{d}z}{\int_{\mathbb{R}^N}|\eta(|\varepsilon_n z|)|^2|w(z)|^2\,\mathrm{d}z}.\end{aligned}$$

By using the dominated convergence theorem in the above formula, we conclude that

$$|\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - y_n| = o_n(1).$$

This contradicts to (5.10). □

Now, we establish the following compactness lemma, which is very important to verify that the solutions of equation (3.1) are the solutions of equation (1.1).

Proposition 5.1. *Let $\varepsilon_n \to 0$. Suppose that $\{u_n\} := \{u_{\varepsilon_n}\}\subset \mathcal{N}_{\varepsilon_n}$ such that $\mathcal{J}_{\varepsilon_n}(u_n)\to c_{\mu_0,V_0}$. Then there exists $\{\tilde{y}_n\}\subset\mathbb{R}^N$ such that by defining $v_n(x) = u_n(x+\tilde{y}_n)$, then $\{v_n\}$ has a convergent subsequence in $\mathbb{Y}_{\mu_0,V_0}$. Furthermore, there exists $y_0\in M$ such that $y_n := \varepsilon_n\tilde{y}_n \to y_0$ in the sense of a subsequence.*

Proof. Since $\langle \mathcal{J}'_{\varepsilon_n}(u_n), u_n\rangle = 0$ and $\mathcal{J}_{\varepsilon_n}(u_n)\to c_{\mu_0,V_0}$, similar to Lemma 3.2, we can deduce that $\{u_n\}$ is uniformly bounded in $\mathbb{X}_{\varepsilon_n}$. Further, one can derive from $m_{\mu_0,V_0}>0$ that $\|u_n\|_{\varepsilon_n}\nrightarrow 0$. Then, by a direct argument (see Lemma 4.2), we conclude that there exist $\tilde{y}_n\subset\mathbb{R}^N$ and R, $\beta>0$ such that

$$\liminf_{n\to\infty}\int_{B_R(\tilde{y}_n)}|u_n|^p\mathrm{d}x \geq \beta. \tag{5.11}$$

Taking $\tilde{u}_n(x) = u_n(x+\tilde{y}_n)$, then $\{\tilde{u}_n\}$ is bounded in $\mathbb{Y}_{\mu_0,V_0}$ due to the fact that $\{u_n\}$ is uniformly bounded in $\mathbb{X}_{\varepsilon_n}$. So we may assume that $\tilde{u}_n \rightharpoonup \tilde{u}$ in $\mathbb{Y}_{\mu_0,V_0}$. One can derive from (5.11) that $u\not\equiv 0$. Observe that there exists a sequence $\{t_n\}\subset(0,\infty)$ such that $\{t_n\tilde{u}_n\}\subset\mathcal{M}_{\mu_0,V_0}$. Fix $\tilde{v}_n = t_n\tilde{u}_n$ and $y_n = \varepsilon_n\tilde{y}_n$. Since $G_{\varepsilon_n}(x,t)\leq F(t)$ for any $x\in\mathbb{R}^N$ and $t\in\mathbb{R}$, by (A_2), (A_4), $\tilde{v}_n\subset\mathcal{M}_{\mu_0,V_0}$ and $\{u_n\}\subset\mathcal{N}_{\varepsilon_n}$, there holds

$$\begin{aligned} c_{\mu_0,V_0} &\leq \mathcal{I}_{\mu_0,V_0}(\tilde{v}_n)\\ &= \frac{t_n^p}{p}\|\tilde{u}_n\|^p_{p,V_0} + \mu_0\frac{t_n^q}{q}\|\tilde{u}_n\|^q_{q,V_0} - \int_{\mathbb{R}^N}F(t_n\tilde{u}_n)\mathrm{d}x\\ &\leq \frac{t_n^p}{p}\|u_n\|^p_{p,\varepsilon_n} + \frac{t_n^q}{q}\|u_n\|^q_{q,\varepsilon_n,\mu_{\varepsilon_n}} - \int_{\mathbb{R}^N}G_\varepsilon(x,t_nu_n)\mathrm{d}x\\ &= \mathcal{J}_{\varepsilon_n}(t_nu_n)\leq\mathcal{J}_{\varepsilon_n}(u_n)\leq c_{\mu_0,V_0}+o_n(1).\end{aligned}$$

Thereby, one has that

$$\mathcal{I}_{\mu_0,V_0}(\tilde{v}_n) \to c_{\mu_0,V_0} \quad \text{and} \quad \{\tilde{v}_n\} \subset \mathcal{M}_{\mu_0,V_0}. \tag{5.12}$$

Obviously, by (5.12), we deduce that $\{\tilde{v}_n\}$ is bounded in $\mathbb{Y}_{\mu_0,V_0}$. then there exists $t_0 \geq 0$ such that $t_n \to t_0$. If $t_0 = 0$, then we can conclude that $\mathcal{I}_{\mu_0,V_0}(\tilde{v}_n) \to 0$, which contradicts to (5.12) due to $c_{\mu_0,V_0} > 0$. Hence, $t_0 > 0$ and $\tilde{v}_n \rightharpoonup \tilde{v} := t_0\tilde{u}$. Applying Lemma 4.4 and (5.12), we have that $\tilde{v}_n \to \tilde{v}$. By this fact, one has that $\tilde{u}_n \to \tilde{u}$ in $\mathbb{Y}_{\mu_0,V_0}$.

Now, we demonstrate that there exists $y_0 \in M$ such that $y_n \to y_0$ up to a subsequence. First we claim that $\{y_n\}$ is bounded. Otherwise, we have that $|y_n| \to \infty$ in the sense of a subsequence. Take $R > 0$ such that $\Lambda \subset B_R$. Then we may assume that $|y_n| \geq 2R$. So, for any $x \in B_{\frac{R}{\varepsilon_n}}$, one has

$$|\,\varepsilon_n x + y_n| \geq |y_n| - |\,\varepsilon_n x| > R. \tag{5.13}$$

Since $\tilde{u}_n \to \tilde{u}$ in $\mathbb{Y}_{\mu_0,V_0}$, it follows from this fact, (5.13) and $\langle \mathcal{J}'_{\varepsilon_n}(u_n), u_n\rangle = 0$ that

$$\begin{aligned}
\|\tilde{u}_n\|^p_{p,V_0} + \mu_0\|\tilde{u}_n\|^q_{q,V_0} &\leq \int_{\mathbb{R}^N} g(\varepsilon_n x + y_n, \tilde{u}_n)\tilde{u}_n \mathrm{d}x \\
&= \int_{B_{\frac{R}{\varepsilon_n}}} g(\varepsilon_n x + y_n, \tilde{u}_n)\tilde{u}_n \mathrm{d}x + \int_{B^c_{\frac{R}{\varepsilon_n}}} g(\varepsilon_n x + y_n, \tilde{u}_n)\tilde{u}_n \mathrm{d}x \\
&\leq \frac{V_0}{k}\|\tilde{u}_n\|^p_{L^p(\mathbb{R}^N)} + o_n(1).
\end{aligned} \tag{5.14}$$

Thereby,

$$\left(1 - \frac{1}{k}\right)\|\tilde{u}_n\|^p_{p,V_0} + \|\tilde{u}_n\|^q_{q,V_0} \to 0.$$

This is a contradiction due to $\tilde{u}_n \to \tilde{u}$ in $\mathbb{Y}_{\mu_0,V_0}$ with $\tilde{u} \not\equiv 0$. Then $\{y_n\}$ is bounded. We may assume that there exists $y_0 \in \overline{\Lambda}$ such that $y_n \to y_0$. Suppose by contradiction that $y_0 \notin \overline{\Lambda}$. Then there exists $R_0 > 0$ such $y_n \in B_{\frac{R_0}{2}}(y_0) \subset \Lambda^c$. Hence for any $x \in B_{\frac{R_0}{\varepsilon_n}}$, we have $\varepsilon_n x + y_n \in \Lambda^c$. Proceeding as (5.14), we can obtain a contradiction. Then, $y_0 \in \overline{\Lambda}$. If $V(y_0) \neq V_0$, then $V_0 < V(y_0)$. By (g_2), (5.12) and Fatou's lemma, we conclude that

$$\begin{aligned}
c_{\mu_0,V_0} &\leq \mathcal{I}_{\mu_0,V_0}(\tilde{v}) \\
&< \liminf_{n\to\infty}\left(\frac{1}{p}\int_{\mathbb{R}^N} |\nabla\tilde{v}_n|^p \mathrm{d}x + \frac{1}{p}\int_{\mathbb{R}^N} V(\varepsilon_n x + y_n)|\tilde{v}_n|^p \mathrm{d}x\right. \\
&\quad \left. + \frac{1}{q}\int_{\mathbb{R}^N} \mu(\varepsilon_n x + y_n)|\nabla\tilde{v}_n|^q \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} \mu(\varepsilon_n x + y_n)V(\varepsilon_n x + y_n)|\tilde{v}_n|^q \mathrm{d}x - \int_{\mathbb{R}^N} F(\tilde{v}_n)\mathrm{d}x\right) \\
&\leq \liminf_{n\to\infty} \mathcal{J}_{\varepsilon_n}(t_n u_n) \leq \liminf_{n\to\infty} \mathcal{J}_{\varepsilon_n}(u_n) = c_{\mu_0,V_0}.
\end{aligned}$$

Obviously, this is a contradiction. Then, we have concluded that $y_0 \in M$ and $y_n \to y_0$. □

We define the function $h_\varepsilon : \mathbb{R}^+ \to \mathbb{R}^+$ by $h_\varepsilon = \sup_{y\in M} |\mathcal{J}_\varepsilon(\Phi_\varepsilon(y)) - c_{\mu_0,V_0}|$. Recalling Lemma 5.1, we have that $h_\varepsilon \to 0$ as $\varepsilon \to 0$. Now, let us introduce the subset of $\mathcal{N}_\varepsilon$ as

$$\tilde{\mathcal{N}_\varepsilon} = \left\{u \in \mathcal{N}_\varepsilon : \mathcal{J}_\varepsilon(u) \leq c_{\mu_0,V_0} + h_\varepsilon\right\}.$$

It is clear that $\Phi_\varepsilon(y) \in \tilde{\mathcal{N}_\varepsilon}$ for any $y \in M$. Then, $\tilde{\mathcal{N}_\varepsilon}$ is not empty.

Lemma 5.3. *Let $\delta > 0$ such that (5.1) holds. Then,*

$$\lim_{\varepsilon\to 0}\sup_{u\in\tilde{\mathcal{N}}_\varepsilon} dist(\beta_\varepsilon(u), M_\delta) = 0.$$

Proof. Let $\varepsilon_n \to 0$. Then, there exists a sequence $\{u_n\} \subset \tilde{\mathcal{N}}_\varepsilon$ such that

$$\sup_{u\in\tilde{\mathcal{N}}_{\varepsilon_n}}\inf_{y\in M_\delta}|\beta_{\varepsilon_n}(u)-y| = \inf_{y\in M_\delta}|\beta_{\varepsilon_n}(u_n)-y| + o_n(1).$$

To complete the proof, we only demand to show that there exists a sequence $\{y_n\} \subset M_\delta$ such that

$$|\beta_{\varepsilon_n}(u_n) - y_n| \to 0 \quad \text{as} \quad n\to\infty. \tag{5.15}$$

We can derive from $\{u_n\} \subset \tilde{\mathcal{N}}_{\varepsilon_n} \subset \mathcal{N}_{\varepsilon_n}$ that

$$c_{\mu_0,V_0} \le \max_{t\ge 0}\mathcal{I}_{\mu_0,V_0}(tu_n) \le \max_{t\ge 0}\mathcal{J}_{\varepsilon_n}(tu_n) \le \mathcal{J}_{\varepsilon_n}(u_n) \le c_{\mu_0,V_0} + h_{\varepsilon_n},$$

which suggests that $\mathcal{J}_{\varepsilon_n}(u_n) \to c_{\mu_0,V_0}$. It follows from Proposition 5.1 that there exists a sequence $\{\tilde{y}_n\} \subset \mathbb{R}^N$ such that $y_n = \varepsilon_n\,\tilde{y}_n \subset M_\delta$. Note that

$$\beta_{\varepsilon_n}(u_n) = y_n + \frac{\int_{\mathbb{R}^N}[\Upsilon(\varepsilon_n x + y_n) - y_n]|u_n(x+\tilde{y}_n)|^2\,\mathrm{d}x}{\int_{\mathbb{R}^N}|u_n(x+\tilde{y}_n)|^2\mathrm{d}x}.$$

Since $u_n(\cdot+\tilde{y}_n)$ is strongly convergent in $\mathbb{Y}_{\mu_0,V_0}$ and $\varepsilon_n\,z + y_n \to y_0 \in M_\delta$, we conclude that $\beta_{\varepsilon_n}(u_n) = y_n + o_n(1)$. This implies that (5.15) holds. □

Combining the above lemmas, we can derive the multiplicity of solutions for the modified equation (3.1).

Theorem 5.1. *Suppose that (A_1)–(A_4) and (f_1)–(f_4) hold. Then for any $\delta > 0$ such that $M_\delta \subset \Lambda$, there exists $\varepsilon_\delta > 0$ such that for any $\varepsilon \in (0,\varepsilon_\delta)$, equation (3.1) admits at least $cat_{M_\delta}(M)$ non-negative solutions.*

Proof. For any $\varepsilon > 0$, let $\alpha_\varepsilon : M \to \mathbb{S}^+_\varepsilon$ be defined as $\alpha_\varepsilon := m_\varepsilon^{-1}(\Phi_\varepsilon(y))$. One can derive from Lemma 5.1 that

$$\lim_{\varepsilon\to 0}\psi_\varepsilon(\alpha_\varepsilon(y)) = \lim_{\varepsilon\to 0}\mathcal{J}_\varepsilon(\Phi_\varepsilon(y)) = c_{\mu_0,V_0} \quad \text{uniformly in} \quad y\in M. \tag{5.16}$$

Further, let us define the subset $\tilde{\mathbb{S}}^+_\varepsilon$ of $\mathbb{S}^+_\varepsilon$ as

$$\tilde{\mathbb{S}}^+_\varepsilon = \{u\in\mathbb{S}^+_\varepsilon : \psi_\varepsilon(u) \le c_{\mu_0,V_0} + h_\varepsilon\},$$

where $h_\varepsilon = \sup_{y\in M}|\psi_\varepsilon(\alpha_\varepsilon(y)) - c_{\mu_0,V_0}| = \sup_{y\in M}|\mathcal{J}_\varepsilon(\Phi_\varepsilon(y)) - c_{\mu_0,V_0}| \to 0$ (see (5.16)). Then, $\alpha_\varepsilon(y) \in \tilde{\mathbb{S}}^+_\varepsilon$ for any $y\in M$. This implies that $\tilde{\mathbb{S}}^+_\varepsilon \ne \emptyset$.

Putting together Lemma 3.3-(*iii*), Lemma 5.1, Lemma 5.2 and Lemma 5.3, we conclude that there exists $\varepsilon_\delta > 0$ such that for any $\varepsilon \in (0,\varepsilon_\delta)$, the following diagram

$$M \xrightarrow{\Phi_\varepsilon} \Phi_\varepsilon(M) \xrightarrow{m_\varepsilon^{-1}} \alpha_\varepsilon(M) \xrightarrow{m_\varepsilon} \Phi_\varepsilon(M) \xrightarrow{\beta_\varepsilon} M_\delta$$

is well defined. From Lemma 5.2, we conclude that the map $\beta_\varepsilon(\Phi_\varepsilon(y)) = y + \theta_\varepsilon(y)$ with $|\theta_\varepsilon(y)| \le \frac{\delta}{2}$ for any $\varepsilon\in(0,\varepsilon_\delta)$ and $y\in M$, here we take ε_δ small enough if necessary. Let $H_\varepsilon(t,y) = y + t\theta_\varepsilon(y)$ for any $t\in[0,1]$ and $y\in M$. Then, H_ε is a homology between the including map $\mathrm{id}: M\to M_\delta$ and $\beta_\varepsilon\circ\Phi_\varepsilon$. By virtue of ([14], Lemma 5.2), one has that for any $\varepsilon\in(0,\varepsilon_\delta)$,

$$\mathrm{cat}_{\alpha_\varepsilon(M)}(\alpha_\varepsilon(M)) \ge \mathrm{cat}_{M_\delta}(M). \tag{5.17}$$

It can be deduced from Proposition 3.2 and ([42], Theorem 2.7) that ψ_ε admits at least $\mathrm{cat}_{\alpha_\varepsilon(M)}(\alpha_\varepsilon(M))$ critical points. Using (5.17) and Proposition 3.1-(iv), we deduce that $\mathcal{J}_\varepsilon$ has at least $\mathrm{cat}_{M_\delta}(M)$ critical points. Since $f(t) \le 0$ for $t \le 0$, then every critical point of $\mathcal{J}_\varepsilon$ is non-negative. Thereby we complete the proof. □

6. Proof of Theorem 1.1

In Sect. 5, we show the multiplicity of solutions of the modified problem (3.1). In the last section, we shall demonstrate the solutions obtained for modified problem are actually solutions of problem (1.1) when ε is small enough.

Inspired by [26,28,29], we establish the following estimates. Since the double-phase operator is non-homogeneous and may be degenerate, we construct a new testing function to obtain the decaying estimates of solutions.

Lemma 6.1. *Let $\varepsilon_n \to 0$. Suppose that $u_n \in \tilde{\mathcal{N}}_{\varepsilon_n}$ is a solution of equation (3.1). Taking $v_n = u_n(\cdot + \tilde{y}_n)$, then there exists a constant $C > 0$ independent of n such that*

$$\|v_n\|_{L^\infty(\mathbb{R}^N)} \le C \quad \textit{for any} \quad n \in \mathbb{N},$$

where $\{\tilde{y}_n\}$ is given by Proposition 5.1. Furthermore, one has

$$\lim_{|x|\to\infty} v_n(x) = 0 \quad \textit{uniformly for} \quad n \in \mathbb{N}.$$

Proof. By the standard Moser iteration method (see ([26],Theorem 3.1)), we have that there exists $C > 0$ such that

$$\|v_n\|_{L^\infty(\mathbb{R}^N)} \le C \quad \text{uniformly for} \quad n \in \mathbb{N}. \tag{6.1}$$

Now we show the decay estimate. For any $R > 0$, we take $0 < r \le \frac{R}{2}$. Further, we introduce the function $\eta \in C^\infty(\mathbb{R}^N)$ such that $\eta = 1$ in B_R^c, $\eta = 0$ in B_{R-r} and $|\nabla\eta| \le \frac{2}{r}$. For any $n \in \mathbb{N}$, let $L > 0$ and $\beta > 1$ to be determined later. Take $v_{L,n}(x) = \min\{v_n(x), L\}$. we denote the functions

$$z_{L,n} = \eta^q v_n v_{L,n}^{p(\beta-1)} \quad \text{and} \quad w_{L,n} = \eta v_n v_{L,n}^{\beta-1},$$

Testing (3.1) with $z_{L,n}$, we conclude that

$$\begin{aligned} &\int_{\mathbb{R}^N} (|\nabla v_n|^{p-2}\nabla v_n \nabla z_{L,n} + \mu(\varepsilon_n x + y_n)|\nabla v_n|^{q-2}\nabla v_n \nabla z_{L,n})\mathrm{d}x \\ &+ \int_{\mathbb{R}^N} V(\varepsilon_n x + y_n)(|v_n|^{p-2}v_n z_{L,n} + \mu(\varepsilon_n x + y_n)|v_n|^{q-2}v_n z_{L,n})\mathrm{d}x \\ &= \int_{\mathbb{R}^N} g(\varepsilon_n x + y_n, v_n) z_{L,n}\mathrm{d}x. \end{aligned} \tag{6.2}$$

By a direct computation, we have

$$\nabla z_{L,n} = q\eta^{q-1} v_n v_{L,n}^{p(\beta-1)}\nabla\eta + \eta^q v_{L,n}^{p(\beta-1)}\nabla v_n + p(\beta-1)\eta^q v_n v_{L,n}^{p(\beta-1)-1}\nabla v_{L,n}. \tag{6.3}$$

One can derive from (6.2) and (6.3) that

$$\begin{aligned}&\int\limits_{\mathbb{R}^N} \eta^q|\nabla v_n|^p v_{L,n}^{p(\beta-1)}\mathrm{d}x + q\int\limits_{\mathbb{R}^N} \eta^{q-1}v_n v_{L,n}^{p(\beta-1)}\nabla\eta|\nabla v_n|^{p-2}\nabla v_n\mathrm{d}x + V_0\int\limits_{\mathbb{R}^N}\eta^q v_n^p v_{L,n}^{p(\beta-1)}\mathrm{d}x\\&+\int\limits_{\mathbb{R}^N}\mu(\varepsilon_n x+y_n)\eta^q|\nabla v_n|^q v_{L,n}^{p(\beta-1)}\mathrm{d}x + q\int\limits_{\mathbb{R}^N}\mu(\varepsilon_n x+y_n)\eta^{q-1}v_n v_{L,n}^{p(\beta-1)}\nabla\eta|\nabla v_n|^{q-2}\nabla v_n\mathrm{d}x\\&+V_0\int\limits_{\mathbb{R}^N}\mu(\varepsilon_n x+y_n)\eta^q v_n^q v_{L,n}^{p(\beta-1)}\mathrm{d}x\\&\le\int\limits_{\mathbb{R}^N} f(v_n)\eta^q v_n v_{L,n}^{p(\beta-1)}\mathrm{d}x.\end{aligned}\tag{6.4}$$

By the Young inequality, there hold that

$$\begin{aligned}&\Bigg|\int\limits_{\mathbb{R}^N}\eta^{q-1}v_n v_{L,n}^{p(\beta-1)}|\nabla v_n|^{p-2}\nabla v_n\nabla\eta\mathrm{d}x\Bigg|\\&\le\frac{1}{2q}\int\limits_{\mathbb{R}^N}\eta^{\frac{q-1}{p-1}p}v_{L,n}^{p(\beta-1)}|\nabla v_n|^p\mathrm{d}x + C\int\limits_{\mathbb{R}^N}v_n^p v_{L,n}^{p(\beta-1)}|\nabla\eta|^p\mathrm{d}x\\&\le\frac{1}{2q}\int\limits_{\mathbb{R}^N}\eta^q v_{L,n}^{p(\beta-1)}|\nabla v_n|^p\mathrm{d}x + C\int\limits_{\mathbb{R}^N}v_n^p v_{L,n}^{p(\beta-1)}|\nabla\eta|^p\mathrm{d}x,\end{aligned}\tag{6.5}$$

here we used the fact $\frac{q-1}{p-1}p \ge q$ due to $q \ge p$. Again by the Hölder inequality, we have

$$\begin{aligned}&\Bigg|\int\limits_{\mathbb{R}^N}\mu(\varepsilon_n x+y_n)\eta^{q-1}v_n v_{L,n}^{p(\beta-1)}\nabla\eta|\nabla v_n|^{q-2}\nabla v_n\mathrm{d}x\Bigg|\\&\le\frac{1}{q}\int\limits_{\mathbb{R}^N}\mu(\varepsilon_n x+y_n)\eta^q v_{L,n}^{p(\beta-1)}|\nabla v_n|^q\mathrm{d}x + C\int\limits_{\mathbb{R}^N}\mu(\varepsilon_n x+y_n)v_n^q v_{L,n}^{p(\beta-1)}|\nabla\eta|^q\mathrm{d}x\\&\le\frac{1}{q}\int\limits_{\mathbb{R}^N}\mu(\varepsilon_n x+y_n)\eta^q v_{L,n}^{p(\beta-1)}|\nabla v_n|^q\mathrm{d}x + C\int\limits_{\mathbb{R}^N}v_n^q v_{L,n}^{p(\beta-1)}|\nabla\eta|^q\mathrm{d}x\end{aligned}\tag{6.6}$$

Putting together (6.4), (6.5) and (6.6), we deduce that

$$\begin{aligned}&\int\limits_{\mathbb{R}^N}\eta^q|\nabla v_n|^p v_{L,n}^{p(\beta-1)}\mathrm{d}x + V_0\int\limits_{\mathbb{R}^N}\eta^q v_n^p v_{L,n}^{p(\beta-1)}\mathrm{d}x\\&\le C\int\limits_{\mathbb{R}^N}v_n^p v_{L,n}^{p(\beta-1)}|\nabla\eta|^p\mathrm{d}x + C\int\limits_{\mathbb{R}^N}v_n^q v_{L,n}^{p(\beta-1)}|\nabla\eta|^q\mathrm{d}x + \int\limits_{\mathbb{R}^N}\eta^q f(v_n)v_n v_{L,n}^{p(\beta-1)}\mathrm{d}x.\end{aligned}\tag{6.7}$$

It follows from (f_1) and (f_2) that

$$f(v_n)\le V_0 v_n^{p-1} + C v_n^{p^*-1}.$$

This fact combined with (6.7) implies that

$$\int_{\mathbb{R}^N} \eta^q |\nabla v_n|^p v_{L,n}^{p(\beta-1)} \mathrm{d}x \le C \int_{\mathbb{R}^N} v_n^p v_{L,n}^{p(\beta-1)} |\nabla \eta|^p \mathrm{d}x + C \int_{\mathbb{R}^N} v_n^q v_{L,n}^{p(\beta-1)} |\nabla \eta|^q \mathrm{d}x + C \int_{\mathbb{R}^N} \eta^q v_n^{p^*} v_{L,n}^{p(\beta-1)} \mathrm{d}x.$$

By the Sobolev inequality and the above formula, we conclude that

$$\begin{aligned} &\|\eta^{\frac{q}{p}} v_n v_{L,n}^{\beta-1}\|_{L^{p^*}(\mathbb{R}^N)}^p \\ &\le C \int_{\mathbb{R}^N} |\nabla(\eta^{\frac{q}{p}} v_n v_{L,n}^{\beta-1})|^p \mathrm{d}x \\ &\le C\beta^p \left(\int_{\mathbb{R}^N} v_n^p v_{L,n}^{p(\beta-1)} |\nabla \eta|^p \mathrm{d}x + \int_{\mathbb{R}^N} v_n^q v_{L,n}^{p(\beta-1)} |\nabla \eta|^q \mathrm{d}x + \int_{\mathbb{R}^N} \eta^q v_n^{p^*} v_{L,n}^{p(\beta-1)} \mathrm{d}x \right) \\ &\le C\beta^p \left(\int_{\mathbb{R}^N} v_n^p v_{L,n}^{p(\beta-1)} |\nabla \eta|^p \mathrm{d}x + \int_{\mathbb{R}^N} v_n^p v_{L,n}^{p(\beta-1)} |\nabla \eta|^q \mathrm{d}x + \int_{\mathbb{R}^N} \eta^q v_n^{p^*} v_{L,n}^{p(\beta-1)} \mathrm{d}x \right) \end{aligned} \tag{6.8}$$

Let $\beta = p^* \frac{t-1}{pt}$ with $t = \frac{(p^*)^2}{p(p^*-p)}$. Then $v_n \in L^{\frac{\beta p t}{t-1}}(\mathbb{R}^N)$ and $\beta > 1$. Applying (6.4), we deduce

$$\left(\int_{|x|>R} (v_n v_{L,n}^{\beta-1})^{p^*} \mathrm{d}x \right)^{\frac{p}{p^*}} \le C\beta^p \left(\left(\frac{1}{r^p} + \frac{1}{r^q} \right) \int_{R>|x|>R-r} (v_n v_{L,n})^p \mathrm{d}x + \int_{|x|>R-r} v_n^{p^*-p} (v_n v_{L,n}^{\beta-1})^p \mathrm{d}x \right).$$

Furthermore, by the Hölder inequality, we conclude that

$$\begin{aligned} \|v_n v_{L,n}^{\beta-1}\|_{L^{p^*}(|x|>R)}^p \le C \left(\frac{1}{r^p} + \frac{1}{r^q} \right) &\left\{ \frac{1}{r^p} \left[\int_{R>|x|>R-r} v_n^{\beta p} \frac{t}{t-1} \mathrm{d}x \right]^{\frac{t-1}{t}} \left[\int_{R>|x|>R-r} \mathrm{d}x \right]^{\frac{1}{t}} \right. \\ &\left. + \left[\int_{|x|>R-r} v_n^{(p^*-p)t} \mathrm{d}x \right]^{\frac{1}{t}} \left[\int_{|x|>R-r} v_n^{\beta p \frac{t}{t-1}} \mathrm{d}x \right] \right\}. \end{aligned}$$

Due to $(p^* - p)t = \frac{(p^*)^2}{p}$ and $v_n \in L^{\frac{(p^*)^2}{p}}(|x| > R - r)$, one has that

$$\|v_n v_{L,n}^{\beta-1}\|_{L^{p^*}(|x|>R)}^p \le C\beta^p \left(1 + \frac{R^{\frac{N}{t}}}{r^p} + \frac{R^{\frac{N}{t}}}{r^q} \right) \left(\int_{|x|>R} v_n^{\beta p \frac{t}{t-1}} \mathrm{d}x \right).$$

Let $L \to \infty$. we can deduce from the Fatou's lemma that

$$\|v_n\|_{L^{\beta p^*}(|x|>R)}^{\beta p} \le C\beta^p \left(1 + \frac{R^{\frac{N}{t}}}{r^p} + \frac{R^{\frac{N}{t}}}{r^q} \right) \|v_n\|_{L^{\beta p \frac{t}{t-1}}(|x|>R-r)}^{\beta p}.$$

This implies that

$$\|v_n\|_{L^{\beta p^*}(|x|>R)} \le C^{\frac{1}{\beta p}} \beta^{\frac{1}{\beta}} \left(1 + \frac{R^{\frac{N}{t}}}{r^p} + \frac{R^{\frac{N}{t}}}{r^q} \right)^{\frac{1}{\beta p}} \|v_n\|_{L^{\beta p \frac{t}{t-1}}(|x|>R-r)}.$$

Let $\mathcal{X} = \frac{p^*(t-1)}{pt}$, $s = \frac{pt}{t-1}$, $\beta = \mathcal{X}^m$ and $r_m = \frac{R}{2^{m+1}}$ for $m = 1, 2, \cdots$. Then, we can derive that

$$\|v\|_{L^{\mathcal{X}^{m+1}s}(|x|>R-r_{m+1})} \leq C^{\mathcal{X}^{-m}} \mathcal{X}^{m\mathcal{X}^{-m}} \left(1 + \frac{R^{\frac{N}{t}}}{r_m^p} + \frac{R^{\frac{N}{t}}}{r_m^q}\right)^{\frac{1}{p\mathcal{X}^m}} \|v_n\|_{L^{\mathcal{X}^s}(|x|>R-r_m)}. \tag{6.9}$$

Since $p > \frac{N}{t}$ and $q > \frac{N}{t}$, we conclude from (6.9) that

$$\|v\|_{L^{\mathcal{X}^{m+1}s}(|x|>R)} \leq C^{\sum_{i=1}^m \mathcal{X}^{-i}} \mathcal{X}^{\sum_{i=1}^m i\mathcal{X}^{-i}} e^{\sum_{i=1}^m \frac{\ln(1+2^{p(i+1)}+2^{q(i+1)})}{p\mathcal{X}^i}} \|v_n\|_{L^{\mathcal{X}^s}(|x|>R-r_1)}. \tag{6.10}$$

Letting $m \to \infty$ in (6.10), there holds that

$$\|v_n\|_{L^\infty(|x|>R)} \leq C\|v_n\|_{L^{p^*}(|x|>\frac{R}{2})}. \tag{6.11}$$

Noting that $v_n \to v$ in $\mathbb{Y}_{\mu_0,V_0}$, by (6.11), we derive that

$$\lim_{|x|\to\infty} v_n(x) = 0 \quad \text{uniformly in} \quad n \in \mathbb{N}.$$

□

At the end of this section, we complete the proof of Theorem 1.1.

Proof of theorem 1.1. First, we shall show that for any $\delta > 0$ such that $M_\delta \subset \Lambda$, there exists $\varepsilon_\delta > 0$ such that for any $\varepsilon \in (0, \varepsilon_\delta)$, if $u_\varepsilon \in \tilde{\mathcal{N}}_\varepsilon$ is a solution of equation (3.1), then

$$|u_\varepsilon(x)| < a \quad \text{for any} \quad x \in \Lambda_\varepsilon^c. \tag{6.12}$$

Arguing by contradiction, there exists $\varepsilon_n \to 0$ and $u_n := u_{\varepsilon_n} \in \tilde{\mathcal{N}}_{\varepsilon_n}$ is a solution of equation (3.1) such that

$$\|u_n\|_{L^\infty(\Lambda_{\varepsilon_n}^c)} \geq a. \tag{6.13}$$

Clearly, from the proof of Proposition 5.1, we have $\mathcal{J}_{\varepsilon_n}(u_n) \to c_{\mu_0,V_0}$. By Proposition 5.1, we have that there exists $\{\tilde{y}_n\} \subset \mathbb{R}^N$. Taking $v_n = u_n(\cdot + \tilde{y}_n)$, then $v_n \to v$ in $\mathbb{Y}_{\mu_0,V_0}$ with $v \neq 0$ and $y_n = \varepsilon_n \tilde{y}_n \to y_0 \in M$. Noting that $\varepsilon_n \tilde{y}_n \to y_0 \in M$, then there exists $r > 0$ such that $B_r(\varepsilon_n \tilde{y}_n) \subset \Lambda$, and for any $R > 0$, there holds that up to a subsequence

$$B_R(\tilde{y}_n) \subset B_{\frac{r}{\varepsilon_n}}(\tilde{y}_n) \subset \Lambda_{\varepsilon_n}. \tag{6.14}$$

In view of Lemma 6.1, we deduce that there exists $R > 0$ large enough such that

$$v_n(x) < a \quad \text{in} \quad B_R^c(0).$$

This implies that

$$u_n(x) < a \quad \text{in} \quad B_R^c(\tilde{y}_n). \tag{6.15}$$

By (6.14), we derive that

$$\Lambda_{\varepsilon_n}^c \subset B_{\frac{r}{\varepsilon_n}}^c(\tilde{y}_n) \subset B_R^c(\tilde{y}_n).$$

By this and (6.15), one can deduce that

$$u_n(x) < a \quad \text{in} \quad \Lambda_{\varepsilon_n}^c.$$

This is a contradiction due to (6.12). Hence, (6.12) holds.

From (6.12), we know that for any $\varepsilon \in (0, \varepsilon_\delta)$, if $u_\varepsilon \in \tilde{\mathcal{N}}_\varepsilon$ is a solution of equation (3.1), then u_ε is a solution of equation (1.1). By this fact and Lemma 5.1, we conclude that equation (1.1) admits at least $\text{cat}_{M_\delta}(M)$ non-negative solutions.

Then, we show that the concentration of solutions. Let $\varepsilon_n \to 0$ and the sequence $\{u_n\} \subset \mathbb{X}_\varepsilon$ be solutions of equation (3.1). By virtue of (g_1), we can obtain that there exists $\gamma \in (0, a)$ such that

$$g_\varepsilon(x, t)t \leq \frac{V_0}{k} t^p \quad \text{for any} \quad x \in \mathbb{R}^N, \ 0 \leq t \leq \gamma. \tag{6.16}$$

Arguing as before, there exists $R > 0$ such that

$$\|u_n\|_{L^\infty(B^c_R(\tilde{y}_n))} < \gamma. \tag{6.17}$$

By a direct way, we can show that

$$\|u_n\|_{L^\infty(B_R(\tilde{y}_n))} \geq \gamma. \tag{6.18}$$

Indeed, if (6.18) is false, it follows from (6.17) that

$$\|u_n\|_{L^\infty(\mathbb{R}^N)} < \gamma.$$

In the light of this fact, (6.16) and $\langle \mathcal{J}'_{\varepsilon_n}(u_n), u_n\rangle = 0$, we have

$$\|u_n\|^p_{p,\varepsilon_n} + \|u_n\|^q_{q,\varepsilon_n,\mu_{\varepsilon_n}} = \int\limits_{\mathbb{R}^N} g_{\varepsilon_n}(x, u_n)u_n \mathrm{d}x \leq \frac{V_0}{k}\int\limits_{\mathbb{R}^N} |u_n|^p \mathrm{d}x,$$

which implies that $\|u_n\|_{\varepsilon_n} = 0$. This contradicts to (6.18).

Let η_{ε_n} be a global maximum point of u_n. One can derive from (6.17) and (6.18) that $\eta_{\varepsilon_n} = \tilde{y}_n + p_n$ with $|p_n| \leq R$. Since $\varepsilon_n \tilde{y}_n \to y_0 \in M$ and $|p_n| \leq R$, we have $\varepsilon_n \eta_{\varepsilon_n} \to y_0$. It follows from the continuity of V that

$$\lim_{n\to\infty} V(\varepsilon_n \eta_{\varepsilon_n}) = V(y_0) = V_0.$$

So far, the proof of Theorem 1.1 is completed. □

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