



Solutions with prescribed mass for critical Schrödinger–Poisson systems concentrating at a potential well

Qi Gao, Xiaoming He and Vicențiu D. Rădulescu

Abstract. This paper is concerned with the existence and multiplicity of normalized solutions to the following Schrödinger–Poisson system:

$$\begin{cases} -\varepsilon^2 \Delta u + V(x)u - \phi|u|^3 u = \lambda u + \mu|u|^{q-2}u + |u|^4 u, & \text{in } \mathbb{R}^3, \\ -\varepsilon^2 \Delta \phi = |u|^5, & \text{in } \mathbb{R}^3, \end{cases}$$

with prescribed mass

$$\int_{\mathbb{R}^3} |u|^2 dx = a^2 \varepsilon^3,$$

where $a > 0$, $\mu > 0$, $q \in (2, \frac{10}{3})$, and $\varepsilon > 0$ is a small parameter. Here, $\lambda \in \mathbb{R}$ arises as a Lagrange multiplier, and the potential $V : \mathbb{R}^3 \rightarrow [0, +\infty)$ is a continuous function satisfying suitable conditions. By combining truncation techniques with some adequate estimates, we establish that, for sufficiently small $\varepsilon > 0$, normalized solutions do exist. Moreover, by employing Ljusternik–Schnirelmann theory, we find a relationship between the number of positive solutions and the topology of the set where the potential V attains its minimum. Our work extends and complements recent contributions of X. Feng [17, 18] (*Z. Angew. Math. Phys.* 2020), to the abstract setting of multiple normalized concentrating solutions. This study seems to be the first work dealing with the existence of multiple normalized semiclassical states for the Sobolev critical Schrödinger–Poisson system coupled with a nonlocal critical term in the whole space $\mathbb{R}^3$.

Mathematics Subject Classification. 35A15, 35B33, 35J20, 35J60.

Keywords. Schrödinger–Poisson system, Normalized solutions, Critical growth, Ljusternik–Schnirelmann category.

1. Introduction and main results

In the past decade, the following type of nonlinear Schrödinger–Poisson systems have attracted much considerable interests:

$$\begin{cases} i \frac{\partial \Psi}{\partial t} + \Delta \Psi - V(x)\Psi + \gamma \phi |\Psi|^{q-2} \Psi + |\Psi|^{p-2} \Psi = 0, & x \in \mathbb{R}^3, \\ -\Delta \phi = |\Psi|^2, & x \in \mathbb{R}^3, \end{cases} \quad (1.1)$$

where $\Psi : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{C}$, $\gamma \in \mathbb{R}$, $p \in [2, 6]$ and V is an external potential. Especially, mathematical physicists are interested in the existence of stationary solutions, $\Psi(t, x) = e^{i\lambda t} u(x)$ in (1.1), where $u : \mathbb{R}^3 \rightarrow \mathbb{R}$ is a real function to be found. Thus, u must verify the following equation

$$\begin{cases} -\Delta u + V(x)u + \lambda u + \gamma \phi u = |u|^{p-2}u, & x \in \mathbb{R}^3, \\ -\Delta \phi = u^2, & x \in \mathbb{R}^3. \end{cases} \quad (1.2)$$

Applying the Lax–Milgram theorem, for each fixed $u \in H^1(\mathbb{R}^3)$, the Poisson equation $-\Delta \phi = |u|^2$ admits a unique weak solution $\phi_u \in D^{1,2}(\mathbb{R}^3)$ and ϕ_u can be expressed explicitly as (e.g. [34])

$$\phi_u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{|u(y)|^2}{|x-y|} dy, \quad \forall x \in \mathbb{R}^3.$$

Thus, equation (1.2) can be expressed as an equivalent single equation, we often omit the factor $\frac{1}{4\pi}$ for simplicity:

$$-\Delta u + V(x)u + \lambda u - \gamma \phi_u u = |u|^{p-2}u, \quad x \in \mathbb{R}^3.$$

It is well known that, the Schrödinger–Poisson equations (1.1), (1.2) are originated from quantum mechanic models and semiconductor theory [37, 39, 40], which can be used to describe the interaction of a charge particle with an electromagnetic field [45]. In the past few decades, the fixed frequency problems have been studied extensively, and there are many results concerning the existence, non-existence, and multiplicity of solutions to the classical Schrödinger–Poisson system (1.2), via the variational methods after the pioneering work of Benci and Fortunato [8], we refer to [2–5, 8–10, 14, 15, 26, 31, 45, 46, 51] and references therein. We also refer to the recent comprehensive survey by Chen and Tang [12] on the existence of normalized solutions for several classes of nonlinear elliptic equations.

In [35], Li, Li and Shi studied the existence of positive solutions to for Eq. (1.2) with $V(x) + 1 \equiv b$ being constant, and the second Poisson equation replaced by $-\Delta \phi = |u|^5$:

$$\begin{cases} -\Delta u + bu + \gamma \phi |u|^3 u = f(u), & x \in \mathbb{R}^3, \\ -\Delta \phi = |u|^5, & x \in \mathbb{R}^3, \end{cases} \quad (1.3)$$

by using variational method which does not require usual compactness conditions, and a key Brezis–Lieb type convergence result, where $b \geq 0$ and $\gamma \in \mathbb{R}$ are parameters, and f is a Sobolev subcritical growth. In [34], they considered the existence, nonexistence and multiplicity of positive radially symmetric solutions to (1.3) with $f(u) = \mu |u|^{p-1}u$, with variational methods.

Recently, Feng [17] studied the existence and concentration of positive ground state solutions to the following Schrödinger–Poisson system with critical nonlinearity

$$\begin{cases} -\varepsilon^2 \Delta u + V(x)u - \phi |u|^3 u = f(u) + |u|^4 u, & x \in \mathbb{R}^3, \\ -\varepsilon^2 \Delta \phi = |u|^5, & x \in \mathbb{R}^3, \end{cases} \quad (1.4)$$

where $\varepsilon > 0$ is a parameter, and f is a Sobolev subcritical growth, by the modified concentration compactness principle and mountain pass theorem. For more results on (1.3), (1.4), see [17, 34, 36] and references therein.

On the other hand, from the physical background described as above, it is more interesting to study (1.3) or (1.4) to prescribe the L^2 -mass, which are often referred to in the literature as normalized solutions, and to regard the frequency λ as a Lagrange multiplier. We remark that, there are few papers concerning the existence of normalized solutions for problems (1.3) and (1.4). In 2023, Meng and He [41] studied the multiplicity of normalized solutions to the following critical Schrödinger–Poisson system

$$\begin{cases} -\Delta u - \phi |u|^3 u = \lambda u + \mu |u|^{q-2}u + |u|^4 u, & x \in \mathbb{R}^3, \\ -\Delta \phi = |u|^5, & x \in \mathbb{R}^3, \\ \int_{\mathbb{R}^3} |u|^2 dx = a^2 \end{cases} \quad (1.5)$$

by employing the genus theory, the $\mu |u|^{q-2}u$ can be L^2 -subcritical regime or L^2 -supercritical perturbation when the parameter $\mu > 0$ large.

In 2024, He, Meng and Radulescu [24] considered (1.5) under the L^2 -subcritical, L^2 -critical and L^2 -supercritical perturbation $\mu |u|^{q-2}u$, respectively, the authors presented several existence, and non-existence results. Moreover, the qualitative behavior of the ground states as $\mu \rightarrow 0^+$ was also discussed.

Very recently, He, Liu and Meng [25] studied the following more general problem than (1.5) with a potential well V :

$$\begin{cases} -\Delta u + V(x)u - \phi|u|^3u = \lambda u + \mu|u|^{q-2}u + |u|^4u, & x \in \mathbb{R}^3, \\ -\Delta \phi = |u|^5, & x \in \mathbb{R}^3, \\ \int_{\mathbb{R}^3} |u|^2 dx = a^2, \end{cases} \quad (1.6)$$

where V is an external potential vanishing at infinity. The authors proposed some mild assumptions on V , combining the Pohozaev manifold, constrained minimization arguments and some analytical skills. The subcritical perturbation $\mu|u|^{q-2}u$ maybe L^2 -subcritical growth: $q \in (2, \frac{10}{3})$; L^2 -critical growth: $q = \frac{10}{3}$ and may be L^2 -supercritical growth: $q \in (\frac{10}{3}, 6)$, respectively. Simultaneously, they proved the exponential decay property of the ground state solution through the Moser iteration.

We also recall that, in past few years, there has been an increasing interest in the study of normalized solutions for the nonlinear Schrödinger equation

$$\begin{cases} -\Delta u = \lambda u + f(u) & \text{in } \mathbb{R}^N, \\ \|u\|_{L^2(\mathbb{R}^N)}^2 = a^2, \end{cases} \quad (1.7)$$

after the pioneer work [27] due to Jeanjean in 1997. Recently, Soave [48] investigated the existence and non-existence of normalized solutions for (1.7) with the power nonlinearity: $f(u) = \mu|u|^{q-2}u + |u|^{p-2}u$, and $2 < q \leq 2 + \frac{4}{N} \leq p \leq 2^*$, $q \neq p$. Specifically, for the range $2 < q \leq 2 + \frac{4}{N} < 2^*$, the author established the presence of two solutions for (1.7): a local minimizer and a Mountain-Pass type solution. Furthermore, Soave demonstrated orbital stability for the ground state set when $2 < q \leq 2 + \frac{4}{N} < 2^*$ and strong instability for standing waves when $q = 2 + \frac{4}{N} < 2^*$. Subsequently, in [49], He studied the existence and non-existence of normalized solutions for (1.4) with $N \geq 3$, $\lambda \in \mathbb{R}$, $p = 2^* = \frac{2N}{N-2}$ and $q \in (2, 2^*)$, employing Pohozaev manifold decomposition methods and Brezis-Nirenberg approach. For more further results for problem (1.7), we refer to Bartsch and Soave [6], Bartsch, Molle, Rizzi and Verzini [7], Jeanjean and Le [29] and Wei and Wu [50], Jeanjean and Lu [30], Jeanjean, Jendrej, Le and Visciglia [28], and references therein.

In [16] Ding and Zhong considered the existence of ground state solution of the following Schrödinger equations with non-constant potential

$$\begin{cases} -\Delta u + V(x)u + \lambda u = g(u) & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = a^2, \end{cases} \quad (1.8)$$

where $N \geq 1$, $a > 0$, V is an external potential. When g is Sobolev critical growth and can be divided into $g(u) = \mu|u|^{q-2}u + |u|^{2^*-2}u$, Xie, Chen and Tan combined the concentration-compactness principle, constrained minimization arguments to obtain the multiple normalized solutions for the problem (1.8). Later, Lancelotti and Molle [33] obtained the existence of bound state normalized solutions for (1.8) with $g(u) = |u|^{q-2}u$, $q \in (2, 2 + \frac{4}{N})$ by proposing a suitable decay assumption on V . For more results on (1.8), see [1, 32, 42, 53] and references therein.

Motivated by [17, 24, 25], a natural question arises, namely, whether we can obtain multiple normalized solutions for (1.4) with nonconstant potential V ? In this paper, we focus on this issue and give complete answers for this question. Concretely speaking, we investigate the following Schrödinger–Poisson system with potential V and combined critical nonlinearities:

$$\begin{cases} -\varepsilon^2 \Delta u + V(x)u - \phi|u|^3u = \lambda u + \mu|u|^{q-2}u + |u|^4u, & \text{in } \mathbb{R}^3, \\ -\varepsilon^2 \Delta \phi = |u|^5, & \text{in } \mathbb{R}^3, \end{cases} \quad (1.9)$$

with prescribed mass

$$\int_{\mathbb{R}^3} |u|^2 dx = a^2 \varepsilon^3. \quad (1.10)$$

By a variable change $z = \varepsilon x$, we see that problem (1.9)-(1.10) is equivalent to the following constrained mass problem:

$$\begin{cases} -\Delta u + V(\varepsilon x)u - \phi_u |u|^3 u = \lambda u + \mu |u|^{q-2} u + |u|^4 u & \text{in } \mathbb{R}^3, \\ \int_{\mathbb{R}^3} u^2 dx = a^2, \end{cases} \quad (1.11)$$

where $a > 0$, $\mu > 0$, $q \in (2, \frac{10}{3})$, and $\varepsilon > 0$ is a small parameter. Here, $\lambda \in \mathbb{R}$ arises as a Lagrange multiplier, and the potential $V : \mathbb{R}^3 \rightarrow [0, +\infty)$ is a continuous function satisfying the following conditions:

(V1) $V \in C(\mathbb{R}^3, \mathbb{R}) \cap L^\infty(\mathbb{R}^3)$, and $V(x) \geq V(0) = 0$ for all $x \in \mathbb{R}^3$.

(V2) $V_\infty = \lim_{|x| \rightarrow +\infty} V(x) > V_0 := \inf_{x \in \mathbb{R}^3} V(x) = 0$.

Hereafter, we define the sets M and M_δ by

$$M = \{x \in \mathbb{R}^N : V(x) = 0\} \text{ and } M_\delta = \{x \in \mathbb{R}^N : \text{dist}(x, M) \leq \delta\},$$

for $\delta > 0$.

Our main result can be formulated as the following:

Theorem 1.1. *Assume that V satisfies (V1)–(V2). Then for each $\delta > 0$, there exist $\varepsilon_0 > 0$, $a_* > 0$, and $V_* > 0$ such that (1.9)–(1.10) admits $\text{cat}_{M_\delta}(M)$ couples $(u_j, \lambda_j) \in H^1(\mathbb{R}^3) \times \mathbb{R}$ of weak solutions for $\varepsilon \in (0, \varepsilon_0)$, $a \in (0, a_*)$, and $|V|_\infty < V_*$, with $\int_{\mathbb{R}^3} |u_j|^2 dx = a^2$, $\lambda_j < 0$, and $J_\varepsilon(u_j) < 0$. Moreover, if u_ε denotes one of these solutions and ξ_ε is a global maximum point of $|u_\varepsilon|$, then*

$$\lim_{\varepsilon \rightarrow 0} V(\varepsilon \xi_\varepsilon) = 0.$$

By virtue of the definition of category from [52], we would like to point out that the category $\text{cat}_X(A)$ of a subset A of a topological space X is defined as the minimal $k \in \mathbb{N}$ such that A is covered by k closed subsets of X which are contractible in X , namely

$$\text{cat}_X(A) = \inf \{k \in \mathbb{N} \cup \{+\infty\} \mid \exists k \text{ contractible closed subsets of } X :$$

$$\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_k \text{ such that } A \subset \cup_{i=1}^k \mathcal{F}_i\},$$

where set $\mathcal{F}$ is called contractible (in X) if and only if $\exists \mathcal{T} : [0, 1] \times X \mapsto X$ such that

$$\mathcal{T}(0, \mathcal{F}) = \text{id}_X \text{ and } \mathcal{T}(1, \mathcal{F}) = \text{one-point set}.$$

- Remark 1.1.** (i) In [17], Feng only studied the existence of semiclassical ground states for problem (1.4), where the potential V satisfies the global condition introduced by Rabinowitz [44]: $V_\infty = \lim_{|x| \rightarrow +\infty} V(x) > V_0 := \inf_{x \in \mathbb{R}^3} V(x) > 0$. Theorem 1.1 improves and complements the main results of [17] in the sense that, we focus on the multiple normalized concentrating solutions of (1.4).
- (ii) Theorem 1.1 complements and improves the previous works made by the authors in [24, 25, 41]. Comparing our paper with [25], we study the *multiplicity of normalized solutions* for the problem with potential V . We emphasize that V is not necessary radial, thus we can't work in radial subspace $H_{rad}^1(\mathbb{R}^3)$ to restore the compactness. In Theorem 1.1 we relate the number of positive normalized solutions with the topology of the set M , where V attains its minimum, by using Ljusternik-Schnirelmann theory.
- (iii) As far as we are aware, we did not find any works on the multiplicity of normalized solutions of Schrödinger–Poisson system with doubly critical growth and potential $V \neq 0$. Our results extend the previous results made by [24, 41] to the nonconstant potential case.

To prove Theorem 1.1 we shall employ the truncation technique from Peral Alonso [21] to make the corresponding energy functional is bounded from below. Moreover, instead of the traditional genus theory, we make use of the Ljusternik-Schnirelmann theory to describe the multiplicity result. By establishing the relationship between the category of some sublevel sets of the energy functional and the category of set M , we shall prove the existence of multiple normalized solutions to problem (1.9)–(1.10). From the

technical point of view, we will face the following three difficulties to overcome. Firstly, the critical growth term in the equation makes the bounded (PS) sequences could not converge. Secondly, since the problem has two critical terms, it becomes complicated to estimate the critical value of mountain pass. Thirdly, in view of the appearance of a nonlocal critical convolution term $\phi|u|^3u$ in Schrödinger–Poisson systems, it would be natural to consider how the interaction between the nonlocal term and the nonlinear term, which will affect the existence of solutions of problem (1.9)–(1.10). In order to overcome these difficulties, we employ the concentration-compactness principle, mountain pass theorem and the energy estimation to obtain the existence and concentration of normalized solutions.

The paper is organized as follows. In Sect. 2, we study the autonomous version of the system, establishing the coercivity and boundedness from below of the energy functional, as well as the compactness of minimizing sequences. This allows us to obtain positive radial solutions for the autonomous problem, which serve as a foundation for analyzing the non-autonomous case.

In Sect. 3, we employ the truncation technique to modify the functional, proving its coercivity, boundedness from below, and the Palais-Smale condition. This approach overcomes the lack of compactness due to the unbounded domain and the critical growth terms.

In Sect. 4, by constructing mappings that link the topology of the potential's minimum set to critical points of the functional, and combining results from the truncated problem, we establish the existence of at least $\text{cat}_{M_\delta}(M)$ normalized solutions. Furthermore, we show that as $\varepsilon \rightarrow 0$, these solutions concentrate around the points where the potential V attains its minimum.

Throughout this paper, unless otherwise specified, we use the following notations:

- Let $H^1(\mathbb{R}^3)$ denote the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm

$$\|u\|_{H^1}^2 = \int_{\mathbb{R}^3} (|\nabla u|^2 + |u|^2) dx.$$

- The Sobolev space $D^{1,2}(\mathbb{R}^3)$ is defined by

$$D^{1,2}(\mathbb{R}^3) = \left\{ u \in L^6(\mathbb{R}^3) : \int_{\mathbb{R}^3} |\nabla u|^2 dx < +\infty \right\},$$

and is endowed with the norm

$$|u|_{D^{1,2}} = \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^{1/2}.$$

- The working space $H_{\text{rad}}^1(\mathbb{R}^3)$ is defined as

$$H_{\text{rad}}^1(\mathbb{R}^3) := \{u \in H^1(\mathbb{R}^3) : u \text{ is radially decreasing}\}.$$

- For any $x \in \mathbb{R}^3$ and $r > 0$, we denote $B_r(x) := \{y \in \mathbb{R}^3 : |y - x| < r\}$ and $B_r = B_r(0)$.
- The letters, $C, C_1, C_2, \dots$ denote positive constants that may vary from line to line. The notation $o_n(1)$ represents a real sequence satisfying $o_n(1) \rightarrow 0$ as $n \rightarrow \infty$.

2. Preliminary

In this section, we summarize some preliminary results, which will be used in the proof of Theorem 1.1.

Lemma 2.1. ([47]) *For any $N \geq 3$, the embedding $H_{\text{rad}}^1(\mathbb{R}^N) \hookrightarrow L^t(\mathbb{R}^N)$ is compact for any $2 < t < 2^*$.*

Lemma 2.2. (*Gagliardo-Nirenberg-Sobolev inequality [43]*) For any $N \geq 3$ and $p \in [2, 2^*]$, there exists a positive constant $C_{N,p}$ depending on N, p such that

$$\int_{\mathbb{R}^N} |u|^p dx \leq C_{N,p} |\nabla u|_2^{p\gamma_p} |u|_2^{p(1-\gamma_p)}, \quad (2.1)$$

where $\gamma_p = \frac{N(p-2)}{2p}$.

Lemma 2.3. ([38]) Let $t, r > 1$ and $0 < \alpha < N$ with $1/t + \alpha/N + 1/r = 2$, $f \in L^t(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$. There exists a sharp constant $C(t, N, \alpha, r)$, independent of f and h , such that

$$\iint_{\mathbb{R}^{2N}} \frac{f(x)h(y)}{|x-y|^\alpha} dx dy \leq C(t, N, \alpha, r) |f|_t |h|_r. \quad (2.2)$$

If $t = r = 2N/(2N - \alpha)$, then

$$C(t, N, \alpha, r) = C(N, \alpha) = \pi^{\alpha/2} \frac{\Gamma(N/2 - \alpha/2)}{\Gamma(N - \alpha/2)} \left\{ \frac{\Gamma(N/2)}{\Gamma(N)} \right\}^{-1+\alpha/N}. \quad (2.3)$$

In this case, equality in (2.2) holds if and only if $f \equiv (\text{constant})h$ and

$$h(x) = A(\gamma^2 + |x - a|^2)^{-(2N-\alpha)/2}$$

for some $A \in \mathbb{C}$, $0 \neq \gamma \in \mathbb{R}$, and $a \in \mathbb{R}^N$.

In the sequel, let S be the best Sobolev constant defined by

$$S := \inf_{u \in D^{1,2}(\mathbb{R}^3) \setminus \{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^3} |u|^6 dx \right)^{1/3}}. \quad (2.4)$$

For a given $u \in H^1(\mathbb{R}^3)$, we deduce by the Lax-Milgram theorem that there exists a unique solution $\phi = \phi_u \in D^{1,2}(\mathbb{R}^3)$ satisfying $-\Delta \phi_u = |u|^5$ in the weak sense. From [34], the function ϕ_u is represented by

$$\phi_u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{|u(y)|^5}{|x-y|} dy. \quad (2.5)$$

We will use $S_{H,L}$ to denote the best constant defined by

$$S_{H,L} := \inf_{u \in D^{1,2}(\mathbb{R}^3) \setminus \{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^3} \phi_u |u|^5 dx \right)^{1/5}}. \quad (2.6)$$

Lemma 2.4. ([20]) The constant $S_{H,L} = S/C(3,1)^{1/5}$ is achieved if and only if

$$u(x) = C \left(\frac{b}{b^2 + |x - a|^2} \right)^{1/2},$$

where $C > 0$ is a constant, $a \in \mathbb{R}^3$ and $b \in \mathbb{R}$ are parameters, S is the best Sobolev constant, and $C(3,1)$ is the constant from Lemma 2.3.

We recall that the function ϕ_u has the following properties:

Lemma 2.5. ([34]) *The following properties hold:*

- (i) $\phi_u \geq 0$ for all $u \in H^1(\mathbb{R}^3)$;
- (ii) $\phi_{tu} = t^5 \phi_u$ for all $t > 0$ and $u \in H^1(\mathbb{R}^3)$;
- (iii) for any $u \in H^1(\mathbb{R}^3)$,

$$|\phi_u|_{D^{1,2}} \leq S^{-1/2} |u|_6^5,$$

and

$$\int_{\mathbb{R}^3} \phi_u |u|^5 dx \leq S^{-1} |u|_6^{10};$$

- (iv) if $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$, then, up to a subsequence, $\phi_{u_n} \rightharpoonup \phi_u$ in $D^{1,2}(\mathbb{R}^3)$ and

$$\int_{\mathbb{R}^3} \phi_{u_n} |u_n|^3 u_n \varphi dx \rightarrow \int_{\mathbb{R}^3} \phi_u |u|^3 u \varphi dx \quad \text{for each } \varphi \in H^1(\mathbb{R}^3).$$

Finally, we state the following concentration-compactness principle for studying the nonlocal critical equation [19], which is essentially due to Lions [39], and Chabrowski [11].

Lemma 2.6. ([19]) *Let $\{u_n\}$ be a bounded sequence in $D^{1,2}(\mathbb{R}^3)$ converging weakly and almost everywhere to some $u \in D^{1,2}(\mathbb{R}^3)$. Suppose $|\nabla u_n|^2 \rightharpoonup \omega$ and $|u_n|^6 \rightharpoonup \zeta$ in the sense of measures. Assume that*

$$\left(\int_{\mathbb{R}^3} \frac{|u_n(y)|^5}{|x-y|} dy \right) |u_n(x)|^5 \rightharpoonup \nu$$

weakly in the sense of measures, where ν is a bounded positive measure on $\mathbb{R}^3$. Then there exist an at most countable set J , a family of points $\{z_j\}_{j \in J} \subset \mathbb{R}^3$, and families of positive numbers $\{\nu_j\}_{j \in J}$, $\{\zeta_j\}_{j \in J}$, and $\{\omega_j\}_{j \in J}$ such that

$$\nu = \left(\int_{\mathbb{R}^3} \frac{|u(y)|^5}{|x-y|} dy \right) |u(x)|^5 + \sum_{j \in J} \nu_j \delta_{z_j}, \quad \sum_{j \in J} \nu_j^{1/5} < \infty; \quad (2.7)$$

$$\omega \geq |\nabla u|^2 + \sum_{j \in J} \omega_j \delta_{z_j}; \quad (2.8)$$

$$\zeta \geq |u|^6 + \sum_{j \in J} \zeta_j \delta_{z_j}; \quad (2.9)$$

and

$$\omega_j \geq S \zeta_j^{1/3}, \quad \omega_j \geq S_{H,L} \nu_j^{1/5}, \quad C(3,1)^{3/5} \zeta_j \geq \nu_j^{3/5}, \quad (2.10)$$

where δ_x is the Dirac mass of mass 1 concentrated at $x \in \mathbb{R}^3$.

Lemma 2.7. ([19]) *Let $\{u_n\} \subset D^{1,2}(\mathbb{R}^3)$ be a sequence in Lemma 2.6 and define*

$$\omega_\infty := \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{|x| \geq R} |\nabla u_n|^2 dx, \quad \zeta_\infty := \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{|x| \geq R} |u_n|^6 dx.$$

Then it follows that

$$\omega_\infty \geq S \zeta_\infty^{1/3}; \quad (2.11)$$

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} |\nabla u_n|^2 dx = \int_{\mathbb{R}^3} d\omega + \omega_\infty; \quad (2.12)$$

and

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} |u_n|^6 dx = \int_{\mathbb{R}^3} d\zeta + \zeta_\infty. \quad (2.13)$$

Lemma 2.8. ([19]) Let $\{u_n\} \subset D^{1,2}(\mathbb{R}^3)$ such that $u_n \rightharpoonup u$ in $D^{1,2}(\mathbb{R}^3)$, $\omega, \omega_\infty, \zeta, \zeta_\infty$ be the bounded nonnegative measures in Lemma 2.6 and Lemma 2.7. Assume that

$$\nu_\infty := \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{|x| \geq R} \left(\int_{\mathbb{R}^3} \frac{|u_n(y)|^5}{|x-y|} dy \right) |u_n(x)|^5 dx.$$

The quantities $\omega_\infty, \zeta_\infty$ and ν_∞ exist and satisfy

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u_n(y)|^5 |u_n(x)|^5}{|x-y|} dy dx = \int_{\mathbb{R}^3} d\nu + \nu_\infty; \quad (2.14)$$

and

$$C_{3,1}^{-\frac{6}{5}} \nu_\infty^{\frac{6}{5}} \leq \zeta_\infty \left(\int_{\mathbb{R}^3} d\zeta + \zeta_\infty \right); \quad (2.15)$$

$$S_{H,L}^2 \nu_\infty^{\frac{2}{5}} \leq \omega_\infty \left(\int_{\mathbb{R}^3} d\omega + \omega_\infty \right). \quad (2.16)$$

3. Truncated functionals and autonomous truncated functionals

3.1. Truncated functionals

In what follows, we consider the functional $J_\varepsilon : H^1(\mathbb{R}^3) \rightarrow \mathbb{R}$ defined by

$$J_\varepsilon(u) := \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} V(\varepsilon x) |u|^2 dx - \frac{1}{10} \int_{\mathbb{R}^3} \phi_u |u|^5 dx - \frac{\mu}{q} \int_{\mathbb{R}^3} |u|^q dx - \frac{1}{6} \int_{\mathbb{R}^3} |u|^6 dx, \quad (3.1)$$

restricted to the sphere in $L^2(\mathbb{R}^3)$ given by

$$S(a) := \left\{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} |u|^2 dx = a^2 \right\}.$$

By assumptions (V1), (2.1), (2.4), and (2.6), we have for all $u \in S(a)$,

$$J_\varepsilon(u) \geq \frac{1}{2} |\nabla u|_2^2 - \frac{1}{10 S_{H,L}^5} |\nabla u|_2^{10} - \frac{\mu}{q} C_q a^{\frac{6-q}{2}} |\nabla u|_2^{\frac{3(q-2)}{2}} - \frac{1}{6 S^3} |\nabla u|_2^6 =: g_a(|\nabla u|_2),$$

where

$$g_a(r) := \frac{1}{2} r^2 - \frac{1}{10 S_{H,L}^5} r^{10} - \frac{\mu}{q} C_q a^{\frac{6-q}{2}} r^{\frac{3(q-2)}{2}} - \frac{1}{6 S^3} r^6, \quad \forall r > 0.$$

Setting $g_a(r) = r^2 h_a(r)$, we define

$$h_a(r) := \frac{1}{2} - \frac{1}{10 S_{H,L}^5} r^8 - \frac{\mu}{q} C_q a^{\frac{6-q}{2}} r^{\frac{3q-10}{2}} - \frac{1}{6 S^3} r^4, \quad \forall r > 0.$$

Now we investigate the properties of $h_a(r)$. Recalling that $q \in (2, \frac{10}{3})$, we observe that $h_a(r) \rightarrow -\infty$ as $r \rightarrow 0^+$ and $h_a(r) \rightarrow -\infty$ as $r \rightarrow +\infty$. We claim that, there exists a unique $r^* > 0$ such that $h_a(r)$

arrives its global maximum at $h_a(r^*) = \max_{r>0} h_a(r)$. Indeed, if there exists $0 < r_1 < r_2$, such that $\max_{r>0} h_a(r) = h_a(r_1) = h_a(r_2)$. Therefore, using

$$h'_a(r) = -\frac{4}{5S_{H,L}^5}r^7 - \frac{\mu(3q-10)}{2q}C_q a^{\frac{6-q}{2}}r^{\frac{3(q-4)}{2}} - \frac{2}{3S^3}r^3,$$

and that $h'_a(r_1) = h'_a(r_2) = 0$, we infer to

$$\frac{4}{5S_{H,L}^5}(r_2^7 - r_1^7) + \frac{2}{3S^3}(r_2^3 - r_1^3) = \frac{\mu(3q-10)}{2q}C_q a^{\frac{6-q}{2}}(r_1^{\frac{3(q-4)}{2}} - r_2^{\frac{3(q-4)}{2}}).$$

Since $r_1 < r_2$, $q \in (2, \frac{10}{3})$, the left-hand side of the last equality is positive, while the right-hand side is negative, which implies a contradiction. Moreover, there exists some $a_* > 0$ such that the maximum value satisfies

$$\max_{r>0} h_a(r) \begin{cases} > 0 & \text{if } a < a_*, \\ = 0 & \text{if } a = a_*, \\ < 0 & \text{if } a > a_*. \end{cases} \quad (3.2)$$

Consequently, $g_a(r)$ possesses the following properties:

- (i) For $a < a_*$, $\lim_{r \rightarrow 0^+} g_a(r) = 0$ and $\lim_{r \rightarrow +\infty} g_a(r) = -\infty$.
- (ii) There exist $0 < R_0 < R_1 < +\infty$, depending on a , such that $g_a(R_0) = g_a(R_1) = 0$.
- (iii) $g_a(r) < 0$ for $r \in (0, R_0) \cup (R_1, +\infty)$, and $g_a(r) > 0$ for $r \in (R_0, R_1)$.

We define a function $\tau : \mathbb{R}^+ \rightarrow [0, 1]$ as a non-increasing C^∞ function as:

$$\tau(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq R_0, \\ 0 & \text{if } x \geq R_1. \end{cases}$$

In what follows, we consider the truncated functional $J_{\varepsilon,T} : H^1(\mathbb{R}^3) \rightarrow \mathbb{R}$ given by

$$\begin{aligned} J_{\varepsilon,T}(u) := & \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} V(\varepsilon x) |u|^2 dx - \frac{\mu}{q} \int_{\mathbb{R}^3} |u|^q dx \\ & - \frac{\tau(|\nabla u|_2)}{10} \int_{\mathbb{R}^3} \phi_u |u|^5 dx - \frac{\tau(|\nabla u|_2)}{6} \int_{\mathbb{R}^3} |u|^6 dx. \end{aligned} \quad (3.3)$$

By (2.1) and (V1), for any $u \in S(a)$, we obtain

$$\begin{aligned} J_{\varepsilon,T}(u) & \geq \frac{1}{2} |\nabla u|_2^2 - \frac{\mu}{q} C_q a^{\frac{6-q}{2}} |\nabla u|_2^{\frac{3(q-2)}{2}} - \frac{\tau(|\nabla u|_2)}{10S_{H,L}^5} |\nabla u|_2^{10} - \frac{\tau(|\nabla u|_2)}{6S^3} |\nabla u|_2^6 \\ & =: g_{a,T}(|\nabla u|_2), \end{aligned} \quad (3.4)$$

where $g_{a,T}(r) := \frac{1}{2}r^2 - \frac{\mu}{q}C_q a^{\frac{6-q}{2}}r^{\frac{3(q-2)}{2}} - \frac{\tau(r)}{10S_{H,L}^5}r^{10} - \frac{\tau(r)}{6S^3}r^6$. It is easy to observe that $g_{a,T}(r)$ has the following properties:

$$\begin{cases} g_{a,T}(r) \equiv g_a(r) & \text{for all } r \in (0, R_0], \\ g_{a,T}(r) \text{ is positive and strictly increasing in } (R_0, +\infty). \end{cases} \quad (3.5)$$

In the sequel, we may assume that $0 < a < a_*$, and choose $R_0 > 0$ small, such that

$$R_0 < \left(\frac{\sqrt{S^{-6} + 4S_{H,L}^{-5}} - S^{-3}}{2S_{H,L}^{-5}} \right)^{\frac{1}{4}}. \quad (3.6)$$

3.2. Autonomous truncated functionals

For any $w \in [0, |V|_\infty]$, we define the functionals $J_w, J_{w,T} : H^1(\mathbb{R}^3) \rightarrow \mathbb{R}$ by

$$J_w(u) := \frac{1}{2}|\nabla u|_2^2 + \frac{w}{2}|u|_2^2 - \frac{1}{10} \int_{\mathbb{R}^3} \phi_u |u|^5 dx - \frac{\mu}{q}|u|_q^q - \frac{1}{6}|u|_6^6, \quad (3.7)$$

and

$$J_{w,T}(u) := \frac{1}{2}|\nabla u|_2^2 + \frac{w}{2}|u|_2^2 - \frac{\mu}{q}|u|_q^q - \frac{\tau(|\nabla u|_2)}{10} \int_{\mathbb{R}^3} \phi_u |u|^5 dx - \frac{\tau(|\nabla u|_2)}{6}|u|_6^6. \quad (3.8)$$

In the sequel, we study the properties of the functional $J_{w,T}$ in (3.8) restricted to $S(a)$, where $w \in [0, |V|_\infty]$ and $a < a_*$.

Lemma 3.1. *If $w \in [0, |V|_\infty]$ and $a < a_*$, then the functional $J_{w,T}(u)$ is bounded from below on $S(a)$ and coercive.*

Proof. By (3.4)–(3.5), we have

$$\begin{aligned} J_{w,T}(u) &\geq \frac{1}{2}|\nabla u|_2^2 + \frac{V_0}{2}|u|_2^2 - \frac{\mu}{q}|u|_q^q - \frac{\tau(|\nabla u|_2)}{10} \int_{\mathbb{R}^3} \phi_u |u|^5 dx - \frac{\tau(|\nabla u|_2)}{6}|u|_6^6 \\ &\geq g_{a,T}(|\nabla u|_2) \\ &\geq \inf_{r>0} g_{a,T}(r) > -\infty. \end{aligned}$$

Moreover, $J_{w,T}(u) \rightarrow \infty$ as $|\nabla u|_2 \rightarrow +\infty$. The desired conclusion follows. $\square$

We now define the minimum value of the functional $J_{w,T}$ on the constraint $S(a)$ by

$$c_{w,T,a} := \inf_{u \in S(a)} J_{w,T}(u).$$

Lemma 3.2. *If $a < a_*$, then there exists $V_* > 0$ such that $c_{w,T,a} < 0$ for all $w < V_*$.*

Proof. Fix $u \in S(a)$. We define the radial dilation $(t * u)(x) = t^{\frac{3}{2}}u(tx)$ for any $t > 0$. It is straightforward to check that if $u \in S(a)$, then $(t * u) \in S(a)$ for any $t > 0$. Furthermore, the map $(t, u) \in \mathbb{R} \times H^1(\mathbb{R}^3) \mapsto (t * u) \in H^1(\mathbb{R}^3)$ is continuous.

By the definition of the function τ and the condition $2 < q < \frac{10}{3}$, we have

$$\begin{aligned} J_{w,T}((t * u)) &\leq \frac{1}{2} \int_{\mathbb{R}^3} |\nabla((t * u))|^2 dx + \frac{wa_*}{2} - \frac{\mu}{q} \int_{\mathbb{R}^3} |(t * u)|^q dx \\ &= \frac{1}{2}t^2 \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{wa_*}{2} - \frac{\mu}{q} t^{\frac{3(q-2)}{2}} \int_{\mathbb{R}^3} |u|^q dx. \end{aligned}$$

Let $B(t) := \frac{1}{2}t^2|\nabla u|_2^2 - \frac{\mu}{q}t^{\frac{3(q-2)}{2}}|u|_q^q$. Since $2 > \frac{3(q-2)}{2}$, there exists $t^* > 0$ small enough such that $B(t^*) < 0$. Denote by $V_* := -\frac{B(t^*)}{a_*} > 0$. Then for all $w < V_*$, we obtain $J_{w,T}((t * u)) < 0$. Thus, $c_{w,T,a} < 0$, and this completes the proof. $\square$

Lemma 3.3. *The following properties of the functional $J_{w,T}$ hold:*

- (i) $J_{w,T} \in C^1(H^1(\mathbb{R}^3), \mathbb{R})$.
- (ii) If $J_{w,T}(u) < 0$, then $|\nabla u|_2 < R_0$, and $J_{w,T}(v) = J_w(v)$ for all v in a small neighborhood of u in $H^1(\mathbb{R}^3)$.

Proof. (i) The differentiability follows by standard arguments. Now we prove (ii). It follows from $J_{w,T}(u) < 0$ and

$$J_{w,T}(u) \geq g_{a,T}(|\nabla u|_2)$$

that $g_{a,T}(|\nabla u|_2) < 0$, which implies that $|\nabla u|_2 < R_0$ by (3.6).

By (i) and $J_{w,T}(u) < 0$, we obtain that $J_{w,T}(v) < 0$ for all v in a small neighborhood of u in $H^1(\mathbb{R}^3)$, which gives that $|\nabla v|_2 < R_0$ and thus $J_{w,T}(v) = J_w(v)$. The proof is complete. $\square$

By the definition of $g_{a,T}(r)$, we consider the auxiliary function

$$h_T(a, r) := \frac{1}{r^2} g_{a,T}(r) = \frac{1}{2} - \frac{\mu}{q} C_q a^{\frac{6-q}{2}} r^{\frac{3q-10}{2}} - \frac{\tau(r)}{10S_{H,L}^5} r^8 - \frac{\tau(r)}{6S^3} r^4, \quad \forall r > 0.$$

Lemma 3.4. *Let $h(a, r) := h_a(r)$ and $(a_2, r_2) \in (0, \infty) \times (0, \infty)$ satisfy $h(a_2, r_2) \geq 0$. Then for any $a_1 \in (0, a_2]$, we have that*

$$h_T(a_1, r_1) \geq 0, \quad \text{if } r_1 \in \left[\frac{a_1}{a_2} r_2, r_2 \right].$$

Proof. Since $a \mapsto h_T(\cdot, r)$ is non-increasing, we can infer to

$$h_T(a_1, r_2) \geq h_T(a_2, r_2) \geq h(a_2, r_2) \geq 0.$$

A direct calculation shows that

$$\begin{aligned} h_T\left(a_1, \frac{a_1}{a_2} r_2\right) &\geq h\left(a_1, \frac{a_1}{a_2} r_2\right) \\ &= \frac{1}{2} - \frac{\mu}{q} C_q a_1^{\frac{6-q}{2}} \left(\frac{a_1}{a_2}\right)^{\frac{3q-10}{2}} r_2^{\frac{3q-10}{2}} - \frac{1}{10S_{H,L}^5} \left(\frac{a_1}{a_2}\right)^8 r_2^8 - \frac{1}{6S^3} \left(\frac{a_1}{a_2}\right)^4 r_2^4 \\ &\geq \frac{1}{2} - \frac{\mu}{q} C_q a_2^{\frac{6-q}{2}} r_2^{\frac{3q-10}{2}} - \frac{1}{10S_{H,L}^5} r_2^8 - \frac{1}{6S^3} r_2^4 \\ &= h(a_2, r_2) \geq 0. \end{aligned}$$

Thus, $h_T\left(a_1, \frac{a_1}{a_2} r_2\right) \geq 0$ and $h_T(a_1, r_2) \geq 0$. It follows from the definition of τ that $h_T(a_1, r_1) \geq 0$ for $r_1 \in \left[\frac{a_1}{a_2} r_2, r_2\right]$. $\square$

Lemma 3.5. *For any $u \in S(a)$ with $|\nabla u|_2 \leq \left(\frac{\sqrt{S^{-6}+4S_{H,L}^{-5}}-S^{-3}}{2S_{H,L}^5}\right)^{\frac{1}{4}}$, we have that*

$$J_{w,T}(u) \geq |\nabla u|_2^2 h_T(a, |\nabla u|_2).$$

Proof. Applying (2.1), we obtain

$$\begin{aligned} J_{w,T}(u) &\geq \frac{1}{2} |\nabla u|_2^2 - \frac{\mu}{q} C_q a^{\frac{6-q}{2}} |\nabla u|_2^{\frac{3(q-2)}{2}} - \frac{\tau(|\nabla u|_2)}{10S_{H,L}^5} |\nabla u|_2^{10} - \frac{\tau(|\nabla u|_2)}{6S^3} |\nabla u|_2^6 \\ &= |\nabla u|_2^2 \left[\frac{1}{2} - \frac{\mu}{q} C_q a^{\frac{6-q}{2}} |\nabla u|_2^{\frac{3q-10}{2}} - \frac{\tau(|\nabla u|_2)}{10S_{H,L}^5} |\nabla u|_2^8 - \frac{\tau(|\nabla u|_2)}{6S^3} |\nabla u|_2^4 \right] \\ &= |\nabla u|_2^2 h_T(a, |\nabla u|_2), \end{aligned}$$

which completes the proof. $\square$

Lemma 3.6. (i) $a \in (0, a_*) \mapsto c_{w,T,a}$ is continuous;

(ii) $\frac{a_1^2}{a_2^2} c_{w,T,a_2} < c_{w,T,a_1} < 0$ for any $0 < a_1 < a_2 < a_*$.

Proof. (i) A standard argument shows that (i) holds.

(ii) Let $\zeta := \frac{a_2}{a_1} > 1$ and $\{u_n\} \subset S(a_1)$ be a minimizing sequence with respect to c_{w,T,a_1} , that is,

$$J_{w,T}(u_n) \rightarrow c_{w,T,a_1} < 0 \quad \text{and} \quad J'_{w,T}(u_n) \rightarrow 0 \quad \text{as } n \rightarrow +\infty. \quad (3.9)$$

As a consequence, there exists $n_0 \in \mathbb{N}$ large such that

$$J_{w,T}(u_n) < 0 \quad \text{for } n \geq n_0. \quad (3.10)$$

By Lemma 3.4 and $h(a_2, R_0) \geq 0$, $h_T(a_1, r) \geq 0$ for any $r \in \left[\frac{a_1}{a_2}R_0, R_0\right]$. Therefore, combining Lemma 3.5 with (3.9), we deduce that

$$|\nabla u_n|_2 < \min \left\{ \left(\frac{\sqrt{S^{-6} + 4S_{H,L}^{-5}} - S^{-3}}{2S_{H,L}^{-5}} \right)^{\frac{1}{4}}, \frac{a_1}{a_2}R_0 \right\} \quad \text{for } n \geq n_0. \quad (3.11)$$

Setting $v_n = \zeta u_n$, then $v_n \in S(a_2)$. By means of (3.11), we get $|\nabla v_n|_2 = \zeta(|\nabla u_n|_2) < R_0$. Hence, $\tau(|\nabla u_n|_2) = \tau(|\nabla v_n|_2) = 1$. Moreover,

$$\begin{aligned} c_{w,T,a_2} &\leq J_{w,T}(v_n) \\ &= \zeta^2 J_{w,T}(u_n) + \frac{(\zeta^2 - \zeta^q)}{q} \mu |u_n|_q^q + \frac{(\zeta^2 - \zeta^6)}{6} |u_n|_6^6 + \frac{(\zeta^2 - \zeta^{10})}{10} \int_{\mathbb{R}^3} \phi_{u_n} |u_n|^5 dx. \end{aligned}$$

We claim that there exist $C_1 > 0$, $C_2 > 0$, $C_3 > 0$ such that

$$|u_n|_q^q > C_1 \quad \text{or} \quad |u_n|_6^6 > C_2 \quad \text{or} \quad \int_{\mathbb{R}^3} \phi_{u_n} |u_n|^5 dx > C_3.$$

Since $\{u_n\} \subset S(a_1)$ be a minimizing sequence with respect to c_{w,T,a_1} , by (3.9) and Lemma 3.3, we have $\{u_n\} \subset S(a)$ in bounded in $W^{1,p}(\mathbb{R}^3)$. Moreover, we get

$$-\Delta u_n + w u_n - \phi_{u_n} |u_n|^3 u_n = \lambda_n u_n + \mu |u_n|^{q-2} u_n + |u_n|^4 u_n + o_n(1), \quad (3.12)$$

we can infer that

$$\int_{\mathbb{R}^3} |\nabla u_n|^2 dx + w a_1^2 = \int_{\mathbb{R}^3} \phi_{u_n} |u_n|^5 dx + \lambda_n a_1^2 + \mu \int_{\mathbb{R}^3} |u_n|^q dx + \int_{\mathbb{R}^3} |u_n|^6 dx + o_n(1). \quad (3.13)$$

Thus, $\lambda_n \rightarrow \lambda_0 \in \mathbb{R}$ is bounded. Note that, the “almost” Pohozaev identity associated to (3.13) reads as:

$$\begin{aligned} &\frac{1}{2} \int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \frac{3w}{2} a_1^2 - \frac{3}{10} \int_{\mathbb{R}^3} \phi_{u_n} |u_n|^5 dx \\ &\quad - \frac{3\lambda_n}{2} a_1^2 - \frac{3\mu}{q} \int_{\mathbb{R}^3} |u_n|^q dx - \frac{1}{2} \int_{\mathbb{R}^3} |u_n|^6 dx = o_n(1). \end{aligned} \quad (3.14)$$

Combining (3.13) and (3.14), we obtain

$$\int_{\mathbb{R}^3} |\nabla u_n|^2 dx = \frac{6}{5} \int_{\mathbb{R}^3} \phi_{u_n} |u_n|^5 dx + 3\mu \left(\frac{1}{2} - \frac{1}{q} \right) \int_{\mathbb{R}^3} |u_n|^q dx + \int_{\mathbb{R}^3} |u_n|^6 dx + o_n(1). \quad (3.15)$$

If $|u_n|_q^q \rightarrow 0$, $|u_n|_6^6 \rightarrow 0$, and $\int_{\mathbb{R}^3} \phi_{u_n} |u_n|^5 dx \rightarrow 0$, then from (3.15), we get

$$\int_{\mathbb{R}^3} |\nabla u_n|^2 dx \rightarrow 0.$$

Consequently, it follows from (3.8) that

$$J_{w,T}(u_n) \rightarrow \frac{w}{2}a_1^2 \geq 0,$$

which is a contradiction. Hence, the claim holds.

We obtain that for large $n \in \mathbb{N}$,

$$c_{w,T,a_2} \leq \zeta^2 J_{w,T}(u_n) - C \quad (C > 0).$$

By (i) of Lemma 3.6 and letting $n \rightarrow +\infty$, we have

$$c_{w,T,a_2} < \zeta^2 c_{w,T,a_1},$$

which implies that

$$\frac{a_1^2}{a_2^2} c_{w,T,a_2} < c_{w,T,a_1},$$

and the conclusion follows. $\square$

Next, we establish the following compactness result for the truncated autonomous functional restricted to the constraint $S(a)$. This result will play a crucial role in proving the existence of minimizers.

Proposition 3.7. *Let $\{u_n\} \subset S(a)$ be a minimizing sequence with respect to $c_{w,T,a}$. Then, going if necessary to a subsequence, either*

- (i) $\{u_n\}$ is strongly convergent in $H^1(\mathbb{R}^3)$, or
- (ii) there exists $\{y_n\} \subset \mathbb{R}^3$ with $|y_n| \rightarrow \infty$ such that the sequence $\tilde{u}_n(x) = u_n(x + y_n)$ is strongly convergent to a function $\tilde{u} \in S(a)$ with $J_{w,T}(\tilde{u}) = c_{w,T,a}$.

Proof. By Lemma 3.1, $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Thus, there exists $u \in H^1(\mathbb{R}^3)$ such that, up to a subsequence, $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$.

We consider the following three cases.

Case 1. If $u \not\equiv 0$ and $|u|_2^2 = b^2 \neq a^2$, then by the weak lower semicontinuity of the norm, we have $b \in (0, a)$. Define $v_n := u_n - u$ and let $d_n := |v_n|_2^2$. By the Brézis-Lieb lemma and Fatou's lemma, we have

$$|u_n|_2^2 = |v_n|_2^2 + |u|_2^2 + o_n(1), \quad |\nabla u_n|_2^2 = |\nabla v_n|_2^2 + |\nabla u|_2^2 + o_n(1),$$

and $|\nabla u|_2^2 \leq \liminf_{n \rightarrow \infty} |\nabla u_n|_2^2$.

Suppose that $|v_n|_2^2 \rightarrow d^2$, then $d^2 = a^2 - b^2$. Since $d_n \in (0, a)$ for n sufficiently large, and combining the properties of τ with Lemma 3.6, we deduce that

$$\begin{aligned} c_{w,T,a} + o_n(1) &= J_{w,T}(u_n) \\ &= \frac{1}{2}|\nabla v_n|_2^2 + \frac{1}{2}w|v_n|_2^2 - \frac{\mu}{q}|v_n|_q^q - \left(\frac{1}{6}|v_n|_6^6 + \frac{1}{10} \int_{\mathbb{R}^3} \phi_{v_n} |v_n|^5 dx \right) \tau(|\nabla u_n|_2) \\ &\quad + \frac{1}{2}|\nabla u|_2^2 + \frac{1}{2}w|u|_2^2 - \frac{\mu}{q}|u|_q^q - \left(\frac{1}{6}|u|_6^6 + \frac{1}{10} \int_{\mathbb{R}^3} \phi_u |u|^5 dx \right) \tau(|\nabla u_n|_2) + o_n(1) \\ &\geq J_{w,T}(v_n) + J_{w,T}(u) + o_n(1) \\ &\geq c_{w,T,d_n} + c_{w,T,b} + o_n(1) \\ &\geq \frac{d_n^2}{a^2} c_{w,T,a} + c_{w,T,b} + o_n(1). \end{aligned}$$

Letting $n \rightarrow +\infty$, we obtain

$$\begin{aligned} c_{w,T,a} &\geq \frac{d^2}{a^2} c_{w,T,a} + c_{w,T,b} \\ &> \frac{d^2}{a^2} c_{w,T,a} + \frac{b^2}{a^2} c_{w,T,a} = c_{w,T,a}, \end{aligned}$$

which is a contradiction. Therefore, $|u|_2^2 = a^2$, i.e., $u \in S(a)$.

Case 2. If $|u_n|_2^2 = |u|_2^2 = a^2$, then we have $u_n \rightarrow u$ in $L^2(\mathbb{R}^3)$. Combined with interpolation in Lebesgue spaces, this gives

$$u_n \rightarrow u \text{ in } L^t(\mathbb{R}^3) \text{ for all } t \in (2, 2^*). \quad (3.16)$$

Note that $|\nabla v_n|_2 \leq |\nabla u_n|_2 < R_0$ for n sufficiently large. Using (2.4), (2.6), and the fact that $h_a(R_0) = 0$, we obtain

$$\begin{aligned} J_{w,T}(v_n) &= J_w(v_n) = \frac{1}{2} |\nabla v_n|_2^2 + \frac{1}{2} w |v_n|_2^2 - \frac{\mu}{q} |v_n|_q^q - \frac{1}{6} |v_n|_6^6 - \frac{1}{10} \int_{\mathbb{R}^3} \phi_{v_n} |v_n|^5 dx \\ &\geq \frac{1}{2} |\nabla v_n|_2^2 - \frac{1}{6S^3} |\nabla v_n|_2^6 - \frac{1}{10S_{H,L}^5} |\nabla v_n|_2^{10} + o_n(1) \\ &= |\nabla v_n|_2^2 \left(\frac{1}{2} - \frac{1}{6S^3} |\nabla v_n|_2^4 - \frac{1}{10S_{H,L}^5} |\nabla v_n|_2^8 \right) + o_n(1) \\ &\geq |\nabla v_n|_2^2 \left(\frac{1}{2} - \frac{1}{6S^3} R_0^4 - \frac{1}{10S_{H,L}^5} R_0^8 \right) + o_n(1) \\ &= |\nabla v_n|_2^2 \frac{\mu}{q} C_q a^{\frac{6-q}{2}} R_0^{\frac{3q-10}{2}} + o_n(1). \end{aligned} \quad (3.17)$$

In view of

$$J_{w,T}(v_n) + J_{w,T}(u) + o_n(1) \leq J_{w,T}(u_n) \rightarrow c_{w,T,a}, \quad (3.18)$$

and since $u \in S(a)$, we have $J_{w,T}(u) \geq c_{w,T,a}$. Then from (3.17) and (3.18), it follows that $|\nabla v_n|_2^2 \rightarrow 0$. Hence, we infer that $u_n \rightarrow u$ in $H^1(\mathbb{R}^3)$, which implies that (i) holds.

Case 3. If $u \equiv 0$, i.e., $u_n \rightarrow 0$ in $H^1(\mathbb{R}^3)$, we claim that there exist $R, \beta > 0$ and a sequence $\{y_n\} \subset \mathbb{R}^3$ such that

$$\int_{B_R(y_n)} |u_n|^2 dx \geq \beta, \quad \text{for all } n. \quad (3.19)$$

Indeed, otherwise by Lions' lemma, $u_n \rightarrow 0$ in $L^t(\mathbb{R}^3)$ for all $t \in (2, 2^*)$. Similar to (3.17), we have

$$J_{w,T}(u_n) = |\nabla u_n|_2^2 \frac{\mu}{q} C_q a^{\frac{6-q}{2}} R_0^{\frac{3q-10}{2}} + o_n(1).$$

This contradicts $J_{w,T}(u_n) \rightarrow c_{w,T,a} < 0$. Hence, (3.19) holds and clearly $|y_n| \rightarrow +\infty$. Now consider $\bar{u}_n(x) = u_n(x + y_n)$. Then $\{\bar{u}_n\} \subset S(a)$ and it is also a minimizing sequence for $c_{w,T,a}$. Moreover, there exists $\bar{u} \in H^1(\mathbb{R}^3) \setminus \{0\}$ such that $\bar{u}_n \rightarrow \bar{u}$ in $H^1(\mathbb{R}^3)$. Following the same arguments as in Cases 1 and 2, we conclude that $\bar{u}_n \rightarrow \bar{u}$ in $H^1(\mathbb{R}^3)$, which implies that (ii) occurs. This completes the proof. $\square$

Lemma 3.8. *The minimum value $c_{w,T,a} < 0$ is attained.*

Proof. By Lemma 3.1, there exists a bounded minimizing sequence $\{u_n\} \subset S(a)$ such that

$$J_{w,T}(u_n) \rightarrow c_{w,T,a} \text{ as } n \rightarrow +\infty.$$

By Proposition 3.7, passing to a subsequence if necessary, there exists $u \in S(a)$ such that $u_n \rightarrow u$ in $H^1(\mathbb{R}^3)$. Since $J_{w,T}$ is continuous, it follows that

$$J_{w,T}(u) = \lim_{n \rightarrow \infty} J_{w,T}(u_n) = c_{w,T,a}.$$

Thus, u attains the minimum value $c_{w,T,a}$. $\square$

Remark 3.9. From Lemma 3.8 we know that the minimizing sequence $\{u_n\}$ constructed satisfying $J_{w,T}(u_n) < 0$ for sufficiently large n , and by Lemma 3.3(ii), we have that $|\nabla u_n|_2 < R_0$, and so, $\tau(|\nabla u_n|_2) = 1$. Consequently, $J_{w,T}(u_n) = J_w(u_n)$ for large n , and in the limit, $J_w(u) = J_{w,T}(u) = c_{w,T,a}$.

4. Existence and multiplicity of solutions

4.1. Existence of solution for (1.11)

In this subsection, we first study the properties of the truncated functional $J_{\varepsilon,T}$ defined in (3.3) restricted to the L^2 -sphere $S(a)$, and then establish the existence of solutions to problem (1.11).

We introduce the following auxiliary functionals:

$$J_{0,T}(u) := \frac{1}{2}|\nabla u|_2^2 - \frac{\mu}{q}|u|_q^q - \left(\frac{1}{6}|u|_6^6 + \frac{1}{10} \int_{\mathbb{R}^3} \phi_u |u|^5 dx \right) \tau(|\nabla u|_2), \quad (4.1)$$

and

$$J_{\infty,T}(u) := \frac{1}{2}|\nabla u|_2^2 + \frac{V_\infty}{2}|u|_2^2 - \frac{\mu}{q}|u|_q^q - \left(\frac{1}{6}|u|_6^6 + \frac{1}{10} \int_{\mathbb{R}^3} \phi_u |u|^5 dx \right) \tau(|\nabla u|_2). \quad (4.2)$$

The corresponding minimum values are defined by

$$c_{0,T,a} := \inf_{u \in S(a)} J_{0,T}(u), \quad c_{\infty,T,a} := \inf_{u \in S(a)} J_{\infty,T}(u).$$

By assumptions (V1)–(V2) and Lemma 3.8, if $V_\infty < V_*$, then both $c_{0,T,a}$ and $c_{\infty,T,a}$ are attained. That is, there exist $u_0, u_\infty \in S(a)$ such that

$$J_{0,T}(u_0) = c_{0,T,a} \quad \text{and} \quad J_{\infty,T}(u_\infty) = c_{\infty,T,a}.$$

Moreover, since $V_0 < V_\infty$, we have

$$c_{0,T,a} < J_{0,T}(u_\infty) < J_{\infty,T}(u_\infty) = c_{\infty,T,a}. \quad (4.3)$$

Now we consider the minimum value for the perturbed functional:

$$c_{\varepsilon,T,a} := \inf_{u \in S(a)} J_{\varepsilon,T}(u).$$

By the properties of $g_{a,T}$ and Lemma 3.1, $c_{\varepsilon,T,a}$ is well-defined.

The following lemma establishes important relationships between the energy levels $c_{\varepsilon,T,a}$, $c_{0,T,a}$, and $c_{\infty,T,a}$.

Lemma 4.1. *There holds that: $\limsup_{\varepsilon \rightarrow 0^+} c_{\varepsilon,T,a} \leq c_{0,T,a} < c_{\infty,T,a}$.*

Proof. The inequality $c_{0,T,a} < c_{\infty,T,a}$ is obvious. By Lemma 3.8, choose $u_0 \in S(a)$ such that $J_{0,T}(u_0) = c_{0,T,a}$. Then,

$$\begin{aligned} c_{\varepsilon,T,a} &\leq J_{\varepsilon,T}(u_0) \\ &= \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u_0|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} V(\varepsilon x) |u_0|^2 dx - \frac{\mu}{q} \int_{\mathbb{R}^3} |u_0|^q dx \\ &\quad - \left(\frac{1}{6} |u_0|_6^6 + \frac{1}{10} \int_{\mathbb{R}^3} \phi_{u_0} |u_0|^5 dx \right) \tau(|\nabla u_0|_2). \end{aligned}$$

Letting $\varepsilon \rightarrow 0^+$ and applying the Lebesgue dominated convergence theorem, we obtain

$$\limsup_{\varepsilon \rightarrow 0^+} c_{\varepsilon, T, a} \leq \limsup_{\varepsilon \rightarrow 0^+} J_{\varepsilon, T}(u_0) = J_{0, T}(u_0) = c_{0, T, a},$$

which completes the proof. $\square$

From Lemma 4.1, there exists $\varepsilon_0 > 0$ such that $c_{\varepsilon, T, a} < c_{\infty, T, a}$ for all $\varepsilon \in (0, \varepsilon_0)$. Arguing as in the proof of Lemma 3.1, we can prove that $J_{\varepsilon, T}$ is coercive on $S(a)$, that is, $J_{\varepsilon, T}(u_n) \rightarrow \infty$ as $|\nabla u_n|_2 \rightarrow \infty$. Then, the sequence $\{|\nabla u_n|_2\}$ is bounded, and $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Thereby, up to a subsequence, there exists $u_\varepsilon \in H^1(\mathbb{R}^3)$ such that

$$u_n \rightharpoonup u_\varepsilon \text{ in } H^1(\mathbb{R}^3) \quad \text{and} \quad u_n(x) \rightarrow u_\varepsilon(x) \text{ a.e. in } \mathbb{R}^3.$$

By a similar argument to the proof of Lemma 3.3, we can establish the next result, whose proof is omitted.

Lemma 4.2. *We have the following conclusions:*

- (i) $J_{\varepsilon, T} \in C^1(H^1(\mathbb{R}^3), \mathbb{R})$.
- (ii) If $J_{\varepsilon, T}(u) < 0$, then $|\nabla u|_2 < R_0$ and $J_{\varepsilon, T}(v) = J_\varepsilon(v)$ for all v in a small neighborhood of u in $H^1(\mathbb{R}^3)$.

The following two lemmas are crucial in proving the Palais-Smale condition for $J_{\varepsilon, T}$ restricted to $S(a)$ at energy levels $c < c_{\infty, T, a} < 0$.

Lemma 4.3. *Assume that (V1)–(V2) hold, and let $\{u_n\}$ be a sequence such that $u_n \rightharpoonup u_\varepsilon$ in $H^1(\mathbb{R}^3)$. If $J_{\varepsilon, T}(u_n) \rightarrow c < c_{\infty, T, a}$, then $u_\varepsilon \not\equiv 0$.*

Proof. We proceed by contradiction. Suppose $u_\varepsilon \equiv 0$. Then

$$c + o_n(1) = J_{\varepsilon, T}(u_n) = J_{\infty, T}(u_n) + \frac{1}{2} \int_{\mathbb{R}^3} (V(\varepsilon x) - V_\infty) |u_n|^2 dx.$$

By (V1)–(V2), for any $\delta > 0$, there exists $R > 0$ such that

$$V(x) \geq V_\infty - \delta \quad \text{for all } |x| \geq R.$$

Therefore, we see that

$$\begin{aligned} c + o_n(1) &= J_{\varepsilon, T}(u_n) \\ &\geq J_{\infty, T}(u_n) + \frac{1}{2} \int_{B_{R/\varepsilon}(0)} (V(\varepsilon x) - V_\infty) |u_n|^2 dx - \frac{\delta}{2} \int_{B_{R/\varepsilon}^c(0)} |u_n|^2 dx. \end{aligned}$$

Since $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$ and $u_n \rightarrow 0$ in $L^2(B_{R/\varepsilon}(0))$, we infer to

$$\int_{B_{R/\varepsilon}(0)} |u_n|^2 dx = o_n(1).$$

Hence,

$$c + o_n(1) \geq J_{\infty, T}(u_n) - \delta C \geq c_{\infty, T, a} - \delta C,$$

for some constant $C > 0$. Since $\delta > 0$ is arbitrary, letting $\delta \rightarrow 0^+$ yields $c \geq c_{\infty, T, a}$, a contradiction. Thus, $u_\varepsilon \not\equiv 0$. $\square$

Lemma 4.4. *Let $\{u_n\} \subset S(a)$ be a $(PS)_c$ sequence of $J_{\varepsilon, T}$ with $c < c_{\infty, T, a} < 0$. If $u_n \rightharpoonup u_\varepsilon$ in $H^1(\mathbb{R}^3)$, then there exists $\beta > 0$ independent of $\varepsilon \in (0, \varepsilon_0)$ such that*

$$\liminf_{n \rightarrow +\infty} |u_n - u_\varepsilon|_2 > \beta.$$

Proof. By (ii) of Lemma 4.2 and $c < c_{\infty, T, a} < 0$, we have $|\nabla u_n|_2 < R_0$, so $\{u_n\}$ is also a $(PS)_c$ sequence for J_ε , i.e.,

$$J_\varepsilon(u_n) \rightarrow c, \quad \|(J_\varepsilon|_{S(a)})'(u_n)\| \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

Define the functional $\Psi : H^1(\mathbb{R}^3) \rightarrow \mathbb{R}$ by

$$\Psi(u) = \frac{1}{2} \int_{\mathbb{R}^3} |u|^2 dx.$$

Note that $S(a) = \Psi^{-1}(a^2/2)$. By Willem [52], there exists $\{\lambda_n\} \subset \mathbb{R}$ such that

$$\|J'_\varepsilon(u_n) - \lambda_n \Psi'(u_n)\|_{H^{-1}(\mathbb{R}^3)} \rightarrow 0 \quad \text{as } n \rightarrow +\infty, \quad (4.4)$$

which states that,

$$-\Delta u_n + V(\varepsilon x)u_n - \mu|u_n|^{q-2}u_n - |u_n|^4u_n - \phi_{u_n}|u_n|^3u_n = \lambda_n u_n + o_n(1), \quad \text{in } H^{-1}(\mathbb{R}^3). \quad (4.5)$$

Since $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$ and $u_n \rightharpoonup u_\varepsilon$, it follows from (4.5) that $\{\lambda_n\}$ is bounded. Passing to a subsequence, we have $\lambda_n \rightarrow \lambda_\varepsilon$. Together with (4.4), we can derive that u_ε is a nontrivial solution of

$$-\Delta u + V(\varepsilon x)u - \mu|u|^{q-2}u - |u|^4u - \phi_u|u|^3u = \lambda_\varepsilon u \quad \text{in } \mathbb{R}^3.$$

Therefore,

$$J'_\varepsilon(u_\varepsilon) - \lambda_\varepsilon \Psi'(u_\varepsilon) = 0 \quad \text{in } H^{-1}(\mathbb{R}^3),$$

and combining with (4.5) we can infer that

$$\|J'_\varepsilon(v_n) - \lambda_\varepsilon \Psi'(v_n)\|_{H^{-1}(\mathbb{R}^3)} \rightarrow 0 \quad \text{as } n \rightarrow +\infty, \quad (4.6)$$

here $v_n := u_n - u_\varepsilon$. Furthermore, by (4.4) we have

$$\begin{aligned} 0 > c_{\infty, T, a} &> \lim_{n \rightarrow +\infty} J_\varepsilon(u_n) \\ &= \lim_{n \rightarrow +\infty} \left(J_\varepsilon(u_n) - \frac{1}{2} \langle J'_\varepsilon(u_n), u_n \rangle + \frac{1}{2} \lambda_n |u_n|_2^2 + o_n(1) \right) \\ &= \lim_{n \rightarrow +\infty} \left[\left(\frac{1}{2} - \frac{1}{q} \right) \mu |u_n|_q^q + \frac{1}{3} |u_n|_6^6 + \frac{2}{5} \int_{\mathbb{R}^3} \phi_{u_n} |u_n|^5 dx + \frac{1}{2} \lambda_n a^2 + o_n(1) \right] \\ &\geq \frac{1}{2} \lambda_\varepsilon a^2, \end{aligned} \quad (4.7)$$

which implies $\lambda_\varepsilon < 0$ for all $\varepsilon \in (0, \varepsilon_0)$.

By (4.6), we have

$$|\nabla v_n|_2^2 + \int_{\mathbb{R}^3} V(\varepsilon x) |v_n|^2 dx - \lambda_\varepsilon |v_n|_2^2 = \mu |v_n|_q^q + |v_n|_6^6 + \int_{\mathbb{R}^3} \phi_{u_n} |v_n|^5 dx + o_n(1). \quad (4.8)$$

Combining with (4.7), we get

$$|\nabla v_n|_2^2 + \int_{\mathbb{R}^3} V(\varepsilon x) |v_n|^2 dx - \frac{2c_{\infty, T, a}}{a^2} |v_n|_2^2 \leq \mu |v_n|_q^q + |v_n|_6^6 + \int_{\mathbb{R}^3} \phi_{v_n} |v_n|^5 dx + o_n(1). \quad (4.9)$$

By (4.9) and the Sobolev inequalities (2.4), (2.6), there exist constants $C_1, C_2 > 0$ such that

$$\begin{aligned} C\|v_n\|^2 &\leq |\nabla v_n|_2^2 - \frac{2c_{\infty,T,a}}{a^2}|v_n|_2^2 \\ &\leq \mu|v_n|_q^q + |v_n|_6^6 + \int_{\mathbb{R}^3} \phi_{v_n}|v_n|^5 dx + o_n(1) \\ &\leq \mu C_q \|v_n\|^q + \frac{1}{S^3} |\nabla v_n|_2^6 + \frac{1}{S_{H,L}^5} |\nabla v_n|_2^{10} + o_n(1) \\ &\leq \mu C_q \|v_n\|^q + C_1 \|v_n\|^6 + C_2 \|v_n\|^{10} + o_n(1). \end{aligned} \quad (4.10)$$

If $u_n \rightharpoonup u_\varepsilon$ in $H^1(\mathbb{R}^3)$, i.e., $v_n \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$, then there exists $C > 0$ independent of ε , up to a subsequence, such that $\|v_n\| \geq C$. Thereby, we have

$$\liminf_{n \rightarrow +\infty} \left(\mu|v_n|_q^q + |v_n|_6^6 + \int_{\mathbb{R}^3} \phi_{v_n}|v_n|^5 dx \right) \geq C \quad (4.11)$$

for some $C > 0$.

If $\liminf_{n \rightarrow +\infty} |v_n|_q^q \geq C_3$ for some $C_3 > 0$ independent of ε , then by (2.1) and (4.11), there exists $\beta > 0$ independent of $\varepsilon \in (0, \varepsilon_0)$ such that

$$\liminf_{n \rightarrow +\infty} |v_n|_2 \geq \beta.$$

Now, if

$$\liminf_{\varepsilon \rightarrow 0^+} \limsup_{n \rightarrow +\infty} |v_n|_q^q = 0 \quad \text{and} \quad \liminf_{\varepsilon \rightarrow 0^+} \limsup_{n \rightarrow +\infty} |v_n|_2 = 0,$$

then by (4.11), we see that

$$\liminf_{\varepsilon \rightarrow 0^+} \limsup_{n \rightarrow +\infty} \left(|v_n|_6^6 + \int_{\mathbb{R}^3} \phi_{v_n}|v_n|^5 dx \right) \geq C.$$

Together with (4.8), this gives

$$\liminf_{\varepsilon \rightarrow 0^+} \limsup_{n \rightarrow +\infty} |\nabla v_n|_2^2 = \liminf_{\varepsilon \rightarrow 0^+} \limsup_{n \rightarrow +\infty} \left(|v_n|_6^6 + \int_{\mathbb{R}^3} \phi_{v_n}|v_n|^5 dx \right).$$

From (2.4) and (2.6), we obtain

$$\liminf_{\varepsilon \rightarrow 0^+} \limsup_{n \rightarrow +\infty} |\nabla v_n|_2^2 \leq \liminf_{\varepsilon \rightarrow 0^+} \limsup_{n \rightarrow +\infty} \left(\frac{1}{S^3} |\nabla v_n|_2^6 + \frac{1}{S_{H,L}^5} |\nabla v_n|_2^{10} \right),$$

which implies

$$R_0 \geq \liminf_{\varepsilon \rightarrow 0^+} \limsup_{n \rightarrow +\infty} |\nabla v_n|_2 \geq \left(\frac{\sqrt{S^{-6} + 4S_{H,L}^{-5}} - S^{-3}}{2S_{H,L}^{-5}} \right)^{\frac{1}{4}},$$

contradicting the assumption $R_0 < \left(\frac{\sqrt{S^{-6} + 4S_{H,L}^{-5}} - S^{-3}}{2S_{H,L}^{-5}} \right)^{\frac{1}{4}}$. This completes the proof. $\square$

At this time, we choose a constant $\rho_0 > 0$ satisfying

$$0 < \rho_0 < \min \left\{ c_{\infty,T,a} - c_{0,T,a}, \frac{\beta^2}{a^2} (c_{\infty,T,a} - c_{0,T,a}) \right\},$$

and have the following compactness result.

Lemma 4.5. *If conditions (V1)-(V2) are satisfied, then $J_{\varepsilon,T}$ satisfies the Palais-Smale condition at level c on $S(a)$ for any $c < c_{0,T,a} + \rho_0$.*

Proof. Let $\{u_n\} \subset S(a)$ be a Palais-Smale sequence for $J_{\varepsilon,T}$ at level c . Then $c < c_{\infty,T,a} < 0$, and by Lemma 4.2 (ii), $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Therefore, passing to a subsequence, we have $u_n \rightharpoonup u_\varepsilon$ in $H^1(\mathbb{R}^3)$, and by Lemma 4.3, $u_\varepsilon \not\equiv 0$.

If $u_n \rightarrow u_\varepsilon$ in $H^1(\mathbb{R}^3)$, the conclusion holds. Now suppose $u_n \not\rightarrow u_\varepsilon$ in $H^1(\mathbb{R}^3)$. Then by Lemma 4.4, for $v_n := u_n - u_\varepsilon$, one has

$$\liminf_{n \rightarrow +\infty} |v_n|_2 \geq \beta.$$

Let $b^2 = |u_\varepsilon|_2^2$, $d_n^2 = |v_n|_2^2$, and assume that $d_n^2 \rightarrow d^2$. Then $d^2 \geq \beta^2 > 0$, and by the Brézis-Lieb lemma [52], we have $a^2 = b^2 + d^2$.

On the other hand, by (V1)-(V2), for any $\tau > 0$, there exists $R > 0$ such that

$$V(x) \geq V_\infty - \tau \quad \text{for all } |x| \geq R.$$

Hence, we have

$$\begin{aligned} c + o_n(1) &= J_{\varepsilon,T}(u_n) \\ &\geq J_{\infty,T}(v_n) + \frac{1}{2} \int_{B_{R/\varepsilon}^c(0)} (V(\varepsilon x) - V_\infty) |v_n|^2 dx - \frac{\tau}{2} \int_{B_{R/\varepsilon}^c(0)} |v_n|^2 dx. \end{aligned}$$

Since $\{v_n\}$ is bounded and $v_n \rightarrow 0$ in $L^2(B_{R/\varepsilon}(0))$, there exists $C > 0$ such that

$$c + o_n(1) \geq J_{\infty,T}(v_n) - \tau C \geq c_{\infty,T,d_n} - \tau C. \quad (4.12)$$

As $\tau > 0$ is arbitrary, we infer that $J_{\varepsilon,T}(u_n) \geq c_{\infty,T,d_n} + o_n(1)$. Since $d_n \in (0, a)$ for large n , we have

$$\begin{aligned} c + o_n(1) &= J_{\varepsilon,T}(u_n) \\ &\geq J_{\varepsilon,T}(v_n) + J_{\varepsilon,T}(u_\varepsilon) \\ &\geq c_{\infty,T,d_n} + c_{0,T,b} + o_n(1). \end{aligned} \quad (4.13)$$

From Lemma 3.6 and (4.13), it follows that

$$c_{0,T,a} + \rho_0 \geq c + o_n(1) \geq \frac{d_n^2}{a^2} c_{\infty,T,a} + \frac{b^2}{a^2} c_{0,T,a}.$$

Taking the limit as $n \rightarrow +\infty$, we obtain

$$\rho_0 \geq \frac{d^2}{a^2} (c_{\infty,T,a} - c_{0,T,a}) \geq \frac{\beta^2}{a^2} (c_{\infty,T,a} - c_{0,T,a}),$$

contradicting the choice of ρ_0 . Therefore, $u_n \rightarrow u_\varepsilon$ in $H^1(\mathbb{R}^3)$, and the Palais-Smale condition holds. $\square$

4.2. Multiplicity of solution for (1.11)

In this subsection, we shall complete the proof of Theorem 1.1. Let u_0 be a positive solution of the autonomous problem

$$\begin{cases} -\Delta u - \phi |u|^3 u = \lambda u + \mu |u|^{q-2} u + |u|^4 u, & x \in \mathbb{R}^3, \\ -\Delta \phi = |u|^5, & x \in \mathbb{R}^3, \\ \int_{\mathbb{R}^3} |u|^2 dx = a^2, \end{cases} \quad (4.14)$$

with $J_{0,T}(u_0) = c_{0,T,a}$. Let $\delta > 0$ be fixed and η be a smooth nonincreasing cut-off function on $[0, +\infty)$ such that

$$\eta(s) = \begin{cases} 1 & \text{if } 0 \leq s \leq \frac{\delta}{2}, \\ 0 & \text{if } s \geq \delta. \end{cases}$$

For any $y \in M$, we define

$$\varphi_{\varepsilon,y}(x) = \eta(|\varepsilon x - y|) u_0 \left(\frac{\varepsilon x - y}{\varepsilon} \right),$$

$$\tilde{\varphi}_{\varepsilon,y}(x) = a \frac{\varphi_{\varepsilon,y}(x)}{|\varphi_{\varepsilon,y}|_2},$$

and the map $\Phi_\varepsilon : M \rightarrow S(a)$ by $\Phi_\varepsilon(y) = \tilde{\varphi}_{\varepsilon,y}$. By construction, $\Phi_\varepsilon(y)$ has compact support for each $y \in M$.

Lemma 4.6. *The function Φ_ε satisfies:*

$$\lim_{\varepsilon \rightarrow 0} J_{\varepsilon,T}(\Phi_\varepsilon(y)) = c_{0,T,a}, \quad \text{uniformly in } y \in M.$$

Proof. Suppose by contradiction that, there exist $\delta_0 > 0$, $\{y_n\} \subset M$, and $\varepsilon_n \rightarrow 0$ such that

$$|J_{\varepsilon_n,T}(\Phi_{\varepsilon_n}(y_n)) - c_{0,T,a}| \geq \delta_0, \quad \forall n \in \mathbb{N}.$$

Making the variable change $z := \frac{\varepsilon_n x - y_n}{\varepsilon_n}$, then the Lebesgue dominated convergence theorem implies the following limits:

$$\begin{aligned} \lim_{n \rightarrow +\infty} \int_{\mathbb{R}^3} |\Phi_{\varepsilon_n,y_n}|^2 dx &= \lim_{n \rightarrow +\infty} \int_{\mathbb{R}^3} |\eta(|\varepsilon_n z|) u_0(z)|^2 dx = \int_{\mathbb{R}^3} |u_0|^2 dx = a^2, \\ \lim_{n \rightarrow +\infty} \int_{\mathbb{R}^3} |\nabla \Phi_{\varepsilon_n}(y_n)|^2 dx &= \lim_{n \rightarrow +\infty} \int_{\mathbb{R}^3} \frac{a^2}{|\varphi_{\varepsilon_n,y_n}|_2^2} |\nabla (\eta(|\varepsilon_n z|) u_0(z))|^2 dx = \int_{\mathbb{R}^3} |\nabla u_0|^2 dx, \\ \lim_{n \rightarrow +\infty} \tau(|\nabla \Phi_{\varepsilon_n,y_n}|_2) \int_{\mathbb{R}^3} |\Phi_{\varepsilon_n}(y_n)|^6 dx \\ &= \lim_{n \rightarrow +\infty} \tau(|\nabla u_0|_2) \int_{\mathbb{R}^3} \left| a \frac{\eta(|\varepsilon_n z|) u_0(z)}{|\varphi_{\varepsilon_n,y_n}|_2} \right|^6 dx \\ &= \tau(|\nabla u_0|_2) \int_{\mathbb{R}^3} |u_0|^6 dx, \\ \lim_{n \rightarrow +\infty} \int_{\mathbb{R}^3} V(\varepsilon_n x) |\Phi_{\varepsilon_n}(y_n)|^2 dx &= \lim_{n \rightarrow +\infty} \int_{\mathbb{R}^3} \frac{a^2 V(\varepsilon_n z + y_n)}{|\varphi_{\varepsilon_n,y_n}|_2^2} |\eta(|\varepsilon_n z|) u_0(z)|^2 dx = 0, \\ \lim_{n \rightarrow +\infty} \tau(|\nabla \Phi_{\varepsilon_n}(y_n)|_2) \int_{\mathbb{R}^3} \phi_{\Phi_{\varepsilon_n}(y_n)} |\Phi_{\varepsilon_n}(y_n)|^5 dx &= \lim_{n \rightarrow +\infty} \tau(|\nabla u_0|_2) \int_{\mathbb{R}^3} \phi_{u_0} |u_0|^5 dx, \\ \lim_{n \rightarrow +\infty} \int_{\mathbb{R}^3} |\Phi_{\varepsilon_n}(y_n)|^q dx &= \lim_{n \rightarrow +\infty} \int_{\mathbb{R}^3} \left| a \frac{\eta(|\varepsilon_n z|) u_0(z)}{|\varphi_{\varepsilon_n,y_n}|_2} \right|^q dx = \int_{\mathbb{R}^3} |u_0|^q dx. \end{aligned}$$

Therefore, we derive that

$$\lim_{n \rightarrow +\infty} J_{\varepsilon_n,T}(\Phi_{\varepsilon_n}(y_n)) = J_{0,T}(u_0) = c_{0,T,a},$$

which contradicts the assumption. $\square$

For any $\delta > 0$, let $R = R(\delta) > 0$ be such that $M_\delta \subset B_R(0)$. We define a retraction map $\chi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ by

$$\chi(x) = \begin{cases} x & \text{for } |x| \leq R, \\ \frac{Rx}{|x|} & \text{for } |x| \geq R. \end{cases}$$

Moreover, we introduce the barycenter map $\beta_\varepsilon : S(a) \rightarrow \mathbb{R}^3$ by

$$\beta_\varepsilon(u) := \frac{\int_{\mathbb{R}^3} \chi(\varepsilon x) |u|^2 dx}{a^2}.$$

Lemma 4.7. *The function Φ_ε satisfies:*

$$\lim_{\varepsilon \rightarrow 0} \beta_\varepsilon(\Phi_\varepsilon(y)) = y, \quad \text{uniformly in } y \in M.$$

Proof. We proceed by contradiction. Suppose there exist $\delta_0 > 0$, $\{y_n\} \subset M$, and $\varepsilon_n \rightarrow 0$ such that

$$|\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - y_n| \geq \delta_0, \quad \forall n \in \mathbb{N}. \quad (4.15)$$

By the definitions of $\Phi_{\varepsilon_n}(y_n)$ and β_{ε_n} , we have

$$\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - y_n = \frac{1}{a^2} \int_{\mathbb{R}^3} (\chi(\varepsilon_n z + y_n) - y_n) |\eta(|\varepsilon_n z|) u_0(z)|^2 dx.$$

By virtue of $\{y_n\} \subset M \subset B_R(0)$, and by the Lebesgue dominated convergence theorem, we have

$$|\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - y_n| \rightarrow 0 \quad \text{as } n \rightarrow +\infty,$$

which contradicts (4.15), and the conclusion follows. $\square$

Following the approach used in the proof of Proposition 3.7, we can now establish the following crucial compactness result.

Proposition 4.8. *Let $\varepsilon_n \rightarrow 0^+$ and $\{u_n\} \subset S(a)$ be such that $J_{\varepsilon_n, T}(u_n) \rightarrow c_{0, T, a}$. Then there exists $\{\tilde{y}_n\} \subset \mathbb{R}^3$ such that the sequence $\tilde{u}_n(x) = u_n(x + \tilde{y}_n)$ contains a convergent subsequence in $H^1(\mathbb{R}^3)$. Moreover, passing to a subsequence, $\tilde{y}_n := \varepsilon_n \tilde{y}_n \rightarrow \tilde{y}$ for some $\tilde{y} \in M$.*

Proof. We divide the proof into several steps.

Step 1. Non-vanishing property. We first prove that there exist $R_0, C > 0$ and $\tilde{y}_n \in \mathbb{R}^3$ such that

$$\int_{B_{R_0}(\tilde{y}_n)} |u_n|^2 dx \geq C, \quad \forall n \in \mathbb{N}.$$

Suppose the above inequality does not hold, then by Lions' lemma, we have $u_n \rightarrow 0$ in $L^t(\mathbb{R}^3)$ for all $t \in (2, 6)$. Next, we show that in fact $u_n \rightarrow 0$ in $L^6(\mathbb{R}^3)$ using the concentration-compactness principle. We divide the proof into three steps.

Step 1-(i). Since $|\nabla u_n|_2 \leq R_0$ for n large enough, by Lemma 2.6, there exist three positive measures $\zeta, \omega, \nu \in \mathcal{M}(\mathbb{R}^3)$ such that

$$|\nabla u_n|^2 \rightharpoonup \omega, \quad |u_n|^6 \rightharpoonup \zeta, \quad \text{and} \quad \int_{\mathbb{R}^3} \phi_{u_n} |u_n|^5 dx \rightharpoonup \nu \quad (4.16)$$

in $\mathcal{M}(\mathbb{R}^3)$, as $n \rightarrow \infty$. Take $\theta_R \in C_0^\infty(\mathbb{R}^3)$ to be a cut-off function such that $\theta_R(x) = 1$ in $B_R(x_j)$, $\theta_R(x) = 0$ in $B_{2R}^c(x_j)$, and $|\nabla \theta_R| \leq \frac{C}{R}$. The boundedness of $\{u_n\}$ implies that $\{\theta_R u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Since $(J_{\varepsilon_n, T}|_{S(a)})'(u_n) \rightarrow 0$, by the Lagrange multiplier rule, there exists a sequence $\{\lambda_n\} \subset \mathbb{R}$ such that

$$J'_{\varepsilon_n, T}(u_n) + \lambda_n u_n \rightarrow 0 \quad \text{in } H^{-1}(\mathbb{R}^3). \quad (4.17)$$

Now we take $\theta_R u_n$ as a test function in (4.17) and let $R \rightarrow 0$ to have

$$\lim_{R \rightarrow 0} \lim_{n \rightarrow \infty} \langle J'_{\varepsilon_n, T}(u_n) + \lambda_n u_n, \theta_R u_n \rangle = 0. \quad (4.18)$$

Therefore, by the absolute continuity of the integral and Hölder's inequality, we infer that

$$\lim_{R \rightarrow 0} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} \lambda_n u_n^2 \theta_R dx = 0, \quad (4.19)$$

$$\lim_{R \rightarrow 0} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} V(\varepsilon_n x) u_n^2 \theta_R dx = 0. \quad (4.20)$$

Using the Hölder inequality, we can derive the following limit estimations:

$$\begin{aligned} & \lim_{R \rightarrow 0} \lim_{n \rightarrow \infty} \left| \int_{\mathbb{R}^3} \nabla u_n \nabla \theta_R u_n dx \right| \\ & \leq \lim_{R \rightarrow 0} \lim_{n \rightarrow \infty} \left(\int_{B_{2R}(x_i)} |\nabla \theta_R u_n|^2 dx \right)^{\frac{1}{2}} \left(\int_{B_{2R}(x_i)} |\nabla u_n|^2 dx \right)^{\frac{1}{2}} \\ & \leq C \lim_{R \rightarrow 0} \left(\int_{B_{2R}(x_i)} |\nabla \theta_R u|^2 dx \right)^{\frac{1}{2}} \\ & \leq C_1 \lim_{R \rightarrow 0} \left(\int_{B_{2R}(x_i)} |\nabla \theta_R|^3 dx \right)^{\frac{1}{3}} \left(\int_{B_{2R}(x_i)} |u|^6 dx \right)^{\frac{1}{6}} \\ & \leq C_2 \lim_{R \rightarrow 0} \left(\int_{B_{2R}(x_i)} |u|^6 dx \right)^{\frac{1}{6}} = 0; \end{aligned} \quad (4.21)$$

and

$$\lim_{R \rightarrow 0} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} \theta_R |u_n|^q dx = 0, \quad (4.22)$$

$$\lim_{R \rightarrow 0} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} \theta_R |\nabla u_n|^2 dx = \lim_{R \rightarrow 0} \int_{\mathbb{R}^3} \theta_R d\omega = \omega(\{x_j\}) = \omega_j, \quad (4.23)$$

$$\lim_{R \rightarrow 0} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} \theta_R \phi_{u_n} |u_n|^5 dx = \lim_{R \rightarrow 0} \int_{\mathbb{R}^3} \theta_R d\nu = \nu(\{x_j\}) = \nu_j, \quad (4.24)$$

$$\lim_{R \rightarrow 0} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} \theta_R |u_n|^6 dx = \lim_{R \rightarrow 0} \int_{\mathbb{R}^3} \theta_R d\zeta = \zeta(\{x_j\}) = \zeta_j. \quad (4.25)$$

It follows from (4.18) to (4.25), taking the limits as $n \rightarrow \infty$ and $R \rightarrow 0$, that we get

$$\omega_j = \nu_j + \zeta_j. \quad (4.26)$$

Step 1-(ii). We show that $\omega_\infty = \zeta_\infty + \nu_\infty$, where ω_∞ , ζ_∞ and ν_∞ come from Lemma 2.7-2.8. Define the cut-off function $\psi \in C_0^\infty(\mathbb{R}^3)$ as $\psi \in [0, 1]$, $\psi \equiv 0$ in $B_{1/2}(0)$, $\psi \equiv 1$ in $\mathbb{R}^3 \setminus B_1(0)$. For any fixed $R > 0$, put

$$\psi_R(x) := \psi\left(\frac{x}{R}\right) = \begin{cases} 0, & |x| \leq \frac{1}{2}R, \\ 1, & |x| \geq R. \end{cases}$$

From the boundedness of $\{u_n\}$ in $H^1(\mathbb{R}^3)$, we see that $\{\psi_R u_n\}$ is also bounded in $H^1(\mathbb{R}^3)$. Hence, one has

$$\begin{aligned} o_n(1) &= \langle I'(u_n), \psi_R u_n \rangle \\ &= \int_{\mathbb{R}^3} \psi_R |\nabla u_n|^2 dx - \int_{\mathbb{R}^3} \psi_R \phi_{u_n} |u_n|^5 dx - \mu \int_{\mathbb{R}^3} \psi_R |u_n|^q dx - \int_{\mathbb{R}^3} \psi_R |u_n|^6 dx. \end{aligned} \quad (4.27)$$

By virtue of the definition of ψ_R , we infer to

$$\int_{\{x \in \mathbb{R}^3 : |x| > R\}} |\nabla u_n|^2 dx \leq \int_{\mathbb{R}^3} \psi_R |\nabla u_n|^2 dx \leq \int_{\{x \in \mathbb{R}^3 : |x| > \frac{1}{2}R\}} |\nabla u_n|^2 dx.$$

Therefore, by Lemma 2.7, we have

$$\lim_{R \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} \psi_R |\nabla u_n|^2 dx = \omega_\infty, \quad (4.28)$$

Similarly, we can derive that

$$\lim_{R \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} \psi_R \phi_{u_n} |u_n|^5 dx = \nu_\infty, \quad (4.29)$$

$$\lim_{R \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} \psi_R |u_n|^6 dx = \zeta_\infty, \quad (4.30)$$

and

$$\lim_{R \rightarrow \infty} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} \psi_R |u_n|^q dx = \lim_{R \rightarrow \infty} \int_{\mathbb{R}^3} \psi_R |u|^q dx = \lim_{R \rightarrow \infty} \int_{|x| > \frac{1}{2}R} \psi_R |u|^q dx = 0. \quad (4.31)$$

Consequently, from (4.27)-(4.31), passing the limit as $n \rightarrow \infty$, and $R \rightarrow \infty$, we obtain

$$\omega_\infty = \zeta_\infty + \nu_\infty.$$

Step 1-(iii). We verify that $\zeta_j + \nu_j = 0$ for any $j \in J$ and $\zeta_\infty + \nu_\infty = 0$. Assume by contradiction that, there exists some $j_0 \in J$ such that $\zeta_{j_0} + \nu_{j_0} > 0$ or $\zeta_\infty + \nu_\infty > 0$. By Step 1, Step 2 and Lemma 2.7-2.8 we have

$$\zeta_{j_0} \leq (S^{-1}\omega_{j_0})^3 = S^{-3}(\zeta_{j_0} + \nu_{j_0})^3, \quad \nu_{j_0} \leq (S_{H,L}^{-1}\omega_{j_0})^5 = S_{H,L}^{-5}(\zeta_{j_0} + \nu_{j_0})^5 \quad (4.32)$$

or

$$\zeta_\infty \leq (S^{-1}\omega_\infty)^3 = S^{-3}(\zeta_\infty + \nu_\infty)^3, \quad \nu_\infty \leq S_{H,L}^{-5}\omega_\infty^{\frac{5}{2}} \left(\int_{\mathbb{R}^3} d\omega + \omega_\infty \right)^{\frac{5}{2}} \leq S_{H,L}^{-5}R_1^6(\zeta_\infty + \nu_\infty)^2. \quad (4.33)$$

We first consider that (4.32) occurs, then it leads to $\zeta_{j_0} + \nu_{j_0} \geq \left(\frac{\sqrt{S^{-6} + 4S_{H,L}^{-5}} - S^{-3}}{2S_{H,L}^{-5}} \right)^{\frac{1}{2}}$.

Otherwise, assume that (4.33) holds, by a direct calculation, we deduce to

$$\zeta_\infty + \nu_\infty \geq \frac{\sqrt{S_{H,L}^{-10}R_1^{12} + 4S^{-3}} - S_{H,L}^{-5}R_1^6}{2S^{-3}}.$$

If the former case occurs, we can derive that

$$\begin{aligned} R_1^2 &\geq \lim_{\rho \rightarrow 0} \lim_{n \rightarrow \infty} \|\nabla u_n\|_2^2 \geq \lim_{\rho \rightarrow 0} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} \varphi_\rho |\nabla u_n|^2 dx = \lim_{\rho \rightarrow 0} \int_{\mathbb{R}^3} \varphi_\rho d\omega = \omega_{j_0} \\ &= \zeta_{j_0} + \nu_{j_0} \geq \left(\frac{\sqrt{S^{-6} + 4S_{H,L}^{-5}} - S^{-3}}{2S_{H,L}^{-5}} \right)^{\frac{1}{2}}, \end{aligned}$$

which contradicts with (3.6). If the last case occurs, then we get

$$R_1^2 \geq \lim_{R \rightarrow \infty} \lim_{n \rightarrow \infty} \|\nabla u_n\|_2^2 \geq \omega_\infty = \zeta_\infty + \nu_\infty \geq \frac{\sqrt{S_{H,L}^{-10} R_1^{12} + 4S^{-3}} - S_{H,L}^{-5} R_1^6}{2S^{-3}},$$

which implies that $R_1^2 \geq \left(\frac{\sqrt{S^{-6} + 4S_{H,L}^{-5}} - S^{-3}}{2S_{H,L}^{-5}} \right)^{\frac{1}{2}}$, which contradicts to (3.6). Thus, we prove that $\zeta_j + \nu_j = 0$

for any $j \in J$ and $\zeta_\infty + \nu_\infty = 0$, i.e. $\zeta_j = \nu_j = 0$ for any $j \in J$ and $\zeta_\infty = \nu_\infty = 0$. These arguments show that $u_n \rightarrow 0$ in $L^6(\mathbb{R}^3)$ and $\int_{\mathbb{R}^3} \phi_{u_n} |u_n|^5 dx \rightarrow 0$. Combined with the convergence of other terms: $|u_n|_q^q \rightarrow 0$ for $q \in (2, 6)$ by the compact embedding, and $\int_{\mathbb{R}^3} V(\varepsilon_n x) |u_n|^2 dx \rightarrow 0$ by the boundedness of V , we arrive at

$$\lim_{n \rightarrow \infty} J_{\varepsilon_n, T}(u_n) = \lim_{n \rightarrow \infty} \left(\frac{1}{2} \|\nabla u_n\|_2^2 + o_n(1) \right) \geq 0,$$

which implies a contradiction with $c_{0,T,a} < 0$. So, the non-vanishing characteristic holds.

Step 2. Translation and convergence. Define $\tilde{u}_n(x) = u_n(x + \tilde{y}_n)$. Then $\{\tilde{u}_n\} \subset S(a)$ and is bounded in $H^1(\mathbb{R}^3)$. Passing to a subsequence, $\tilde{u}_n \rightharpoonup \tilde{u}$ in $H^1(\mathbb{R}^3)$ for some $\tilde{u} \in H^1(\mathbb{R}^3)$, with $\tilde{u} \not\equiv 0$ by the non-vanishing property.

We now prove strong convergence. Since $\{u_n\} \subset S(a)$ and

$$c_{0,T,a} \leq J_{0,T}(\tilde{u}_n) = J_{0,T}(u_n) \leq J_{\varepsilon_n, T}(u_n) + o_n(1) \rightarrow c_{0,T,a},$$

which implies that $J_{0,T}(\tilde{u}_n)$. By Proposition 3.7, $\tilde{u}_n \rightarrow \tilde{u}$ in $H^1(\mathbb{R}^3)$ and $\tilde{u} \in S(a)$.

Step 3. Boundedness of the translation sequence. Let $\hat{y}_n = \varepsilon_n \tilde{y}_n$. We claim that $\{\hat{y}_n\}$ is bounded. Suppose by contradiction that, for some subsequence, $|\hat{y}_n| \rightarrow +\infty$.

By condition (V2), for any $\delta > 0$, there exists $R > 0$ such that $V(x) \geq V_\infty - \delta$ for all $|x| \geq R$. For large n , we have $|\hat{y}_n| \geq 2R$, and for $x \in B_{R/\varepsilon_n}(0)$, we have $|\varepsilon_n x + \hat{y}_n| \geq |\hat{y}_n| - \varepsilon_n |x| \geq R$, so,

$$V(\varepsilon_n x + \hat{y}_n) \geq V_\infty - \delta.$$

Therefore,

$$\begin{aligned} \int_{\mathbb{R}^3} V(\varepsilon_n x + \hat{y}_n) |\tilde{u}_n|^2 dx &= \int_{B_{R/\varepsilon_n}(0)} V(\varepsilon_n x + \hat{y}_n) |\tilde{u}_n|^2 dx + \int_{B_{R/\varepsilon_n}^c(0)} V(\varepsilon_n x + \hat{y}_n) |\tilde{u}_n|^2 dx \\ &\geq (V_\infty - \delta) \int_{B_{R/\varepsilon_n}(0)} |\tilde{u}_n|^2 dx - |V|_\infty \int_{B_{R/\varepsilon_n}^c(0)} |\tilde{u}_n|^2 dx. \end{aligned}$$

Since $\tilde{u}_n \rightarrow \tilde{u}$ strongly in $L^2(\mathbb{R}^3)$, we have,

$$\int_{B_{R/\varepsilon_n}(0)} |\tilde{u}_n|^2 dx \rightarrow |\tilde{u}|_2^2 = a^2 \quad \text{and} \quad \int_{B_{R/\varepsilon_n}^c(0)} |\tilde{u}_n|^2 dx \rightarrow 0.$$

Therefore, we get,

$$\liminf_{n \rightarrow \infty} \int_{\mathbb{R}^3} V(\varepsilon_n x + \widehat{y}_n) |\widetilde{u}_n|^2 dx \geq (V_\infty - \delta) a^2.$$

Since $\delta > 0$ is arbitrary, we obtain,

$$\liminf_{n \rightarrow \infty} \int_{\mathbb{R}^3} V(\varepsilon_n x + \widehat{y}_n) |\widetilde{u}_n|^2 dx \geq V_\infty a^2.$$

Now, using the strong convergence $\widetilde{u}_n \rightarrow \widetilde{u}$ in $H^1(\mathbb{R}^3)$ and the continuity of the nonlocal terms, we have,

$$\begin{aligned} c_{0,T,a} &= \lim_{n \rightarrow \infty} J_{\varepsilon_n,T}(u_n) \\ &= \lim_{n \rightarrow \infty} \left(\frac{1}{2} |\nabla \widetilde{u}_n|_2^2 + \frac{1}{2} \int_{\mathbb{R}^3} V(\varepsilon_n x + \widehat{y}_n) |\widetilde{u}_n|^2 dx \right. \\ &\quad \left. - \frac{\mu}{q} |\widetilde{u}_n|_q^q - \frac{\tau(|\nabla \widetilde{u}_n|_2)}{6} |\widetilde{u}_n|_6^6 - \frac{\tau(|\nabla \widetilde{u}_n|_2)}{10} \int_{\mathbb{R}^3} \phi_{\widetilde{u}_n} |\widetilde{u}_n|^5 dx \right) \\ &\geq \frac{1}{2} |\nabla \widetilde{u}|_2^2 + \frac{V_\infty}{2} a^2 - \frac{\mu}{q} |\widetilde{u}|_q^q - \frac{1}{6} |\widetilde{u}|_6^6 - \frac{1}{10} \int_{\mathbb{R}^3} \phi_{\widetilde{u}} |\widetilde{u}|^5 dx \\ &= J_{\infty,T}(\widetilde{u}) \geq c_{\infty,T,a}, \end{aligned}$$

which contradicts $c_{\infty,T,a} > c_{0,T,a}$. Hence, $\{\widehat{y}_n\}$ is bounded.

Step 4. Identification of the limit point. Since $\{\widehat{y}_n\}$ is bounded, passing to a subsequence, $\widehat{y}_n \rightarrow \widehat{y} \in \mathbb{R}^3$. By the continuity of V and the strong convergence $\widetilde{u}_n \rightarrow \widetilde{u}$ in $H^1(\mathbb{R}^3)$, we have,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} V(\varepsilon_n x + \widehat{y}_n) |\widetilde{u}_n|^2 dx = V(\widehat{y}) |\widetilde{u}|_2^2 = V(\widehat{y}) a^2.$$

Therefore,

$$\begin{aligned} c_{0,T,a} &= \lim_{n \rightarrow \infty} J_{\varepsilon_n,T}(u_n) \\ &= \frac{1}{2} |\nabla \widetilde{u}|_2^2 + \frac{V(\widehat{y})}{2} a^2 - \frac{\mu}{q} |\widetilde{u}|_q^q - \frac{1}{6} |\widetilde{u}|_6^6 - \frac{1}{10} \int_{\mathbb{R}^3} \phi_{\widetilde{u}} |\widetilde{u}|^5 dx \\ &= J_{V(\widehat{y}),T}(\widetilde{u}) \geq c_{V(\widehat{y}),T,a}. \end{aligned}$$

Since $c_{V(\widehat{y}),T,a} > c_{0,T,a}$ if $V(\widehat{y}) > 0$ (by the strict monotonicity in Lemma 3.6(ii)), and $V \geq 0$, we must have $V(\widehat{y}) = 0$, i.e., $\widehat{y} \in M$. $\square$

Let $\gamma : [0, +\infty) \rightarrow [0, +\infty)$ be a positive function such that $\gamma(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$, and define

$$\widetilde{S}(a) = \{u \in S(a) : J_{\varepsilon,T}(u) \leq c_{0,T,a} + \gamma(\varepsilon)\}. \quad (4.34)$$

By virtue of Lemma 4.6, the function $\gamma(\varepsilon) = \sup_{y \in M} |J_{\varepsilon,T}(\Phi_\varepsilon(y)) - c_{0,T,a}|$ satisfies $\gamma(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Therefore, $\Phi_\varepsilon(y) \in \widetilde{S}(a)$ for all $y \in M$.

Lemma 4.9. *Let $\delta > 0$ and $M_\delta = \{x \in \mathbb{R}^N : \text{dist}(x, M) \leq \delta\}$. Then*

$$\lim_{\varepsilon \rightarrow 0} \sup_{u \in \widetilde{S}(a)} \inf_{z \in M_\delta} |\beta_\varepsilon(u) - z| = 0.$$

Proof. Let $\varepsilon_n \rightarrow 0$ and $u_n \in \tilde{S}(a)$ such that

$$\inf_{z \in M_\delta} |\beta_{\varepsilon_n}(u_n) - z| = \sup_{u_n \in \tilde{S}(a)} \inf_{z \in M_\delta} |\beta_{\varepsilon_n}(u_n) - z| + o_n(1).$$

Based on the above equality, it suffices to find a sequence $\{z_n\} \subset M_\delta$ such that

$$\lim_{n \rightarrow +\infty} |\beta_{\varepsilon_n}(u_n) - z_n| = 0 \quad (4.35)$$

Since $u_n \in \tilde{S}(a)$, we have

$$c_{0,T,a} \leq J_{0,T}(u_n) \leq J_{\varepsilon_n,T}(u_n) \leq c_{0,T,a} + \gamma(\varepsilon_n), \quad \forall n \in \mathbb{N}$$

thus,

$$J_{\varepsilon_n,T}(u_n) \rightarrow c_{0,T,a}$$

According to Proposition 4.8, there exists $\{\tilde{z}_n\} \subset \mathbb{R}^3$ such that $z_n = \varepsilon_n \tilde{z}_n \rightarrow z$ for some $z \in M$ and $\tilde{v}_n(x) = u_n(x + \tilde{z}_n)$ is strongly convergent to some $\tilde{v} \in H^1(\mathbb{R}^3) \setminus \{0\}$ with $\tilde{v} \neq 0$. Then, $\{z_n\} \subset M_\delta$ for n large enough and

$$\beta_{\varepsilon_n}(u_n) = z_n + \frac{\int_{\mathbb{R}^3} (\chi(\varepsilon_n z + z_n) - z_n) |\tilde{v}_n|^2 dx}{a^2},$$

implying that

$$\beta_{\varepsilon_n}(u_n) - z_n = \frac{\int_{\mathbb{R}^3} (\chi(\varepsilon_n z + z_n) - z_n) |\tilde{v}_n|^2 dx}{a^2} \rightarrow 0 \text{ as } n \rightarrow +\infty$$

and the proof of this lemma is completed. $\square$

5. Proof of Theorem 1.1

In the sequel, we will divide the proof into two parts.

Part I: Multiplicity of solutions. We set $\varepsilon \in (0, \varepsilon_0)$. Then by Lemmas 4.6, 4.7 and 4.9, it is easy to see that the continuous mappings below are well defined for $\varepsilon \in (0, \varepsilon_0)$:

$$M \xrightarrow{\Phi_\varepsilon} \tilde{S}(a) \xrightarrow{\beta_\varepsilon} M_\delta.$$

From Lemma 4.7, we choose a function $z(\varepsilon, y)$ with $|z(\varepsilon, y)| < \frac{\delta}{2}$ uniformly in $y \in M$, such that

$$\beta_\varepsilon(\Phi_\varepsilon(y)) = y + z(\varepsilon, y) \quad \text{for all } \varepsilon \in (0, \varepsilon_0), y \in M.$$

We define $\mathcal{H}(t, y) = y + (1 - t)z(\varepsilon, y)$. Then $\mathcal{H} : [0, 1] \times M \rightarrow M_\delta$ is continuous. Obviously, $\mathcal{H}(0, y) = \beta_\varepsilon(\Phi_\varepsilon(y))$, $\mathcal{H}(1, y) = y$ for all $y \in M$. That is, $\mathcal{H}(t, y)$ is a homotopy between $\beta_\varepsilon \circ \Phi_\varepsilon$ and the inclusion map $\text{id} : M \rightarrow M_\delta$. From [13], it is easy to check that

$$\text{cat}(\tilde{S}(a)) \geq \text{cat}_{M_\delta}(M).$$

Arguing as in Lemmas 3.1 and 4.5, we know that $J_{\varepsilon,T}$ is bounded from below on $S(a)$ and satisfies the $(\text{PS})_c$ condition for levels $c \in (c_{0,T,a}, c_{0,T,a} + r(\varepsilon))$. Then, the Ljusternik-Schnirelmann theory of critical points (see [22] and [52]) asserts that $J_{\varepsilon,T}$ admits at least $\text{cat}_{M_\delta}(M)$ critical points on $S(a)$. By (ii) of Lemma 4.2, we know that the critical points of $J_{\varepsilon,T}$ are in fact critical points of J_ε . Thus J_ε also has at least $\text{cat}_{M_\delta}(M)$ critical points on $S(a)$.

Part II: The concentration behavior of maximum points of the solutions.

Let u_ε be a solution of (1.1) with $J_{\varepsilon,T}(u_\varepsilon) \leq c_{0,T,a} + \gamma(\varepsilon)$, where $\gamma(\varepsilon)$ was defined in (4.34). Arguing as in the proof of Proposition 4.8, for each sequence $\varepsilon_n \rightarrow 0$, there exists $\hat{y}_n \in \mathbb{R}^3$ such that $\hat{y}_n := \varepsilon_n \tilde{y}_n \rightarrow \hat{y}$ with $\hat{y} \in M$ and $\tilde{u}_n(x) = u_{\varepsilon_n}(x + \hat{y}_n)$ is strongly convergent to $\tilde{u} \in H^1(\mathbb{R}^3)$ with $\tilde{u} \neq 0$. As $\tilde{u}_n$ is a solution of

$$-\Delta \tilde{u}_n + V(\varepsilon_n x + \hat{y}_n) \tilde{u}_n - \phi_{\tilde{u}_n} |\tilde{u}_n|^3 \tilde{u}_n = \lambda_n \tilde{u}_n + \mu |\tilde{u}_n|^{q-2} \tilde{u}_n + |\tilde{u}_n|^4 \tilde{u}_n \quad \text{in } \mathbb{R}^3,$$

with

$$\limsup_{n \rightarrow \infty} \lambda_n \leq \frac{\rho_1 + c_{0,T,a}}{a^2} < 0,$$

the convergence $\tilde{u}_n \rightarrow \tilde{u}$ in $H^1(\mathbb{R}^3)$ permits us to apply the same arguments found in [18] to obtain that

$$\lim_{|x| \rightarrow +\infty} \tilde{u}_n(x) = 0, \quad \text{uniformly in } n.$$

That is, for given $\tau^* > 0$, there exist $\bar{R} > 0$ and $\bar{n} \in \mathbb{N}$ such that

$$|\tilde{u}_n(x)| \leq \tau^* \quad \text{for } |x| \geq \bar{R} \text{ and } n \geq \bar{n}.$$

Next we claim that $|\tilde{u}_n|_\infty \rightarrow 0$. Indeed, otherwise we must have $\tilde{u}_n \rightarrow 0$ in $H^1(\mathbb{R}^3)$, which is impossible because $|\tilde{u}_n|_2^2 = a^2$ for all $n \in \mathbb{N}$. Now, let us fix $\tau^* > 0$ such that $|\tilde{u}_n|_\infty \geq 2\tau^*$, and let $z_n \in \mathbb{R}^3$ satisfy $|\tilde{u}_n(z_n)| = |\tilde{u}_n|_\infty$ for all $n \in \mathbb{N}$. Finally, let us consider $\xi_n \in \mathbb{R}^3$ such that $|u_{\varepsilon_n}(\xi_n)| = |u_{\varepsilon_n}|_\infty$ for all $n \in \mathbb{N}$. Then we have

$$\xi_n = z_n + \tilde{y}_n \quad \text{and} \quad \lim_{n \rightarrow +\infty} V(\varepsilon_n \xi_n) = \lim_{n \rightarrow +\infty} V(\varepsilon_n z_n + \varepsilon_n \tilde{y}_n) = V(\hat{y}) = 0.$$

The proof of Theorem 1.1 is now complete. $\square$

Acknowledgements

The authors would like to thank the anonymous referees for carefully reading this manuscript and giving valuable suggestions and comments. This work was supported by the National Natural Science Foundation of China (12171497, 11771468, 11971027). V.D. Rădulescu was supported by the AGH University of Kraków under grant no. 16.16.420.054, funded by the Polish Ministry of Science and Higher Education.

Author contributions X.H. and V.R. coordinated the preparation of this paper and Q.G. wrote the initial version of the manuscript text. All authors reviewed the manuscript.

Availability of data and materials No datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors declare that there is no conflict of interest.

Open Access. This article is licensed under a Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International License, which permits any non-commercial use, sharing, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if you modified the licensed material. You do not have permission under this licence to share adapted material derived from this article or parts of it. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by-nc-nd/4.0/>.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

- [1] Alves, C.O., Ji, C.: Normalized solutions for the Schrödinger equations with L^2 -subcritical growth and different types of potentials. *J. Geom. Anal.* **32**(5(165)), 34 (2022)
- [2] Ambrosetti, A.: On Schrödinger–Poisson systems. *Milan J. Math.* **76**, 257–274 (2008)

- [3] Ambrosetti, A., Ruiz, D.: Multiple bound states for the Schrödinger–Poisson problem. *Commun. Contemp. Math.* **10**, 391–404 (2008)
- [4] Azzollini, A., Pomponio, A.: Bound state solutions for the nonlinear Schrödinger–Paxwell equations. *J. Math. Anal. Appl.* **345**, 90–108 (2008)
- [5] Azzollini, A., Aveniab, P., Vaira, G.: Generalized Schrödinger–Mewton system in dimension $\mathbb{R}^3$: critical case. *J. Math. Anal. Appl.* **449**, 531–552 (2017)
- [6] Bartsch, T., Soave, N.: A natural constraint approach to normalized solutions of nonlinear Schrödinger equations and systems. *J. Funct. Anal.* **272**(12), 4998–5037 (2017)
- [7] Bartsch, T., Molle, R., Rizzi, M., Verzini, G.: Normalized solutions of mass supercritical Schrödinger equations with potential. *Comm. Partial Differ. Equ.* **46**(9), 1729–1756 (2021)
- [8] Benci, V., Fortunato, D.: An eigenvalue problem for the Schrödinger–Maxwell equations. *Topol. Methods Nonlinear Anal.* **11**, 283–293 (1998)
- [9] Benci, V., Fortunato, D.: Solitary waves of the nonlinear Klein–Gordon equation coupled with Maxwell equations. *Rev. Math. Phys.* **14**, 409–420 (2002)
- [10] Cerami, G., Vaira, G.: Positive solutions for some non-autonomous Schrödinger–Poisson systems. *J. Differ. Equ.* **248**, 521–543 (2010)
- [11] Chabrowski, J.: Concentration-compactness principle at infinity and semilinear elliptic equations involving critical and subcritical Sobolev exponents. *Calc. Var. Partial. Differ. Equ.* **3**, 493–512 (1995)
- [12] Chen, S., Tang, X.: A comprehensive review on the existence of normalized solutions for four classes of nonlinear elliptic equations. *Opusc. Math.* **45**, 739–763 (2025)
- [13] Cingolani, S., Lazzo, M.: Multiple semiclassical standing waves for a class of nonlinear Schrödinger equations. *Topol. Methods Nonlinear Anal.* **10**, 1–13 (1997)
- [14] D’Aprile, T., Mugnai, D.: Solitary waves for nonlinear Klein–Gordon–Maxwell and Schrödinger–Maxwell equations. *Proc. Roy. Soc. Edinburgh Sect. A* **134**, 893–906 (2004)
- [15] D’Aprile, T., Mugnai, D.: Non-existence results for the coupled Klein–Gordon–Maxwell equations. *Adv. Nonlinear Stud.* **4**, 307–322 (2004)
- [16] Ding, Y., Zhong, X.: Normalized solution to the Schrödinger equation with potential and general nonlinear term: mass super-critical case. *J. Differ. Equ.* **334**, 194–215 (2022)
- [17] Feng, X.: Ground state solutions for a class of Schrödinger–Poisson systems with partial potential. *Z. Angew. Math. Phys.* **71**(1), 16 (2020)
- [18] Feng, X.: Existence and concentration of ground state solutions for doubly critical Schrödinger–Poisson-type systems. *Z. Angew. Math. Phys.* **71**(5), 25, 153 (2020)
- [19] Gao, F., da Silva, E., Yang, M., Zhou, J.: Existence of solutions for critical Choquard equations via the concentration compactness method. *Proc. Roy. Soc. Edinb. Sect. A* **150**(2), 921–954 (2020)
- [20] Gao, F., Yang, M.: Brezis–Nirenberg type critical problem for nonlinear Choquard equation. *Sci China Math* **61**, 1219–1242 (2018)
- [21] Azorero, J.P.G., Alonso, I.P.: Multiplicity of solutions for elliptic problems with critical exponent or with a nonsymmetric term. *Trans. Am. Math. Soc.* **323**, 877–895 (1991)
- [22] Ghoussoub, N.: *Duality and Perturbation Methods in Critical Point Theory*. Cambridge University Press, Cambridge, UK (1993)
- [23] Gilbarg, D., Trudinger, N.S.: *Elliptic Partial Differential Equations of Second Order*, Classics in Mathematics, Springer-Verlag, Berlin, (2001)
- [24] He, X., Meng, Y., Rădulescu, V.D.: Prescribed mass solutions for Schrödinger–Poisson systems with combined critical nonlinearities, (2024) In Preprint
- [25] He, X., Liu, W., Meng, Y.: Normalized solutions for the critical Schrödinger–Poisson systems with potentials. *Adv. Differ. Equ.* **31**, 459–512 (2026)
- [26] Ikoma, N., Miyamoto, Y.: Stable standing waves of nonlinear Schrödinger equations with potentials and general nonlinearities. *Calc. Var. Partial. Differ. Equ.* **59**(2), 31, 48 (2020)
- [27] Jeanjean, L.: Existence of solutions with prescribed norm for semilinear elliptic equations. *Nonlinear Anal. TMA* **28**(10), 1633–1659 (1997)
- [28] Jeanjean, L., Jendrej, J., Le, T.T., Visciglia, N.: Orbital stability of ground states for a Sobolev critical Schrödinger equation. *J. Math. Pures Appl.* **164**, 158–179 (2022)
- [29] Jeanjean, L., Le, T.T.: Multiple normalized solutions for a Sobolev critical Schrödinger equation. *Math. Ann.* **384**(1–2), 101–138 (2021)
- [30] Jeanjean, L., Lu, S.-S.: Nonradial normalized solutions for nonlinear scalar field equations. *Nonlinearity* **32**(12), 4942–4966 (2019)
- [31] Jin, P., Yang, H., Zhou, X.: Normalized solutions for Schrödinger equations with critical Sobolev exponent and perturbations of Choquard terms. *Bull. Math. Sci.* **15**(2), 36 (2025)

- [32] Jin, Z., Zhang, W.: Normalized solutions for nonlinear Schrödinger equation involving potential and Sobolev critical exponent. *J. Math. Anal. Appl.* **535**, 128161 (2024)
- [33] Lancelotti, S., Molle, R.: Normalized positive solutions for Schrödinger equations with potentials in unbounded domains. *Proc. Roy. Soc. Edinb. Sect. A* **153**, 1023–1045 (2023)
- [34] Li, F., Li, Y., Shi, J.: Existence and multiplicity of positive solutions to Schrödinger–Poisson type systems with critical nonlocal term. *Calc. Var. Partial. Differ. Equ.* **56**(5), 17, 134 (2017)
- [35] Li, F., Li, Y., Shi, J.: Existence of positive solutions to Schrödinger–Poisson type systems with critical exponent. *Commun. Contem. Math.* **16**, 28 (2014). <https://doi.org/10.1142/S0219199714500369>
- [36] Li, N., He, X.: Existence and multiplicity results for some Schrödinger–Poisson system with critical growth. *J. Math. Anal. Appl.* **488**(2), 35 (2020)
- [37] Lieb, E.H.: Thomas-Fermi and related theories and molecules. *Rev. Mod. Phys.* **53**, 603–641 (1981)
- [38] Lieb, E., Loss, M.: *Analysis*, Graduate Studies in Mathematics, vol. 14. American Mathematical Society, Providence (2001)
- [39] Lions, P.L.: Solutions of Hartree–Fock equations for Coulomb systems. *Commun. Math. Phys.* **109**, 33–97 (1987)
- [40] Markowich, P., Ringhofer, C., Schmeiser, C.: *Semiconductor Equations*. Springer-Verlag, New York (1990)
- [41] Meng, Y., He, X.: Normalized solutions for the Schrödinger–Poisson system with doubly critical growth. *Topol. Methods Nonlinear Anal.* **62**, 509–534 (2023)
- [42] Molle, R., Riey, G., Verzini, G.: Normalized solutions to mass supercritical Schrödinger equations with negative potential. *J. Differ. Equ.* **333**, 302–331 (2022)
- [43] Nirenberg, L.: On elliptic partial differential equations. *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **13**, 115–162 (1959)
- [44] Rabinowitz, P.H.: On a class of nonlinear Schrödinger equations. *Z. Angew. Math. Phys.* **43**, 270–291 (1992)
- [45] Ruiz, D.: The Schrödinger–Poisson equation under the effect of a nonlinear local term. *J. Funct. Anal.* **237**, 655–674 (2006)
- [46] Shu, M., Wen, L., Yang, H.: Normalized solutions for planar Schrödinger–Poisson equation with mixed nonlinearities. *Bull. Math. Sci.* **15**(3), 24 (2025)
- [47] Strauss, W.A.: Existence of solitary waves in higher dimensions. *Comm. Math. Phys.* **55**, 149–162 (1977)
- [48] Soave, N.: Normalized ground states for the NLS equation with combined nonlinearities. *J. Differ. Equ.* **269**(9), 6941–6987 (2020)
- [49] Soave, N.: Normalized ground states for the NLS equation with combined nonlinearities: the Sobolev critical case. *J. Funct. Anal.* **279**(6), 43, 108610 (2020)
- [50] Wei, J., Wu, Y.: Normalized solutions for Schrödinger equations with critical Sobolev exponent and mixed nonlinearities. *J. Funct. Anal.* **283**(6), 109574 (2022)
- [51] Wei, L., Song, Y.: Normalized solutions for critical Schrödinger equations involving $(2, q)$ -Laplacian. *Opusc. Math.* **45**(5), 685–716 (2025)
- [52] Willem, M.: *Minimax Theorems*. Birkhäuser, Boston (1996)
- [53] Xie, Z., Chen, J., Tan, Y.: Multiple normalized solutions to Schrödinger equations in $\mathbb{R}^N$ with critical growth and potential. *J. Fixed Point Theory Appl.* **26**(4), 41 (2024)

Qi Gao and Xiaoming He
College of Science
Minzu University of China
Beijing 100081
China

e-mail: gaoqi0910@163.com

Xiaoming He

e-mail: xmhe923@muc.edu.cn

Vicențiu D. Rădulescu
Faculty of Applied Mathematics
AGH University of Kraków
al. Mickiewicza 30
30-059 Kraków
Poland

Vicențiu D. Rădulescu
Faculty of Electrical Engineering and Communication
Brno University of Technology
Technická 3058/10
61600 Brno
Czech Republic

Vicențiu D. Rădulescu
“Simion Stoilow” Institute of Mathematics of the Romanian Academy
P.O. Box 1-764
014700 Bucharest
Romania

Vicențiu D. Rădulescu
Scientific Research Center
Baku Engineering University
AZ0102 Baku
Azerbaijan
e-mail: radulescu@agh.edu.pl

(Received: December 21, 2025; revised: March 29, 2026; accepted: April 27, 2026)