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APPROXIMATIONS OF THE HETEROCLINIC ORBITS NEAR A DOUBLE-ZERO BIFURCATION WITH SYMMETRY OF ORDER TWO. APPLICATION TO A LIÉNARD EQUATION

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A dynamical system possessing an equilibrium point with two zero eigenvalues is considered. We assume that a degenerate Bogdanov-Takens bifurcation with symmetry of order two is present and, in the parameter space, a curve of double heteroclinic bifurcation values emerges from the codimension two bifurcation point. Using a blow-up transformation and a perturbation method, we obtain second-order approximations both for the heteroclinic orbits and for the curve of heteroclinic bifurcation values. Applications of our results for the truncated normal form and for the Liénard equation are presented. Some numerical simulations illustrating the accuracy of our results are performed.

Keywords: heteroclinic orbit, double-zero bifurcation with symmetry of order two, regular perturbation method, Liénard equation.

1. Introduction

The double-zero bifurcation with $\mathbb{Z}_2$ -symmetry is a codimension two bifurcation of equilibria of a family of planar vector fields that are invariant under a central symmetry. The Poincaré normal form of such a family is [Khorosov, 1979], [Chow *et al.*, 1994]:

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= \mu_1 x + \mu_2 y + a_3 x^3 + b_3 x^2 y + \sum_{k\geq 2}(a_{2k+1}x^{2k+1} + b_{2k+1}x^{2k}y),\end{aligned} \tag{1}$$

where $\mu_1,\ \mu_2,\ a_k, b_k$ are real parameters and $a_3 b_3 \neq 0$.

Remark that using transformations of time and variables, system (1) with $a_3 b_3 < 0$ can be written as

$$\begin{aligned}\dot{x} &= y,\\ \dot{y} &= \beta_1 x + \beta_2 y + x^3 - x^2 y + O(|(x,y)|^5),\end{aligned} \tag{2}$$

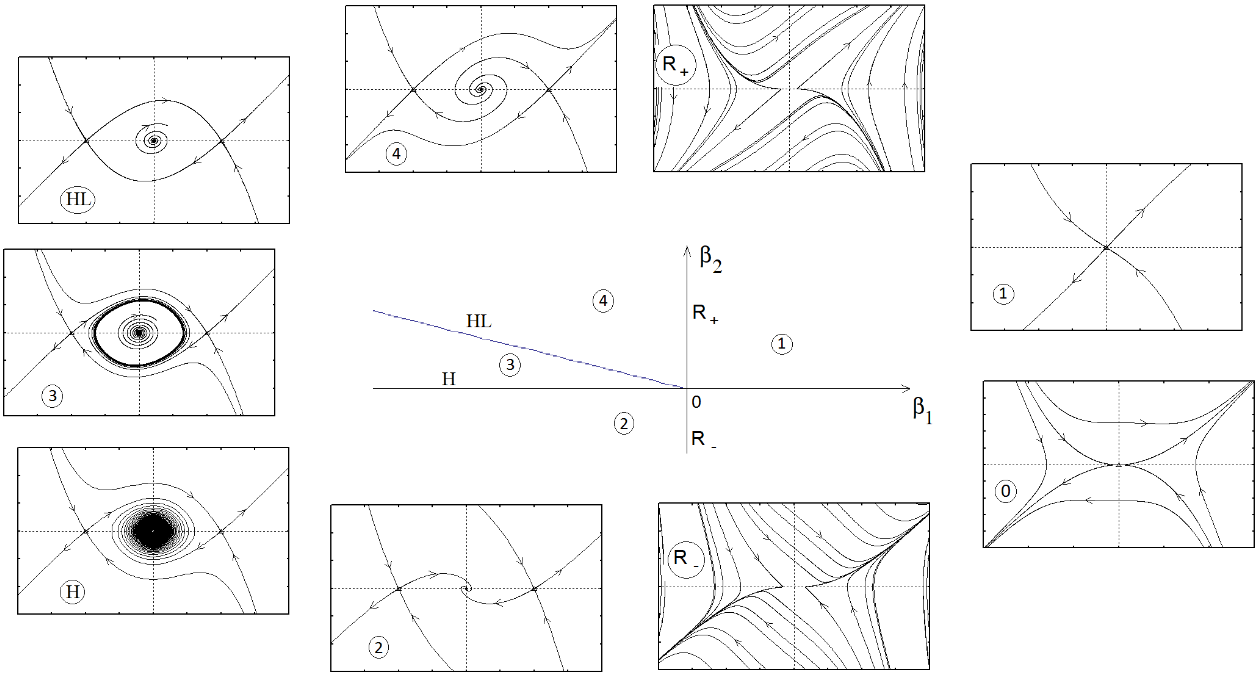

Fig. 1. The bifurcation diagram for the double-zero bifurcation with symmetry of order two, for system 2.

The parametric portrait around the origin of the parameter plane (β_1, β_2) includes four bifurcation curves at which pitchfork, Hopf, double heteroclinic bifurcation occur [Chow *et al.*, 1994], as follows from Theorem 1 bellow.

Theorem 1. [Chow *et al.*, 1994] *The parametric portrait of system (2) is divided in 9 different strata by the origin and the following curves:*

(i) $R_+ = \{\beta \mid \beta_1 = 0, \beta_2 > 0\}$; $\quad R_- = \{\beta \mid \beta_1 = 0, \beta_2 < 0\}$;
(ii) $H = \{\beta \mid \beta_2 = 0, \beta_1 < 0\}$;
(iii) $HL = \{\beta \mid \beta_2 = -\frac{1}{5}\beta_1 + O(\beta_1^{3/2}), \beta_1 < 0\}$;

The bifurcation diagram for (2) around the origin is given in Figure 1.

Thus, the following bifurcations are present in the (β_1, β_2) - parameter plane of system (2):

(i) a pitchfork bifurcation of the origin for $(\beta_1, \beta_2) \in R_+ \cup R_-$: while in region (1) (for $\beta_1 > 0$) the origin is the only equilibrium point (a saddle), when $\beta_1 < 0$, together with the origin which becomes a focus, two new equilibria $\left(\pm\sqrt{-\beta_1}, 0\right)$ appear; these two equilibria are hyperbolic saddles;

(ii) a supercritical Hopf bifurcation of the origin for $(\beta_1, \beta_2) \in H$;

(iii) a double heteroclinic connection between the two saddles, for $(\beta_1, \beta_2) \in HL$.

While expressions for the pitchfork and Hopf bifurcation values are easy to derive analytically, the heteroclinic bifurcation values can be determined only numerically.

In most cases, the homoclinic and heteroclinc orbits of continuous dynamical systems are difficult to find analytically. Several asymptotic methods such as the regular perturbation method, the elliptic averaging method, the elliptic Lindstedt-Poincaré method, the hyperbolic perturbation method, allow to detect the presence of these orbits. Such results can be found in [Belhaq *et al.*, 2000], [Belhaq & Lakrad, 2000], [Chen & Chen, 2009], [Chen *et al.*, 2009].

There exist some bifurcations of equlibria which involve the presence in the parameter space, near the bifurcation value, of parameters corresponding to homoclinic or heteroclinic orbits. In such cases, asymptotic prediction of the curve of heteroclinic bifurcation values can be obtained. The double-zero bifurcation is such an example. Thus, for the nondegenerated Bogdanov-Takens bifurcation, a strata of homoclinic bifurcation values emerges at the codimension two bifurcation point. Predictors of the homoclinic orbits in this case can be found in [Al-Hdaibat *et al.*, 2016], [Kuznetsov *et. al*, 2014], [Kuznetsov *et. al*, 2015]. For a degenerate Bogdanov-Takens bifurcation, asymptotic approximations of homoclinics near a double-zero bifurcation point with symmetry of order two are found in [Rocşoreanu & Sterpu, 2017]. Approximations of heteroclinic connections in the 1:3 and 1:4 resonance problems are done in [Fahsi & Belhaq, 2012], [Chung *et al.*, 2014], [Qin *et al.*, 2016].

The paper is organized as follows. In Section 2, we derive explicit first and second order approximations of double heteroclinic solutions of system (3), using the regular perturbation method, in terms of hyperbolic functions. In addition, we find a more accurate approximation of the bifurcation curve HL corresponding to parameters at which such solutions exist. In Section 3, the results obtained in Section 2 are used to obtain approximations of heteroclinic orbits for the truncated normal form (2) of double-zero bifurcation with symmetry of order two. Some numerical simulations illustrating the efficiency of our theoretical results are performed. In Section 4, our theoretical results from Section 2 are applied to obtain approximations of heteroclinic orbits for the Liénard system. Finally, some conclusions are formulated.

2. First and second order approximations of heteroclinic orbits for the parametric-dependent normal form by regular perturbation method

In (1), consider $\mu_1,\ \mu_2$ as bifurcation parameters and the coefficients $a_i,\ b_i$, depending on them. As around $(\mu_1,\mu_2)=(0,0)$, we have $a_3=a+a_{10}\mu_1+a_{01}\mu_2+O(|\mu|^2)$, $b_3=-b-b_{10}\mu_1-b_{01}\mu_2+O(|\mu|^2)$, $a_5=c+O(|\mu|)$, $b_5=-d+O(|\mu|)$, system (1) is written into the form

$$\begin{aligned}\dot{x}&=y,\\ \dot{y}&=\mu_1x+\mu_2y+ax^3-bx^2y+g\left(\mu_1,\mu_2,x,y\right)+...,\end{aligned}\tag{3}$$

where $g\left(\mu_1,\mu_2,x,y\right)=(a_{10}\mu_1+a_{01}\mu_2)x^3-(b_{10}\mu_1+b_{01}\mu_2)x^2y+cx^5-dx^4y$ and $\mu_1,\ \mu_2$ are real parameters, while $a,b,c,d,a_{10},a_{01},b_{10},b_{01}$ are real constants and $a>0,\ b>0$.

Following the lines in [Chow *et al.*, 1994], we apply a blow-up transformation for system (3):

$$\begin{aligned}x&=\frac{\varepsilon}{\sqrt{a}}\,u,\qquad y=\frac{\varepsilon^2}{\sqrt{a}}\,v,\\ \mu_1&=-\varepsilon^2,\qquad \mu_2=\frac{b}{a}\varepsilon^2\,\theta,\end{aligned}\tag{4}$$

and consider $\varepsilon\geq 0$ and $s=\varepsilon t$ the new time. Thus, system (3) becomes:

$$\begin{aligned}\frac{du}{ds}&=v,\\ \frac{dv}{ds}&=-u+u^3+\frac{b}{a}\varepsilon v(\theta-u^2)+g_1+O\left(\varepsilon^4\right)\end{aligned}\tag{5}$$

where $g_1=\frac{\varepsilon^2}{a^2}u^3\left(-aa_{10}+ba_{01}\theta+cu^2\right)+\frac{\varepsilon^3}{a^2}u^2v\left(ab_{10}-bb_{01}\theta-du^2\right)$.

For $\varepsilon=0$, system (5) is Hamiltonian

$$\begin{aligned}\dot{u}&=v,\\ \dot{v}&=-u+u^3,\end{aligned}\tag{6}$$

with the first integral

$$H(u,v)=\frac{v^2}{2}+\frac{u^2}{2}-\frac{u^4}{4}.$$

This system possesses three equilibrium points, namely $E_0=(0,0)$ (which is a nonlinear center), $E_+=(1,0)$ and $E_-=(-1,0)$ (which are hyperbolic saddles). The level curve $H(u,v)=0$ contains a pair of heteroclinic orbits between the two saddles.

For $\varepsilon\neq 0$ and each θ, the ε-perturbed system (5) still possesses three equilibria, one of them being the origin (which is either an attractor or a repeller) and two saddles. It also has two heteroclinic orbits, symmetric with respect to the origin of the phase plane.

One of these orbits parameterized by ε is given by

$$
\begin{aligned}
u &= u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + ... + \varepsilon^k u_k + ... \\
v &= v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + ... + \varepsilon^k v_k + ... \\
\text{with } \theta &= \theta_0 + \varepsilon \theta_1 + \varepsilon^2 \theta_2 + ... + \varepsilon^k \theta_k + ...
\end{aligned} \tag{7}
$$

where k stands for the order of approximation. The other heteroclicnic orbit can be obtained by symmetry with respect to the origin. We require that $\lim_{s\to\infty} u(s)$ and $\lim_{s\to-\infty} u(s)$ are finite and

$$
\lim_{s\to\infty} v(s) = \lim_{s\to-\infty} v(s) = 0. \tag{8}
$$

We also fix that $u(0) = 0$. Next, we replace (7) into system (5) and collect the ε^k terms.

For $k = 0$, we get:

$$
\begin{aligned}
\dot{u}_0 &= v_0, \\
\dot{v}_0 &= -u_0 + u_0^3,
\end{aligned}
$$

that is the hamiltonian system (6). Its solution satisfying the initial conditions $u_0(0) = 0$, $v_0(0) = 1/\sqrt{2}$ can be written as

$$
\begin{aligned}
u_0(s) &= \tanh \frac{s}{\sqrt{2}}, \\
v_0(s) &= \frac{1}{\sqrt{2}} \left(1 - \tanh^2 \frac{s}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} \operatorname{sech}^2 \frac{s}{\sqrt{2}}.
\end{aligned} \tag{9}
$$

Remark that we chose $v_0(0)$, such that $H(u_0(0), v_0(0)) = 0$.

For $k = 1, 2, 3$, we get the linear non-homogeneous systems

$$
\begin{aligned}
\dot{u}_k &= v_k, \\
\dot{v}_k &= -(1 - 3u_0^2)u_k + h_k(s),
\end{aligned} \tag{10}
$$

with $h_1(s) = \frac{b}{a} v_0(\theta_0 - u_0^2)$,

$$
\begin{aligned}
h_2(s) &= 3u_0u_1^2 + \frac{b}{a} v_0(\theta_1 - 2u_0u_1) + \frac{b}{a} v_1(\theta_0 - u_0^2) + \frac{u_0^3}{a}\left(-a_{10} + \frac{b}{a} a_{01}\theta_0 + \frac{c}{a} u_0^2\right), \\
h_3(s) &= u_1^3 + 6u_0u_1u_2 + \frac{b}{a} v_2(\theta_0 - u_0^2) + \frac{b}{a} v_1(\theta_1 - 2u_0u_1) + \frac{b}{a} v_0(\theta_2 - u_1^2 - 2u_0u_2) \\
&\quad + \frac{u_0^3}{a^2}\left(a_{01}\theta_1 b + 2cu_0u_1\right) + \frac{3u_0^2u_1}{a^2}\left(-a_{10}a + a_{01}b\theta_0 + cu_0^2\right) + \frac{u_0^2 v_0}{a^2}\left(ab_{10} - bb_{01}\theta_0 - du_0^2\right).
\end{aligned}
$$

The homogenous part of (10) leads to the second order equation $\ddot{u}_k = -(1 - 3u_0^2)u_k, k = 1, 2, 3$. It has the particular independent solutions:

$$
\begin{aligned}
\varphi_1(s) &= \operatorname{sech}^2 \frac{s}{\sqrt{2}}, \\
\varphi_2(s) &= 3s\sqrt{2} \operatorname{sech}^2 \frac{s}{\sqrt{2}} + 2\sinh(s\sqrt{2}) + 6\tanh \frac{s}{\sqrt{2}},
\end{aligned}
$$

with the Wronskian $W(\varphi_1, \varphi_2) = 8\sqrt{2}$.

By the variation of constants method we obtain the general solution of (10):

$$
\begin{aligned}
u_k(s) &= c_{1k}(s)\varphi_1(s) + c_{2k}(s)\varphi_2(s), \\
v_k(s) &= c_{1k}(s)\varphi_1'(s) + c_{2k}(s)\varphi_2'(s),
\end{aligned}
$$

with

$$
c_{1k}(s) = c_{1k} - \frac{1}{8\sqrt{2}} \int h_k(s)\varphi_2(s)ds, \quad c_{2k}(s) = c_{2k} + \frac{1}{8\sqrt{2}} \int h_k(s)\varphi_1(s)ds
$$

for $k = 1, 2, 3$.

Next we compute the values for $c_{1k},\ c_{2k},\ \theta_{k-1},\ k = 1, 2, ...$

For $k = 1$, we obtain

$$\lim_{s\to\infty} v_1(s) = \left(60\sqrt{2}ac_{21} + b\,(5\theta_0 - 1)\right)\infty,$$

$$\lim_{s\to-\infty} v_1(s) = \left(60\sqrt{2}ac_{21} - b\,(5\theta_0 - 1)\right)\infty.$$

Since $\lim_{s\to\infty} v_1(s)$ and $\lim_{s\to-\infty} v_1(s)$ are finite, we get

$$c_{21} = 0 \text{ and } \theta_0 = \frac{1}{5}.$$

The condition $u_1(0) = 0$ gives $c_{11} + \frac{\sqrt{2}}{20} = 0$, so we get

$$c_{11} = -\frac{\sqrt{2}}{20}\frac{b}{a}.$$

Thus, u_1 and v_1 are completely determined, into the form:

$$\begin{aligned} u_1(s) &= \frac{\sqrt{2}}{5}\frac{b}{a}\ln\cosh\frac{s}{\sqrt{2}}\cdot\operatorname{sech}^2\frac{s}{\sqrt{2}}, \\ v_1(s) &= -\frac{1}{5}\frac{b}{a}\left[-1 + 2\ln\cosh\frac{s}{\sqrt{2}}\right]\operatorname{sech}^2\frac{s}{\sqrt{2}}\tanh\frac{s}{\sqrt{2}}. \end{aligned} \tag{11}$$

Similarly, for $k = 2$, we have

$$\lim_{s\to\infty} v_2(s) = \left(12a\sqrt{2}c_{22} + b\theta_1\right)\infty,$$

$$\lim_{s\to-\infty} v_2(s) = \left(12a\sqrt{2}c_{22} - b\theta_1\right)\infty.$$

As $\lim_{s\to\infty} v_2(s)$ and $\lim_{s\to-\infty} v_2(s)$ are finite, we get

$$c_{22} = \theta_1 = 0,$$

while from $u_2(0) = 0$ we obtain $c_{12} = 0$. Thus, the non-homogenous system (10), with $k = 2$, has the solution

$$\begin{aligned} u_2(s) = {} & \frac{1}{300a^2}\operatorname{sech}^2\frac{s}{\sqrt{2}}\{3\sqrt{2}s(25c - 2b^2) + \tanh\frac{s}{\sqrt{2}}[75aa_{10} - 15a_{01}b + 16b^2 - 175c \\ & + 15\cosh(\sqrt{2}s)(5aa_{10} - a_{01}b - 5c) + 24b^2\ln\cosh\frac{s}{\sqrt{2}} - 24b^2\ln^2\cosh\frac{s}{\sqrt{2}}]\}, \\ v_2(s) = {} & -\frac{\sqrt{2}}{600a^2}\operatorname{sech}^4\frac{s}{\sqrt{2}}\{-75a\ a_{10} + 15a_{01}b - 14b^2 + 200c - 72b^2\ln\cosh\frac{s}{\sqrt{2}} \\ & + 48b^2\ln^2\cosh\frac{s}{\sqrt{2}} + -\cosh(\sqrt{2}s)[5(15aa_{10} - 3a_{01}b - 2b^2 + 20c) \\ & - 48b^2\ln\cosh\frac{s}{\sqrt{2}} + 24b^2\ln^2\cosh\frac{s}{\sqrt{2}}] + 3\sqrt{2}s(25c - 2b^2)\sinh(\sqrt{2}s)\}. \end{aligned} \tag{12}$$

From system (10) with $k = 3$, $u_3(s), v_3(s)$ and θ_2 can be obtained. As we intend to find only second order approximations of the heteroclinics and of the heteroclinic bifurcation curve, only the value of θ_2 is necessary, so the expressions of $u_3(s), v_3(s)$ will be omitted here. As $\lim_{s\to\infty} v_3(s)$ and $\lim_{s\to-\infty} v_3(s)$ are finite, we get:

$$\theta_2 = \frac{75a(35a_{10}b + 7bb_{01} + 15d) - b\left(525a_{01}b - 16b^2 + 2425c\right) - 2625a^2b_{10}}{13125a^2b}. \tag{13}$$

Thus, the second order approximation of the curve containing heteroclinic bifurcation values is also completely determined. The above expressions were simplified using the software Mathematica [Wolfram, 2010].

The following results are thus obtained.

Theorem 2. *The second order approximations of the heteroclinic orbits in the normal form (3) are:*

$$x(t) = \frac{1}{\sqrt{a}}\left[\varepsilon u_0(\varepsilon t) + \varepsilon^2 u_1(\varepsilon t) + \varepsilon^3 u_2(\varepsilon t)\right], \tag{14}$$
$$y(t) = \frac{1}{\sqrt{a}}\left[\varepsilon^2 v_0(\varepsilon t) + \varepsilon^3 v_1(\varepsilon t) + \varepsilon^4 v_2(\varepsilon t)\right],$$

where u_0, u_1, u_2 and v_0, v_1, v_2 are given by (9), (11), (12) above.

Theorem 3. *The second order approximation of the curve of double heteroclinic bifurcation values is*

$$\mu_2 = -\frac{b}{a}\mu_1\left[\frac{1}{5} - \theta_2\mu_1\right], \tag{15}$$

where θ_2 is given by (13).

3. Second order aproximation of heteroclinic orbits for the truncated normal form

In this section we apply the results from Section 2 in the particular case of the truncated system (2)

$$\dot{x} = y, \tag{16}$$
$$\dot{y} = \beta_1 x + \beta_2 y + x^3 - x^2 y,$$

The blow-up transformation $x = \varepsilon u,\ y = \varepsilon^2 v,\ \beta_1 = -\varepsilon^2,\ \beta_2 = \varepsilon^2\theta$ maps (16) into

$$\frac{du}{ds} = v, \tag{17}$$
$$\frac{dv}{ds} = -u + u^3 + \varepsilon v(\theta - u^2).$$

In this case the second order approximation of the heteroclinic orbit reads

$$x(t) = \varepsilon u_0(\varepsilon t) + \varepsilon^2 u_1(\varepsilon t) + \varepsilon^3 u_2(\varepsilon t), \tag{18}$$
$$y(t) = \varepsilon^2 v_0(\varepsilon t) + \varepsilon^3 v_1(\varepsilon t) + \varepsilon^4 v_2(\varepsilon t),$$

where the expressions of $u_i, v_i,\ i = 1, 2$, simplify as follows:

$$u_1(s) = \frac{\sqrt{2}}{5}\ln\cosh\frac{s}{\sqrt{2}}\operatorname{sech}^2\frac{s}{\sqrt{2}},$$
$$v_1(s) = -\frac{1}{5}\left[-1 + 2\ln\cosh\frac{s}{\sqrt{2}}\right]\operatorname{sech}^2\frac{s}{\sqrt{2}}\tanh\frac{s}{\sqrt{2}}$$
$$u_2(s) = \frac{1}{150}\operatorname{sech}^2\frac{s}{\sqrt{2}}\left[-3\sqrt{2}s + \tanh\frac{s}{\sqrt{2}}\left(8 + 12\ln\cosh\frac{s}{\sqrt{2}} - 12\ln^2\cosh\frac{s}{\sqrt{2}}\right)\right],$$
$$v_2(s) = \frac{1}{300}\sqrt{2}\operatorname{sech}^4\frac{s}{\sqrt{2}}\left[7 + 36\ln\cosh\frac{s}{\sqrt{2}} - 24\ln^2\cosh\frac{s}{\sqrt{2}}\right.$$
$$\left. + \cosh\frac{s}{\sqrt{2}}\left(-5 - 24\ln\cosh\frac{s}{\sqrt{2}} + 12\ln^2\cosh\frac{s}{\sqrt{2}}\right) + \frac{6s}{\sqrt{2}}\sinh(\sqrt{2}s)\right].$$

In addition, the second order approximation (15) of the curve of heteroclinic bifurcation values reads in this particular case:

$$\beta_2 = -\frac{1}{5}\beta_1 + \frac{16}{13125}\beta_1^2.$$

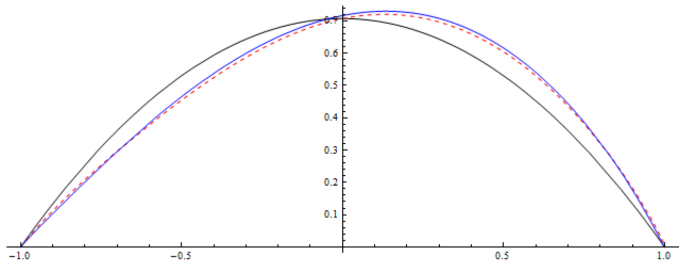

Fig. 2. Hamiltonian, first and second order approximations of the heteroclinic orbit for $\varepsilon = 1$.

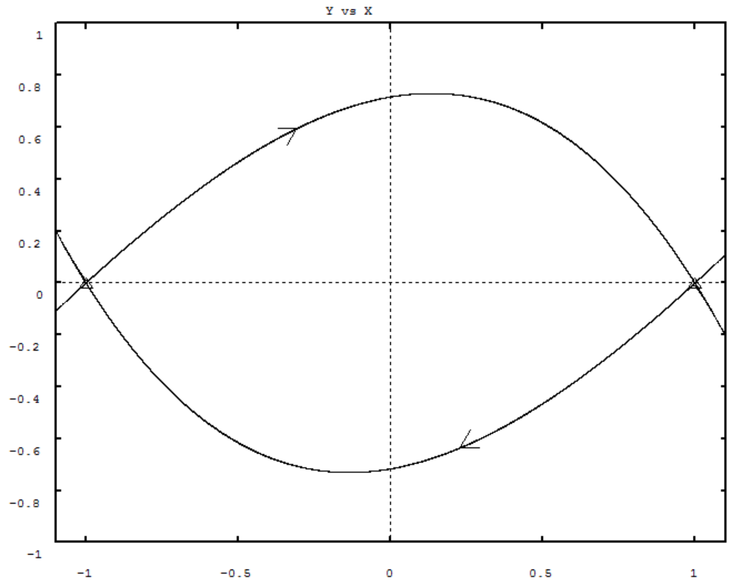

Fig. 3. Heteroclinic orbits for $\varepsilon = 1$ obtained numerically.

As an example, for $\varepsilon = 1$ we get $\beta_1 = -1$. From the above formula the first approximation of β_2 is 0.2, while the second approximation is $\beta_2 = 0.201219$. The numerically obtained bifurcation values are $\beta_1 = -1,\ \beta_2 = 0.20115$.

In Fig. 2, the Hamiltonian (black line), first (red dotted line) and second order approximations (blue line) of the heteroclinic orbit with $y > 0$ are represented. Remark that in Fig. 2 there is small difference between the first and second order approximation of the heteroclinic orbit. These two approximations almost collide with the heteroclinic orbit obtained numericaly by the Runge-Kutta method, plotted in Fig. 3, using [Ermentrout, 2002].

For bigger values of ε, the second order approximated orbit is much more accurate than the first order one. For instance, as $\varepsilon = 1.8$ we obtain $\beta_1 = -3.24$, and $\beta_2 = 0.648$ for the first approximation, while $\beta_2 = 0.660797$ for the second one. The numerical parameter values are $\beta_1 = -3.24,\ \beta_2 = 0.659238$.

The heteroclinic orbits for $\varepsilon = 1.8$ are represented in Fig. 4. As we see in this figure, the second order approximation of the heteroclinic orbit almost collides with the numerical computed one (for this reason they were drawn separately), while the first order approximation is quite different.

4. Double-zero bifurcation for a generalized Liénard equation

As an application of the above presented technique the following generalized Liénard equation is studied

$$\ddot{x} + c_1 x + c_3 x^3 = \delta \left(m_0 - m_1 x^2 - m_2 \dot{x}^2\right) \dot{x}, \tag{19}$$

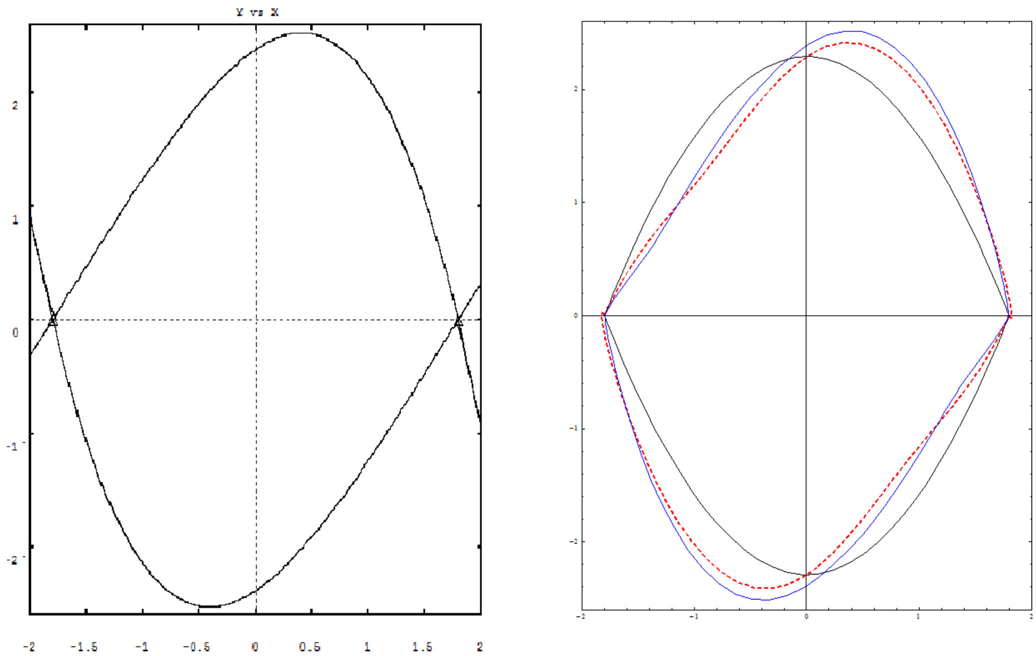

Fig. 4. Heteroclinic orbits for of system (16) for $\varepsilon = 1.8$: (i) numerical approximation; (ii) Hamiltonian, first and second order approximations.

where the dot over quantities stands for differentiation with respect to the time τ.

Denoting $x_1 = x, x_2 = \dot{x}$, equation (19) transforms into the system

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -c_1 x_1 + \delta m_0 x_2 - c_3 x_1^3 - \delta m_1 x_1^2 x_2 - \delta m_2 x_2^3, \end{aligned} \tag{20}$$

This system is invariant with respect to the symmetry $(x_1, x_2) \longrightarrow (-x_1, -x_2)$.

A bifurcation study of the above system as three parameters vary is done in [Khibnik *et. al*, 1998].

System (20) has an unique equilibrium point $E_0 = (0,0)$ for $c_1 c_3 \geq 0$, while for $c_1 c_3 < 0$ two additional equilibria $E_1 = \left(\sqrt{-\frac{c_1}{c_3}}, 0\right)$ and $E_2 = \left(-\sqrt{-\frac{c_1}{c_3}}, 0\right)$ appear.

Using techniques from dynamical system theory [Kuznetsov, 2004], the stability of these equilibria is established as follows. The equilibrium E_0 is a hyperbolic saddle for $c_1 < 0$, while for $c_1 > 0$ it is an attractor for $\delta m_0 < 0$, a repeller for $\delta m_0 > 0$, and nonhyperbolic of Hopf type for $\delta m_0 = 0$.

The symmetric equilibria E_1 and E_2 are hyperbolic saddles for $c_1 > 0$, while for $c_1 < 0$, these equilibria are attractors for $\delta m_0 + \delta m_1 \frac{c_1}{c_3} < 0$, repellers for $\delta m_0 + \delta m_1 \frac{c_1}{c_3} < 0$, and nonhyperbolic of Hopf type for $\delta m_0 + \delta m_1 \frac{c_1}{c_3} = 0$.

As $c_1 = 0$, and $\delta m_0 = 0$, the equilibrium E_0 is nonhyperbolic, with two zero eigenvalues. In this case the normal form corresponds to a double-zero with $\mathbb{Z}_2$-symmetry singularity. This implies the presence of homoclinic and heteroclinic orbits.

In [Chen & Chen, 2009], for certain values of the parameters, approximations of homoclinic or heteroclinic solutions were find using the hyperbolic perturbation method.

In the following, we first obtain the normal form of system (20) close to the origin E_0, then using our results from Section 2, we derive second order approximations for the curve of heteroclinic bifurcation values and for the double heteroclinic connections.

In order to derive the normal form, we have to eliminate the term $\delta m_2 x_2^3$. To this aim, we consider first a new time t, such that

$$d\tau = \left(1 + t_1 x_1^2 + t_2 x_2^2\right) dt, \tag{21}$$

where t_1, t_2 will be determined later. System (20) written with the new time t becomes

$$\begin{aligned}\frac{dx_1}{dt} &= \left(1 + t_1 x_1^2 + t_2 x_2^2\right) x_2, \\ \frac{dx_2}{dt} &= \left(1 + t_1 x_1^2 + t_2 x_2^2\right)\left(-c_1 x_1 + \delta m_0 x_2 - c_3 x_1^3 - \delta m_1 x_1^2 x_2 - \delta m_2 x_2^3\right).\end{aligned} \tag{22}$$

Next, we perform the variables transformation

$$\begin{aligned}y_1 &= x_1 \\ y_2 &= \left(1 + t_1 x_1^2 + t_2 x_2^2\right) x_2\end{aligned} \tag{23}$$

with the local inverse

$$\begin{aligned}x_1 &= y_1 \\ x_2 &= y_2 - t_1 y_1^2 y_2 - t_2 y_1 y_2^2 + t_1^2 y_1^4 y_2 + 3 t_1 t_2 y_1^3 y_2^2 + 2 t_2^2 y_1^2 y_2^3 + O\left(|y|^7\right).\end{aligned} \tag{24}$$

System (22) transforms into

$$\begin{aligned}\frac{dy_1}{dt} &= y_2, \\ \frac{dy_2}{dt} &= -c_1 y_1 + \delta m_0 y_2 - (c_3 + 2c_1 t_1) y_1^3 - (\delta m_1 + 3 c_1 t_2 - \delta m_0 t_1)\, y_1^2 y_2 \\ &\quad + 2\,(t_1 + \delta m_0 t_2)\, y_1 y_2^2 + (t_2 - \delta m_2)\, y_2^3 + O\left(|y|^5\right).\end{aligned} \tag{25}$$

Choosing t_1, t_2 to anihilate the last two third-order terms in (25), we get

$$t_2 = \delta m_2, \quad t_1 = -\delta^2 m_0 m_2.$$

Thus, the time transformation (21) is completely determined, and system (25) reads

$$\begin{aligned}\frac{dy_1}{dt} &= y_2, \\ \frac{dy_2}{dt} &= -c_1 y_1 + \delta m_0 y_2 + a_3 y_1^3 + b_3 y_1^2 y_2 + a_5 y_1^5 + b_5 y_1^4 y_2 + O\left(|y|^5\right).\end{aligned} \tag{26}$$

The above system is Poincaré normal for for the Liénard system up to five-order terms. Here the bifurcation parameters are $\mu_1 = -c_1$ and $\mu_2 = \delta m_0$. In addition, we have

$$\begin{aligned}a_3 &= -c_3 + 2c_1 \delta^2 m_0 m_2 = -c_3 - 2\mu_1 \mu_2 \delta m_2, \\ b_3 &= -\delta m_1 - 3 c_1 \delta m_2 - \delta^3 m_0^2 m_2 = -\delta m_1 + 3\delta m_2 \mu_1 - \delta m_2 \mu_2^2, \\ a_5 &= 2 c_3 \delta^2 m_0 m_2 - c_1 \delta^4 m_0^2 m_2^2 = 2 c_3 \delta m_2 \mu_2 + \delta^2 m_2^2 \mu_1 \mu_2^2, \\ b_5 &= -3 c_3 \delta m_2 + \delta^3 m_0 m_1 m_2 = -3 c_3 \delta m_2 + \delta^2 m_1 m_2 \mu_2.\end{aligned}$$

At the bifurcation values $-c_1 = 0$ and $\delta m_0 = 0$, we obtain $a_3 = -c_3$, $b_3 = -\delta m_1$, $a_5 = 0$, $b_5 = -3 c_3 \delta m_2$.

Consequently, as $\delta m_1 c_3 \neq 0$, a double-zero bifurcation with symmetry of order two is present in the two-dimensional system associated to the Liénard equation (19).

In the case $\delta m_1 c_3 < 0$, we are in the hypothesis of Theorem 1. Taking into account the expressions of $a_3,\ b_3,\ a_5,\ b_5$, the coefficients involved in the computations in Section 2 are

$$a = -c_3,\ a_{10} = 0,\ a_{01} = 0,\ b = \delta m_1,\ b_{10} = -3\delta m_2,\ b_{01} = 0,\ c = 0,\ d = 3 c_3 \delta m_2.$$

Thus, the equations of second order approximations of the heteroclinic orbits in the variables y_1 and y_2 are obtained as in Theorem 2:

$$\begin{aligned}y_1(t) &= \frac{1}{\sqrt{-c_3}}\left(\varepsilon u_0(\varepsilon t) + \varepsilon^2 u_1(\varepsilon t) + \varepsilon^3 u_2(\varepsilon t)\right), \\ y_2(t) &= \frac{1}{\sqrt{-c_3}}\left(\varepsilon u_0(\varepsilon t) + \varepsilon^2 u_1(\varepsilon t) + \varepsilon^3 u_2(\varepsilon t)\right),\end{aligned} \tag{27}$$

where the functions u_i, v_i have the expressions (9), (11), (12) given in Section 2.

Applying Theorem 3, the curve of second order approximation of the heteroclinic bifurcation values (15) is written as

$$c_3 m_0 + c_1 m_1 \left(\frac{1}{5} + c_1 \theta_2 \right) = 0, \tag{28}$$

where

$$\theta_2 = \frac{1}{13125} \left(\frac{16\delta^2 m_1^2}{c_3^2} + \frac{4500 m_2}{m_1} \right).$$

For, example, consider $c_1 = 1,\ c_3 = -2,\ m_1 = 3,\ m_2 = 4$, and $\delta = 1.5$. Using the first approximation of the curve of heteroclinic bifurcation values, we obtain $m_0 = 0.3$, while using the second approximation (28) we obtain $m_0 = 0.99497$. This last value is very close to the numerically obtained value, namely $m_0 = 1.01464$. For the above values, in [Chen & Chen, 2009] using the hyperbolic perturbation method, it is found the approximated bifurcation value $m_0 = 0.98571$, which is less accurate than our second approximation. It is shown once again that the second approximation is good also far away from the bifurcation point.

5. Conclusions

The present study concerns with the heteroclinic orbits corresponding to a double-zero bifurcation with symmetry of order two. Using the regular perturbation method, we obtained the second order approximations of the curve of heteroclinic bifurcation values (Theorem 3) and of the heteroclinic orbits (Theorem 2). The numerical simulations from Section 3 show the accuracy of our approximations. An application was done in Section 4 for a generalized Liénard equation. Our theoretical results are in good accordance with the numerical ones.

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